SOME $L_p$ INEQUALITIES FOR THE FAMILY OF B-OPERATORS

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Abstract. Let $P_n$ be the class of polynomials of degree at most $n$ and $B_n$ be a class of operators that map $P_n$ into itself. For every $P \in P_n$ and $B \in B_n$, we investigate on $|z| = 1$, the dependence of $||B[P(R \cdot)] - B[P(r \cdot)]||_q$ on $||P||_q$, for every $R > r \geq 1$ and $q \geq 1$.

1. Introduction

Let $P_n$ be the class of polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most $n$ with complex coefficients. For $P \in P_n$, define

$$||P||_q := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q} \quad \text{and} \quad ||P||_{\infty} := \max_{|z|=1} |P(z)|.$$

Rahman [6] (see also Rahman and Schmeisser [8, p. 538]) introduced a class $B_n$ of operators $B$ that map $P \in P_n$ into itself. That is, the operator $B$ carries $P \in P_n$ into

(1)  
$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left( \frac{n z}{2} \right) \frac{P'(z)}{1!} + \lambda_2 \left( \frac{n z}{2} \right)^2 \frac{P''(z)}{2!},$$

where $\lambda_0$, $\lambda_1$ and $\lambda_2$ are real or complex numbers such that all the zeros of

(2)  
$$U(z) := \lambda_0 + C(n, 1) \lambda_1 z + C(n, 2) \lambda_2 z^2, \quad C(n, r) = \frac{n!}{r!(n-r)!},$$

lie in the half plane

(3)  
$$|z| \leq \left| z - \frac{n}{2} \right|$$

and observed:

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Theorem 1.1. If $P \in P_n$, then for $|z| \geq 1$,
\begin{equation}
|B[P]|_\infty \leq |B[E_n]|_\infty |P|_\infty,
\end{equation}
where $E_n(z) := z^n$.

As an improvement of (4), Shah and Liman [9] proved the following:

Theorem 1.2. If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $|z| \geq 1$
\begin{equation}
|B[P]|_\infty \leq \frac{1}{2} \left\{ |B[E_n]| + |\lambda_0| \right\} |P|_\infty.
\end{equation}
The result is sharp and equality holds for a polynomial whose all zeros lie on the unit disk.

Recently, Shah and Liman [10] extended the above results to the $L_p$ norm by proving the following more general results:

Theorem 1.3. If $P \in P_n$, then for every $R \geq 1$, $q \geq 1$ and $|z| = 1,$
\begin{equation}
B[P(R \cdot)]_q \leq |B[E_n(R \cdot)]|_q |P|_q,
\end{equation}
where $B \in B_n$ and $E_n(z) := z^n$. The result is best possible and equality holds for $P(z) = \alpha z^n, \alpha \neq 0$.

Theorem 1.4. Let $P \in P_n$ be such that $P(z) \neq 0$ in $|z| < 1$, then for every $R \geq 1$, $q \geq 1$ and $|z| = 1$,
\begin{equation}
B[P(R \cdot)]_q \leq \frac{|B[E_n(R \cdot)]|_q + |\lambda_0|_q}{1 + E_n}_q |P|_q,
\end{equation}
where $B \in B_n$ and $E_n(z) := z^n$. The result is best possible and equality holds for the polynomial $P(z) = \alpha z^n + \beta$, where $|\alpha| = |eta|$.  

2. Statement and Proof of Results

For the proofs of these theorems, we need the following lemmas. The first lemma is a special case of a result due to Aziz and Zargar [2].

Lemma 2.1. If $P(z)$ is a polynomial of degree $n$ having all zeros in $|z| \leq 1$, then for $R > r \geq 1$ and $|z| = 1$,
\begin{equation}
|P(Rz)| > |P(rz)|.
\end{equation}

The next lemma follows from Corollary 18.3 of [5, p.86].

Lemma 2.2. If all the zeros of a polynomial $P(z)$ of degree $n$ lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[P](z)$ also lie in the circle $|z| \leq 1$.

Lemma 2.3. If $P(z)$ is a polynomial of degree $n$ having all zeros in $|z| \geq 1$ and $Q(z) = z^n P \left( \frac{1}{z} \right)$, then for every real or complex number $\alpha$ with $|\alpha| \leq 1$ and $R > r \geq 1$,
\begin{equation}
|B[P(R \cdot)] - \alpha B[P(r \cdot)]| \leq |B[Q(R \cdot)] - \alpha B[Q(r \cdot)]|.
\end{equation}
The result is trivial if $R = r$. So we assume that $R > r$. Since $P(z)$ has all zero in $|z| \geq 1$, therefore all the zeros of $Q(z) = z^n P(1/z)$ lie in $|z| \leq 1$. By maximum modulus principle, $|Q(z)| \leq |P(z)|$ for $|z| \leq 1$, and in particular, $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. By Rouches theorem, it follows that for $\alpha$ with $|\alpha| \leq 1$ all the zeros of $F(z) = P(z) - \beta Q(z)$ lie in $|z| \leq 1$, for every $\beta$ with $|\beta| > 1$. Applying Lemma 2.1 to $F(z)$, we get for $|z| = 1, R > r \geq 1$

$$|F(rz)| < |F(Rz)|.$$ 

Since all the zeros of $F(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, by Rouches theorem it follows that all the zeros of

$$F(Rz) - \alpha F(rz)$$

lie in $|z| < 1$. Since $B$ is a linear operator (see [6, sec. 5]), it follows by Lemma 2.2, that all zeros of

$$H(z) := B[F(Rz) - \alpha F(rz)]$$

$$= \{B[P(Rz)] - \alpha B[P(rz)] \} - \beta \{B[Q(Rz)] - \alpha B[Q(rz)]\}$$

lie in $|z| < 1$. This gives, for $|z| \geq 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |B[Q(Rz)] - \alpha B[Q(rz)]|.$$ 

For, if this is not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$, such that

$$|B[P(Rz)] - \alpha B[P(rz)]|_{z=z_0} > |B[Q(Rz)] - \alpha B[Q(rz)]|_{z=z_0}.$$ 

We take

$$\beta = \frac{\{B[P(Rz)] - \alpha B[P(rz)]\}_{z=z_0}}{\{B[Q(Rz)] - \alpha B[Q(rz)]\}_{z=z_0}},$$

so that $|\beta| > 1$. With this value of $z$, $H(z) = 0$, for $|z| \geq 1$. This is a contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Hence the proof of lemma is complete.

\textbf{Lemma 2.4.} If $P \in \mathcal{P}_n$, then for every $\alpha$ with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$|B[P(R \cdot)] - \alpha B[P(r \cdot)]| \leq |B[E_n(R \cdot) - \alpha B[E_n(r \cdot)]]| ||P||_\infty.$$ 

\textbf{Proof.} Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. By Rouches theorem, it follows that all the zeros of polynomial $F(z) = P(z) - \zeta z^n M$ lie in $|z| < 1$ for every real or complex number $\zeta$ with $|\zeta| > 1$. Therefore, it follows from Lemma 2.1, that for $R > r \geq 1$, and $|z| = 1$,

$$|F(rz)| < |F(Rz)|.$$ 

Since all the zeros of polynomial $F(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, again making use of Rouches theorem, we conclude that all the zeros of the polynomial $F(Rz) - \alpha F(rz)$ lie in $|z| < 1$, for every real or complex number $\alpha$ with $|\alpha| \leq 1$. By Lemma 2.2, the polynomial

$$B[F(Rz) - \alpha F(rz)] = (B[P(Rz)] - \alpha B[P(rz)]) - \zeta (R^n - \alpha r^n) B[z^n] M,$$
has all the zeros in open unit disc for every real or complex number $\zeta$ with $|\zeta| > 1$. This implies similarly, as in the case of Lemma 2.3, for $|z| \geq 1$ and $R > r \geq 1$,

\[(11) \quad |B[P(Rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]|M.\]

This gives the desired result. \hfill \Box

**Lemma 2.5.** Let $P_n$ denote the linear space of polynomials

$$P(z) = a_0 + \cdots + a_n z^n$$

of degree $n$ with complex coefficients, normed by $\|P\| = \max |P(e^{i\theta})|$, $0 < \theta \leq 2\pi$. Define the linear functional $\mathcal{L}$ on $P_n$ as

$$\mathcal{L}: P \mapsto l_0 a_0 + l_1 a_1 + \cdots + l_n a_n,$$

where $l_j$'s are complex numbers. If the norm of the functional is $N$, then

\[(12) \quad Z^2 \sum_{k=0}^{n} |a_k e^{ik\theta}| d\theta \leq \int_{0}^{2\pi} \Theta \left( \frac{\sum_{k=0}^{n} a_k e^{ik\theta}}{N} \right) d\theta,
\]

where $\Theta(t)$ is a non-decreasing convex function of $t$.

The above lemma is due to Rahman [6].

In this paper, we prove some results which generalize the above theorems and thereby obtain compact generalizations of many polynomial inequalities as well. In fact, we prove:

**Theorem 2.6.** If $P \in P_n$, then for every $\alpha$, with $|\alpha| \leq 1$, $R > r \geq 1$, $q \geq 1$ and $|z| = 1$,

$$\|B[P(Rz)] - \alpha B[P(rz)]\|_{q} \leq |B[E_{n}(R z) - \alpha B[E_{n}(r z)]| \|P\|_{q}$$

The result is best possible and equality holds for $P(z) = az^n$, $a \neq 0$.

**Proof.** Let $M = \max_{|z|=1} |P(z)|$, then by Lemma 2.4, we have, for $|z| \geq 1$ and $R > r \geq 1$,

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]|M. \quad \text{(14)}$$

This in particular gives for every $\theta, 0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]| \leq |R^n - \alpha r^n| \left| \lambda_0 + \frac{n^2}{2} \lambda_1 + \frac{n^3(n-1)}{8} \lambda_2 \right| M. \quad \text{(15)}$$

Since $B$ is a linear operator (see [6, sec. 5]), therefore

$$\Lambda = B[P(Rz)] - \alpha B[P(rz)]$$

is a bounded linear operator on $P_n$. Thus in view of (15), the norm of the bounded linear functional

$$\mathcal{L}: P \to \{B[P(Rz)] - \alpha B[P(rz)]\}_{\theta=0}$$
is
\[ |R^α - α^n| \left( \lambda_0 + \frac{n^2}{2}λ_1 + \frac{n^3(n - 1)}{8}λ_2 \right). \]

Hence by Lemma 2.5, for every $q \geq 1$, we have,
\[
\int_0^{2π} |B[P(Re^{iθ})] - αB[P(re^{iθ})]|^q dθ \leq \left|R^n - α^n \right| \left( \lambda_0 + \frac{n^2}{2}λ_1 + \frac{n^3(n - 1)}{8}λ_2 \right) \int_0^{2π} |P(e^{iθ})|^q dθ.
\]

From this inequality, (13) follows immediately and this completes the proof of Theorem 2.6.

Remark 2.7. For $α = 0$, Theorem 2.6 reduces to Theorem 1.3.

The following corollary immediately follows from Theorem 2.6, when we let $q \to ∞$.

\textbf{Corollary 2.8.} If $P \in \mathcal{P}_n$, then for every real or complex number $α$ with $|α| \leq 1$, $R > r \geq 1$ and $|z| = 1$,
\[
||B[P(R \cdot)] - αB[P(r \cdot)]||_∞ \leq ||B[E_n(R \cdot) - αB[E_n(r \cdot)]|| ||P||_∞.
\]

Or, equivalently,
\[
(16) \quad ||B[P(R \cdot)] - αB[P(r \cdot)]||_∞ \leq |R^n - α^n| \left( \lambda_0 + \frac{n^2}{2}λ_1 + \frac{n^3(n - 1)}{8}λ_2 \right) ||P||_∞.
\]

The result is best possible and equality holds for $P(z) = az^n$, $a ≠ 0$.

Remark 2.9. Theorem 1.1 is a special case of Corollary 2.8, when we take $α = 0$.

Also, If we choose $α = 0$ and $λ_0 = 0 = λ_2$ in (16), which is possible, as it can be easily verified that in this case all the zeros of $U(z)$ defined by (2) lie in (3), we get,

\textbf{Corollary 2.10.} If $P \in \mathcal{P}_n$, then for every $R \geq 1$, $q \geq 1$ and $|z| = 1$,
\[
||P'||_q \leq nR^{n-1}||P||_q.
\]

This in particular for $R = 1$, gives,
\[
||P'||_q \leq n||P||_q, \text{ for } q \geq 1.
\]

which is an inequality due to Zygmund [11].
Lemma 2.11. If $P \in \mathcal{P}_n$, then for every $\alpha$ with $|\alpha| \leq 1$, $R > r \geq 1$, $q \geq 1$ and $0 \leq \theta, \beta < 2\pi$, 

\[
(17) \quad \int_0^{2\pi} \int_0^{2\pi} \left| (B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]) + e^{i\beta} (B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]) \right|^q d\theta d\beta 
\leq 2\pi \left| B[E_n(R \cdot) - \alpha B[E_n(r \cdot)] \right| + |1 - \alpha||\lambda_0|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.
\]

Proof. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. By Rouches theorem, it follows that all the zeros of polynomial $F(z) = P(z) - \zeta M$ lie in $|z| \geq 1$, for every real or complex number $\zeta$ with $|\zeta| > 1$. Applying Lemma 2.3 to the polynomial $F(z)$ and using the fact that $B$ is a linear operator, it follows that for every real or complex number $\alpha$ with $|\alpha| \leq 1$, $R > r \geq 1$, 

\[
(18) \quad |B[F(Rz)] - \alpha B[F(rz)]| \leq |B[G(Rz)] - \alpha B[G(rz)]|
\]

for $|z| \geq 1$, where 

$$G(z) = z^n F(1/\bar{z}) = Q(z) - z^n \bar{\zeta} M.$$ 

Using the fact that $B[1] = \lambda_0$, we get from (18), 

\[
(19) \quad |B[P(Rz)] - \alpha B[P(rz)] - \zeta(1 - \alpha)\lambda_0 M| 
\leq |B[Q(Rz)] - \alpha B[Q(rz)] - \bar{\zeta}(R^n - \alpha^n)B[z^n]M|.
\]

Now choosing argument of $\zeta$ such that 

$$|B[Q(Rz)] - \alpha B[Q(rz)] - \bar{\zeta}(R^n - \alpha^n)B[z^n]M|$$

is possible by (9), we get from (19), for $|\zeta| > 1$ and $|z| \geq 1$, 

\[
|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]| 
\leq |\zeta||R^n - \alpha^n||B[z^n] | + |1 - \alpha||\lambda_0| \max_{|z|=1} |P(z)|.
\]

Letting $|\zeta| \rightarrow 1$, we obtain 

$$|B[P(Rz)] - \alpha B[P(rz)]| + |B[Q(Rz)] - \alpha B[Q(rz)]|$$

\leq (|R^n - \alpha^n||B[z^n] | + |1 - \alpha||\lambda_0|) \max_{|z|=1} |P(z)|.
\]

This in particular gives for every $\theta$, $0 \leq \theta < 2\pi$ and $R > r \geq 1$, 

$$|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]| + |B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]|$$

\leq (|R^n - \alpha^n||B[e^{im\theta}] | + |1 - \alpha||\lambda_0|) \max_{|z|=1} |P(z)|.
\]

Thus for every $\beta$ with $0 \leq \beta < 2\pi$, we have
If \( (20) \)

Therefore, by Lemma 2.5, it follows that

is a bounded linear operator on \( \mathcal{P}_n \). From this the desired result follows.

Next we prove:

**Theorem 2.12.** If \( P \in \mathcal{P}_n \) and \( P(z) \neq 0 \) in \( |z| < 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \), \( q \geq 1 \), \( R > r \geq 1 \) and \( |z| = 1 \),

\[
\| B[P(R \cdot)] - \alpha B[P(r \cdot)] \|_q \leq \frac{|B[E_n(R \cdot)] - \alpha B[E_n(r \cdot)]| + |1 - \alpha||\lambda_0|}{\|1 + E_n\|_q} \|P\|_q.
\]

Or, equivalently,

\[
(21) \quad \| B[P(R \cdot)] - \alpha B[P(r \cdot)] \|_q \\
\leq \frac{|R^n - \alpha r^n| |\lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}| + |1 - \alpha||\lambda_0|}{\|1 + E_n\|_q} \|P\|_q,
\]

where \( B \in \mathcal{B}_n \). The result is sharp and equality holds for a polynomial \( P(z) = az^n + b, |a| = |b| \).
Proof. Since $P(z) \neq 0$ in $|z| < 1$, by Lemma 2.3, we have for each $\theta$, $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]| \leq |B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]|.$$  

Also for every real $\theta$ and $t \geq 1$, it can be easily verified that $|1 + te^{i\theta}| \geq |1 + e^{i\theta}|$ and therefore for every $q \geq 1$,

$$\int_0^{2\pi} |1 + te^{i\theta}|^q d\theta \geq \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta. \tag{22}$$

Now, taking $t = \frac{|B[Q(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|}{|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|} \geq 1$ and using inequality (22), we have

$$\int_0^{2\pi} \int_0^{2\pi} |(B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]) + e^{i\beta}(B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})])|^q d\beta d\theta$$

$$= \int_0^{2\pi} \int_0^{2\pi} |B(P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q \times$$

$$\times |1 + e^{i\beta}B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]|^q \times$$

$$\times |B(P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q d\beta d\theta$$

$$\geq \int_0^{2\pi} \left\{ |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q \int_0^{2\pi} |1 + e^{i\beta}|^q d\beta \right\} d\theta$$

$$= \int_0^{2\pi} |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q d\theta \int_0^{2\pi} |1 + e^{i\beta}|^q d\beta.$$

Inequality (23) in conjunction with Lemma 2.11, gives

$$\int_0^{2\pi} |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q d\theta$$

$$\leq 2\pi \left[ |R^n - \alpha r^n| \lambda_0 + \frac{n^2}{2} \lambda_1 + \frac{n^3(n-1)}{8} \lambda_2 + |1 - \alpha||\lambda_0| \right]^q \int_0^{2\pi} |1 + e^{i\beta}|^q d\beta \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

Or, equivalently,

$$\|B[P(R \cdot)] - \alpha B[P(r \cdot)]\|_q \leq \frac{|B[E_n(R \cdot) - \alpha B[E_n(r \cdot)]| + |1 - \alpha||\lambda_0|}{1 + E_n} \|P\|_q.$$  

This completes the proof of Theorem 2.12. \hfill \square

Remark 2.13. If we choose $\alpha = 0$ in (21), we obtain Theorem 1.4. Also Theorem 1.2 easily follows from Theorem 2.12, if we make $\alpha = 0$ and $q \rightarrow \infty$. 
Further, if we choose \( \alpha = 0 \) and \( \lambda_0 = \lambda_2 = 0 \), \( R = 1 \) in (21) which is possible, we get the following inequality:

\[
\|P'\|_q \leq \frac{n}{\|1 + z^n\|_q} \|P\|_q,
\]

for every \( q \geq 1 \), which is a result due to de Bruijn [3]. On the other hand, for \( \alpha = 0 \) and \( \lambda_1 = \lambda_2 = 0 \), we have the following:

If \( P \in \mathcal{P}_n \) be such that \( P(z) \neq 0 \) in \( |z| < 1 \), then for every \( R > 1 \), \( q \geq 1 \) and \( |z| = 1 \),

\[
\|P(R \cdot)\|_q \leq \frac{R^n + 1}{\|1 + z^n\|_q} \|P\|_q.
\]

An inequality proved by Ankeny and Rivlin [1] is a special case of this inequality when we let \( q \to \infty \). Also for \( q \to \infty \), Theorem 2.12 yields the following:

**Corollary 2.14.** If \( P \in \mathcal{P}_n \) and \( P(z) \neq 0 \) in \( |z| < 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \), \( R > r \geq 1 \) and \( |z| = 1 \),

\[
\|B[P(R \cdot) - \alpha B(P(r \cdot))]\|_q \leq \frac{[B[E_n(R \cdot) - \alpha B(E_n(r \cdot))] + |1 - \alpha||\lambda_0|]}{2} \|P\|_\infty,
\]

where \( B \in \mathcal{B}_n \). The result is sharp and equality holds for a polynomial \( P(z) = \alpha z^n + b \), \( |\alpha| = |b| \).

If we choose \( r = 1 \), \( \lambda_1 = 0 = \lambda_2 \) in (21), we get the following:

**Corollary 2.15.** If \( P \in \mathcal{P}_n \) and \( P(z) \neq 0 \) in \( |z| < 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \), \( R > r \geq 1 \) and \( |z| = 1 \),

\[
\|P(R \cdot) - \alpha P\|_q \leq \frac{|R^n - \alpha| + |1 - \alpha|}{\|1 + z^n\|_q} \|P\|_q.
\]

This is a compact generalization of a result of Shah and Liman [10, Corollary 1].

A polynomial \( P(z) \) is said to be self-inversive if \( P(z) = u Q(z) \), \( |u| = 1 \), where \( Q(z) = z^n P(1/z) \). It is known [4] that, if \( P \in \mathcal{P}_n \) is a self-inversive polynomial, then for every \( q \geq 1 \),

\[
\|P'\|_q \leq \frac{n}{\|1 + z^n\|_q} \|P\|_q.
\]

We next present the following result for the class of self-inversive polynomials:

**Theorem 2.16.** If \( P \in \mathcal{P}_n \) is self-inversive, then for every \( q \geq 1 \), \( R > r \geq 1 \) and \( |z| = 1 \),

\[
\|B[P(R \cdot)] - \alpha B[P(r \cdot)]\|_q \leq \frac{[B[E_n(R \cdot) - \alpha B(E_n(r \cdot))] + |1 - \alpha||\lambda_0|]}{\|1 + E_n\|_q} \|P\|_q.
\]
The result is sharp and equality holds for \( P(z) = z^n + 1 \).

\textbf{Proof.} Since \( P(z) \) is a self inversive polynomial, therefore for all \( z \in C, |z| \geq 1 \), we have
\[
|B[P(Rz)] - \alpha B[P(rz)]| = |B[Q(Rz)] - \alpha B[Q(rz)]|.
\]
This in particular gives, for \( 0 \leq \theta < 2\pi \),
\[
|B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]| = |B[Q(Re^{i\theta})] - \alpha B[Q(re^{i\theta})]|.
\]
Proceeding similarly as in the case of Theorem 2.12, we get
\[
\int_0^{2\pi} |B[P(Re^{i\theta})] - \alpha B[P(re^{i\theta})]|^q d\theta
\leq \frac{2\pi \left[ |R^n - r^n| \left| \lambda_0 + \frac{n^2}{2} \lambda_1 + \frac{n^3(n-1)}{6} \lambda_2 \right| + |1 - \alpha||\lambda_0| \right]^q}{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta}.
\]
Or, equivalently,
\[
\|B[P(R \cdot)] - \alpha B[P(r \cdot)]\|_q \leq \frac{|B[E_n(R \cdot) - \alpha B[E_n(r \cdot)]| + |1 - \alpha||\lambda_0|}{\|1 + E_n\|_q} \|P\|_q.
\]
Hence the result is proved. \( \square \)

The above inequality of Dewan and Govil [4] and many such results follow as special cases from Theorem 2.16.

Further, if we make \( q \to \infty \) in inequality (25), we get the following:

\textbf{Corollary 2.17.} If \( P \in \mathcal{P}_n \) is self inversive, then for every \( R > r \geq 1 \), and \( |z| = 1 \),
\[
\|B[P(R \cdot)] - \alpha B[P(r \cdot)]\|_\infty \leq \frac{|B[E_n(R \cdot) - \alpha B[E_n(r \cdot)]| + |1 - \alpha||\lambda_0|}{2} \|P\|_\infty,
\]
where \( B \in \mathcal{B}_n \). The result is sharp and equality holds for a polynomial \( P(z) = z^n + 1 \).

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