A CLASS OF FINSLER METRICS WITH ISOTROPIC MEAN BERWALD CURVATURE

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Abstract. In this paper, we find a condition on \((\alpha, \beta)\)-metrics under which the notions of isotropic \(S\)-curvature, weakly isotropic \(S\)-curvature and isotropic mean Berwald curvature are equivalent.

1. Introduction

The \(S\)-curvature is introduced by Shen for a comparison theorem on Finsler manifolds [8]. Recent studies show that the \(S\)-curvature plays a very important role in Finsler geometry [11, 12]. A Finsler metric \(F\) is said to have isotropic \(S\)-curvature if \(S = (n + 1)cF\), where \(c = c(x)\) is a scalar function on an \(n\)-dimensional manifold \(M\).

Taking twice vertical covariant derivatives of the \(S\)-curvature gives rise the mean Berwald curvature. A Finsler metric \(F\) with vanishing mean Berwald curvature is called weakly Berwald metric. In [1], Bácsó and Yoshikawa studied some weakly Berwald metrics. Also, \(F\) is called to have isotropic mean Berwald curvature if \(E = \frac{n+1}{2}cF^{-1}h\), for some scalar function \(c\) on \(M\), where \(h\) is the angular metric. It is easy to see that every Finsler metric of isotropic \(S\)-curvature is of isotropic mean Berwald curvature. Now, is the equation \(S = (n + 1)cF\) equivalent to the equation \(E = \frac{n+1}{2}cF^{-1}h\)?

Recently, Cheng and Shen proved that a Randers metric \(F = \alpha + \beta\) is of isotropic \(S\)-curvature if and only if it is of isotropic mean Berwald curvature [2]. Then Xiang and Cheng extended this equivalency to the Finsler metric \(F = \alpha^{-m}(\alpha + \beta)^{m+1}\) for every real constant \(m\), including Randers metric [13]. In [7] Lee and Lee proved that this notions are equivalent for the Finsler metrics in the form \(F = \alpha + \alpha^{-1}\beta^2\).

All of above metrics are special Finsler metrics so-called \((\alpha, \beta)\)-metrics. An \((\alpha, \beta)\)-metric is a scalar function on \(TM\) defined by \(F := \alpha\phi(s), s = \beta/\alpha\) where \(\phi = \phi(s)\) is a \(C^\infty\) on \((-b_0, b_0)\) with certain regularity, \(\alpha\) is a Riemannian metric and \(\beta\) is a 1-form on a manifold \(M\). A natural question arises:

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\end{itemize}
Is being of isotropic $S$-curvature equivalent to being of isotropic mean Berwald curvature for $(\alpha, \beta)$-metrics?

In [6] Deng and Wang found the formula of the $S$-curvature of homogeneous $(\alpha, \beta)$-metrics. Then Cheng and Shen classified $(\alpha, \beta)$-metrics of isotropic $S$-curvature [3].

Let $F = \alpha \phi(s)$ be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n$, where $s = \frac{\beta}{n}$. $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. For an $(\alpha, \beta)$-metric, put

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\Delta = 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'',$$
$$\Xi = \frac{(b^2Q + s)\Phi}{\Delta^2}.$$

Using the same method as in [3], we give an affirmative answer to the above question for almost all $(\alpha, \beta)$-metrics. More precisely, we prove the following.

**Theorem 1.1.** Let $F = \alpha \phi(s)$ be an $(\alpha, \beta)$-metric, where $s = \frac{\beta}{n}$. Suppose that $\Xi$ is not constant. Then $F$ is of isotropic $S$-curvature if and only if it is of isotropic mean Berwald curvature.

It is remarkable that if $\Xi = 0$, then $F$ reduces to a Riemannian metric. But, in general, it is still an open problem if Theorem 1.1 is true when $\Xi$ is a constant.

**Example 1.2.** The above mentioned $(\alpha, \beta)$-metric correspond to $\phi = 1 + s$, $\phi = (1 + s)^{m+1}$ and $\phi = 1 + s^2$, respectively. Using a Maple program shows that for all these metrics $\Xi$ is not constant.

## 2. Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an $n$-dimensional manifold $M$. There is a notion of distortion $\tau = \tau(x, y)$ on $TM$ associated with a volume form $dV = \sigma(x)dx$, which is defined by

$$\tau(x, y) = \ln \sqrt{\det(g_{ij}(x, y))} \sigma(x).$$

Then the $S$-curvature is defined by

$$S(x, y) = \frac{d}{dt}\left[\tau(c(t), \dot{c}(t))\right]_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ [5, 10]. From the definition, we see that the $S$-curvature $S(x, y)$ measures the rate of change in the distortion on $(T_xM, F_x)$ in the direction $y \in T_xM$. 
Let \( G = y^r \frac{\partial}{\partial x^r} - 2G^i \frac{\partial}{\partial y^i} \) denote the spray of \( F \) and \( dV_{BH} = \sigma(x) dx \) be the Busemann-Hausdorff volume form on \( M \), where the spray coefficients \( G^i \) are defined by

\[
G^i(x, y) := \frac{1}{4} g^{il}(x, y) \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^j} (x, y) y^k - \frac{\partial [F^2]}{\partial x^i} (x, y) \right\}, \quad y \in T_x M.
\]

Then the S-curvature is given by

\[
S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} (\ln \sigma).
\]

The mean Berwald curvature \( E = E_{ij} dx^i \otimes dx^j \) is given by

\[
E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}.
\]

**Definition 2.1.** Let \((M, F)\) be an \(n\)-dimensional Finsler manifold. Then

(a) \( F \) is of isotropic \( S \)-curvature if \( S = (n + 1) cF \),

(b) \( F \) is of weak isotropic \( S \)-curvature if \( S = (n + 1) cF + \eta \),

(c) \( F \) is of isotropic mean Berwald curvature if \( E = \frac{n+1}{c} F - h \),

where \( c = c(x) \) is a scalar function on \( M \), \( \eta = \eta(x) y^i \) is a 1-form on \( M \) and \( h \) is the angular metric [9].

Consider the \((\alpha, \beta)\)-metric \( F = \alpha \phi \left( \frac{x}{\alpha} \right) \) where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1-form on a manifold \( M \). For an \((\alpha, \beta)\)-metric, put

\[
r_{ij} := \frac{1}{2} (b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2} (b_{ij} - b_{ji}),
\]

\[
r_j := b_j r_{ij}, \quad s_j := b_j s_{ij}, \quad r_{ii} := r_{ij} y^i, \quad s_{ii} := s_{ij} y^i, \quad r_0 := r_{ij} y^i, \quad s_0 := s_{ij} y^i.
\]

Let \( \tilde{G}^i \) denote the spray coefficients of \( \alpha \). We have the following formula for the spray coefficients \( G^i \) of \( F \) [5]:

\[
G^i = \tilde{G}^i + \alpha Q s^i_0 + \Theta \left\{ -2Q \alpha s_0 + r_{00} \right\} \frac{y^i}{\alpha} + \Psi \left\{ -2Q \alpha s_0 + r_{00} \right\} b^i,
\]

where \( s^i_j := a^{ih} s_{hj} \), \( s^i_0 := s^i_j y^j \) and \( r_{00} := r_{ij} y^i y^j \). In [3], Cheng-Shen found the \( S \)-curvature as follows

\[
S = \left\{ 2 \Psi - \frac{f'(b)}{b f(b)} \right\} (r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2 \Delta^2} (r_{00} - 2 \alpha Q s_0),
\]

where

\[
Q = \frac{\phi'}{\phi - s \phi'}, \quad \Delta = 1 + s Q + (b^2 - s^2) Q', \quad \Psi = \frac{Q'}{2 \Delta} \\
\Phi = -(Q - s Q') \left\{ n \Delta + 1 + s Q \right\} - (b^2 - s^2) (1 + s Q) Q''.
\]

Recently, Cheng and Shen characterized \((\alpha, \beta)\)-metrics with isotropic \( S \)-curvature and proved the following.
Lemma 2.2 (3). Let \( F = \alpha \phi(\beta/\alpha) \) be an \((\alpha, \beta)\)-metric on an \(n\)-manifold. Then, \( F \) is of isotropic \(S\)-curvature \( S = (n + 1)cF \), if and only if one of the following holds

(i) \( \beta \) satisfies

\[
    r_{ij} = \varepsilon \left\{ b^2 a_{ij} - b_i b_j \right\}, \quad s_j = 0,
\]

where \( \varepsilon = \varepsilon(x) \) is a scalar function, and \( \phi = \phi(s) \) satisfies

\[
    \Phi = -2(n + 1)k \frac{\phi \Delta^2}{b^2 - s^2},
\]

where \( k \) is a constant. In this case, \( c = k\varepsilon \).

(ii) \( \beta \) satisfies

\[
    r_{ij} = 0, \quad s_j = 0.
\]

In this case, \( c = 0 \).

It is remarkable that Cheng, Wang and Wang proved that the condition \( \Phi = 0 \) characterizes the Riemannian metrics among \((\alpha, \beta)\)-metrics [4]. Hence, in the continue, we suppose that \( \Phi \neq 0 \).

3. Proof of Theorem 1.1

First, we find the formula of mean Berwald curvature of \((\alpha, \beta)\)-metrics. After a long and tedious computation, we obtain the following.

Proposition 3.1. Let \( F = \alpha \phi(\beta/\alpha) \) be an \((\alpha, \beta)\)-metric. Put \( \Omega := \frac{\phi}{\Phi^2} \). Then the mean Berwald curvature of \( F \) is given by the following

\[
    E_{ij} = C_1 b_i b_j + C_2 (b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5 (r_{i0} b_j + r_{j0} b_i) + C_6 (r_{i0} y_j + r_{j0} y_i) + C_7 r_{ij} + C_8 (s_i b_j + s_j b_i) + C_9 (s_i y_j + s_j y_i) + C_{10} (r_i b_j + r_j b_i) + C_{11} (r_i y_j + r_j y_i),
\]

where

\[
    C_1 := \frac{1}{2\alpha^3 \Delta^2} \left\{ \Phi \alpha \Omega'' s_0 + 2\alpha \Delta^2 \Psi''r_0 - \Delta^2 \Omega''r_0 + 2\Delta^2 \alpha \Omega'' Q s_0 + 4\Delta^2 \alpha \Omega' Q s_0 + 2\alpha \Delta^2 \Psi'' s_0 \right\},
\]

\[
    C_2 := -\frac{1}{2\alpha^4 \Delta^2} \left\{ 2\alpha \Delta^2 \Psi'' s_0 - 2\Omega' \Delta^2 r_0 + 2\Omega' \Delta^2 \alpha Q s_0 - \Delta^2 \Omega'' s_0 + 2\Delta^2 \alpha \Omega' Q s_0 + 4\Delta^2 \alpha \Omega' Q' s_0 s + 2\alpha \Delta^2 \Psi' r_0 + 2\alpha \Delta^2 \Psi' s_0 r_0 + 2\alpha \Delta^2 \Psi' s_0 s + \Phi \alpha Q' s_0 + \Phi \alpha Q'' s_0 s \right\},
\]

\[
    C_3 := \frac{1}{4\alpha^5 \Delta^2} \left\{ 4\Delta^2 s^2 Q'' s_0 - 2\Delta^2 s^2 \Omega'' r_0 + 12\alpha \Delta^2 \Psi' s_0 + 12\alpha \Delta^2 s_0 + 4\alpha \Delta^2 \Psi' s_0 r_0 + 4\alpha \Delta^2 \Psi' s_0 s + 8 \Delta^2 s^2 Q' \Omega' s_0 + 2\Phi \alpha Q'' s_0 s^2 - 10\Omega' \Delta^2 s_0 + 12\Omega' \Delta^2 s_0 s_0 + 6\Phi \alpha Q' s_0 s - 3\Phi r_0 \right\},
\]
The formula of mean Berwald curvature of Randers metrics and Kropina metrics computed from Proposition 3.1 coincides with the one computed in [1].

It is easy to see that $F$ is of isotropic mean Berwald curvature if and only if $F$ is of weak isotropic $S$-curvature. Hence, we consider an $(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$ with weak isotropic $S$-curvature, $S = (n+1)cF + \eta$, where $\eta = \eta_t(x)y^t$ is a 1-form on underlying manifold $M$. Using the same method used in [3], one can obtain that the condition that $F$ is of weak isotropic $S$-curvature $S = (n+1)cF + \eta$ is equivalent to the following equation

$$\tag{6} \alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Qs_0) - 2\Psi(r_0 + s_0) = -(n+1)cF + \tilde{\theta},$$

where

$$\tag{7} \tilde{\theta} := -\frac{f'(b)}{b f(b)}(r_0 + s_0) - \eta.$$

To simplify the equation (6), we choose special coordinates $\psi: (s, u^A) \rightarrow (y^i)$ as follows

$$\tag{8} y^1 = \frac{s}{\sqrt{b^2 - s^2} \tilde{\alpha}}, \quad y^A = u^A,$$

where

$$\tilde{\alpha} = \sqrt{\sum_{A=2}^n (u^A)^2}.$$

Then

$$\tag{9} \alpha = \frac{b}{\sqrt{b^2 - s^2} \tilde{\alpha}}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2} \tilde{\alpha}}.$$

Fix an arbitrary point $x$. Take a local coordinate system at $x$ as in (8). We have

$$r_1 = br_{11}, \quad r_A = br_{1A},$$

$$s_1 = 0, \quad s_A = bs_{1A}.$$
Let
\[ r_{10} := \sum_{A=2}^{n} r_{1A} y^A, \quad \bar{s}_{10} := \sum_{A=2}^{n} s_{1A} y^A, \quad \bar{r}_{00} := \sum_{A,B=2}^{n} r_{AB} y^A y^B, \]
\[ \bar{r}_{0} := \sum_{A=2}^{n} r_{A} y^A, \quad \bar{s}_{0} := \sum_{A=2}^{n} s_{A} y^A. \]

Put
\[ \tilde{\theta} = t_i y^i - \eta y^i. \]

Then \( t_i \) are given by
\[ t_1 = -\frac{f'(b)}{f(b)} r_{11}, \quad t_A = -\frac{f'(b)}{f(b)} (r_{1A} + s_{1A}). \]

From (8), we have
\[ r_{0} = \frac{s b r_{11}}{\sqrt{b^2 - s^2}} \bar{\alpha} + b \bar{r}_{10}, \quad s_{0} = \bar{s}_{0} = b \bar{s}_{10}, \]
and
\[ r_{00} = \frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + 2 \frac{s \bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00}, \]
\[ \tilde{\theta} = t_1 \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} - \frac{f'(b)}{f(b)} \bar{r}_{10} - \frac{f'(b)}{f(b)} \bar{s}_{10} - \eta. \]

Substituting (11), (12) and (13) into (6) and by using (9), we find that (6) is equivalent to the following equations:
\[ \frac{\Phi}{2\Delta^2} (b^2 - s^2) \bar{r}_{00} = -\left\{ s \left( \frac{s \Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n + 1) c b^2 \phi - s b t_1 \right\} \bar{\alpha}^2, \]
\[ \left( \frac{s \Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2 Q + s) \frac{\Phi}{\Delta^2} s_{1A} + b \eta_A - b t_A = 0. \]
\[ \eta_1 = 0. \]

Let
\[ \Upsilon := \left[ \frac{s \Phi}{\Delta^2} - 2\Psi b^2 \right]' . \]

We see that \( \Upsilon = 0 \) if and only if
\[ \frac{s \Phi}{\Delta^2} - 2\Psi b^2 = b^2 \mu, \]
where \( \mu = \mu(x) \) is independent of \( s \).

Let us suppose that \( \Xi = \frac{(b^2 Q + s) \phi}{\Delta^2} \) is not constant. Now we shall divide the proof into two cases:

(i) \( \Upsilon = 0 \) and (ii) \( \Upsilon \neq 0 \).
3.1. \( \Upsilon = 0 \). First, note that \( \Upsilon = 0 \) implies that
\[
\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2 \mu,
\]
where \( \mu = \mu(x) \) is a function on \( M \) independent of \( s \). First, we prove the following.

**Lemma 3.2.** Let \((M,F)\) be an \( n \)-dimensional Finsler manifold. Suppose that \( F = \alpha \phi(\beta/\alpha) \) be an \((\alpha,\beta)\)-metric and \( \Upsilon = 0 \). If \( F \) has weak isotropic \( S \)-curvature, \( S = (n + 1)cF + \eta \), then \( \beta \) satisfies
\[
\frac{\Phi}{\Delta^2} = \nu + (k - \epsilon b^2)\mu - (n + 1)c\phi,
\]
where \( \nu = \nu(x) \). If \( s_0 \neq 0 \), then \( \phi \) satisfies the following additional ODE:
\[
\frac{\Phi}{\Delta^2}(Qb^2 + s) = b^2(\mu + \lambda),
\]
where \( \lambda = \lambda(x) \).

**Proof.** Since \( \Phi \neq 0 \) and \( \tilde{r}_{00} \) and \( \tilde{\alpha} \) are independent of \( s \), it follows from (14) and (15) that in a special coordinate system \((s, y^a)\) at a point \( x \), the following relations hold
\[
r_{AB} = k\delta_{AB},
\]
\[
s \left( \frac{s\Phi}{2\Delta^2} - 2\Psi b^2 \right) r_{11} + (n + 1)c\epsilon b^2 \phi + k \frac{\Phi}{2\Delta^2} (b^2 - s^2) = bst_1,
\]
\[
\left( \frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) (r_{1A} + s_{1A}) - (b^2 Q + s) \frac{\Phi}{\Delta^2} s_{1A} - bt_A = -b\eta_A,
\]
where \( k = k(x) \) is independent of \( s \). Let
\[
r_{11} = -(k - \epsilon b^2).
\]
Then (18) holds. By (17), we have
\[
\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 = b^2 \mu - \frac{s\Phi}{2\Delta^2}.
\]
Then (22) and (23) become
\[
b(k - \epsilon s^2) \frac{\Phi}{2\Delta^2} = st_1 + sb\mu(k - \epsilon b^2) - (n + 1)c\epsilon b^2 \phi.
\]
\[
b^2 \mu (r_{1A} + s_{1A}) - \frac{\Phi}{\Delta^2} (Qb^2 + s) s_{1A} - bt_A = -b\eta_A.
Letting $t_1 = bv$ in (24) we get (19). Now, suppose that $s_0 \neq 0$. Rewrite (25) as
\[
\left\{ b^2 \mu - \frac{\Phi}{\Delta^2} (Qb^2 + s) \right\} s_{1A} = bt_A - b\eta_A - b^2 \mu r_{1A}.
\]
We can see that there is a function $\lambda = \lambda(x)$ on $M$ such that
\[
\mu b^2 - \frac{\Phi}{\Delta^2} (Qb^2 + s) = - b^2 \lambda.
\]
This gives (20).

\[\square\]

**Lemma 3.3** ([3]). Let $F = \alpha \phi(\beta/\alpha)$ be an $(\alpha, \beta)$-metric. Assume that
\[
\phi \neq k_1 \sqrt{1 + k_2 s^2 + k_3 s}
\]
for any constants $k_1 > 0, k_2$ and $k_3$. If $\Upsilon = 0$, then $b = \text{constant}$.

An $(\alpha, \beta)$-metric is called Randers-type if $\phi = k_1 \sqrt{1 + k_2 s^2 + k_3 s}$ for any constants $k_1 > 0, k_2$ and $k_3$. Now, we consider the equivalency of the notions weak isotropic $S$-curvature and isotropic $S$-curvature for a non-Randers type $(\alpha, \beta)$-metric.

**Lemma 3.4.** Let $F = \alpha \phi(\beta/\alpha)$ be a non-Randers type $(\alpha, \beta)$-metric. Suppose that $\Xi$ is not constant and $\Upsilon = 0$. Then $F$ is of weak isotropic $S$-curvature if and only if $F$ is of isotropic $S$-curvature.

**Proof.** It is sufficient to prove that if $F$ is of weak isotropic $S$-curvature, then $F$ is of isotropic $S$-curvature. By $db = (r_0 + s_0)/b$ and Lemma 3.3, we have
\[
r_0 + s_0 = 0.
\]
Then by the formula of $S$-curvature of an $(\alpha, \beta)$-curvature, we get
\[
S = -\alpha^{-1} \frac{\Phi}{2\Delta^2} \left\{ r_{00} - 2\alpha Qs_0 \right\}.
\]
By Lemma 3.2,
\[
r_{00} = (k - \varepsilon s^2) \alpha^2 + \frac{2s}{b^2} r_0 \alpha.
\]
Then
\[
S = -(k - \varepsilon s^2) \frac{\Phi}{2\Delta^2} \alpha + \frac{\Phi}{b^2 \Delta^2} (b^2 Q + s) s_0.
\]
By (19), we have
\[
S = -s \left\{ \nu + (k - \varepsilon b^2) \mu \right\} \alpha + \frac{\Phi}{b^2 \Delta^2} (b^2 Q + s) s_0 + (n + 1) \nu \alpha.
\]
Since $S = (n + 1) \nu F + \eta$, then by (26) we obtain the following
\[
- s \left\{ \nu + (k - \varepsilon b^2) \mu \right\} \alpha + \frac{\Phi}{b^2 \Delta^2} (b^2 Q + s) s_0 = \eta.
\]
Letting $y^i = \delta b^i$ for a sufficiently small $\delta > 0$ yields
\[ -\delta \left\{ \nu + (k - \varepsilon b^2)\mu \right\} b^2 = \delta \eta_i b^i. \]

It is easy to see that in the special coordinate $\eta_i b^i = 0$, hence in general $\eta_i b^i = 0$. We conclude that
\[ \nu + (k - \varepsilon b^2)\mu = 0. \]

Then (27) reduces to
\[ \frac{\Xi}{b^2} s_0 = \eta. \]

If $s_0 \neq 0$, then from the last equation, we obtain that $\Xi$ is constant, which is excluded here. Hence, we have $s_0 = 0$. Thus by (29), we conclude that $\eta = 0$ and $F$ has isotropic $S$-curvature $S = (n + 1)cF$.

3.2. $\Upsilon \neq 0$. Here, we consider the case when $\phi = \phi(s)$ satisfies
\[ \Upsilon \neq 0 \]

We need the following two lemmas. The proofs mainly follow the proof of Lemma 6.1 and Lemma 6.2 in [3], respectively. Thus we omit the proofs.

**Lemma 3.5.** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold. Assume that $\Upsilon \neq 0$. Suppose that $F$ has weak isotropic $S$-curvature, $S = (n + 1)cF + \eta$. Then
\[ r_{ij} = k a_{ij} - \varepsilon b_i b_j - \lambda (s_i b_j + s_j b_i), \]
where $\lambda = \lambda(x), k = k(x)$ and $\varepsilon = \varepsilon(x)$ are scalar functions of $x$ and
\[ -2s(k - \varepsilon b^2)\Psi + (k - \varepsilon s^2) \frac{\Phi}{2\Delta^2} + (n + 1)c\phi - s\nu = 0, \]
where
\[ \nu := -\frac{f'(b)}{bf(b)}(k - \varepsilon b^2). \]

If in addition $s_0 \neq 0$, i.e., $s_{A_o} \neq 0$ for some $A_o$, then
\[ -2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda \left( \frac{s\Phi}{\Delta^2} - 2\Psi b^2 \right) = \delta, \]
where
\[ \delta := -\frac{f'(b)}{bf(b)}(1 - \lambda b^2) - \frac{\eta_{A_o}}{s_{A_o}}. \]

**Lemma 3.6.** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric. Suppose that $\phi = \phi(s)$ satisfies (30) and $\phi \neq k_1 \sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, $k_2$ and $k_3$. If $F$ has weak isotropic $S$-curvature, then
\[ r_j + s_j = 0. \]
Proposition 3.7. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric. Suppose that
$\phi = \phi(s)$ satisfies (30) and $\phi \neq k_1\sqrt{1 + k_2s^2 + k_3s}$ for any constants $k_1 > 0$, $k_2$
and $k_3$. Suppose that $\Xi$ is not constant. If $F$ is of weak isotropic $S$-curvature, $S = (n + 1)cF + \eta$, then
\begin{equation}
    r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j), \quad s_j = 0,
\end{equation}
where $\varepsilon = \varepsilon(x)$ is a scalar function on $M$ and $\phi = \phi(s)$ satisfies
\begin{equation}
    \varepsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = -(n + 1)c\phi.
\end{equation}

Proof. Contracting (31) with $b^i$ yields
\begin{equation}
    r_j + s_j = (k - \varepsilon b^2)b_j + (1 - \lambda b^2)s_j.
\end{equation}
By Lemma 3.6, $r_j + s_j = 0$. It follows from (38) that
\begin{equation}
    (1 - \lambda b^2)s_j + (k - \varepsilon b^2)b_j = 0.
\end{equation}
Contracting (39) with $b^j$ yields
\begin{equation}
    (k - \varepsilon b^2)b^2 = 0.
\end{equation}
We get
\begin{equation}
    k = \varepsilon b^2.
\end{equation}
Then (31) is reduced to
\begin{equation}
    r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j) - \lambda(b_is_j + b_js_i).
\end{equation}
By (33),
\begin{equation}
    \nu = 0.
\end{equation}
Then (32) is reduced to (37).
We claim that $s_0 = 0$. Suppose that $s_0 \neq 0$. By (39), we conclude that
\begin{equation}
    \lambda = \frac{1}{b^2}.
\end{equation}
By (35),
\begin{equation}
    \delta = -\frac{\eta A_o}{s A_o}.
\end{equation}
It follows from (34) that
\begin{equation}
    \frac{(b^2Q + s)\Phi}{\Delta^2} = \frac{bn A_o}{s A_o},
\end{equation}
which implies that $\Xi$ is constant. This is impossible by the assumption on
non-constancy of $\Xi$. Therefore, $s_j = 0$. This completes the proof. \qed

By Proposition 3.7 and Lemma 2.2, we have the following.

Corollary 3.8. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type $(\alpha, \beta)$-metric.
Suppose that $\Upsilon \neq 0$ and $\Xi$ is not constant. Then $F$ is of weak isotropic $S$-

curvature, if and only if it is of isotropic $S$-curvature.
REFERENCES


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