COMPLETELY PRIME IDEAL RINGS AND THEIR EXTENSIONS

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Abstract. Let $R$ be a ring and let $I \neq R$ be an ideal of $R$. Then $I$ is said to be a completely prime ideal of $R$ if $R/I$ is a domain and is said to be completely semiprime if $R/I$ is a reduced ring.

In this paper, we introduce a new class of rings known as completely prime ideal rings. We say that a ring $R$ is a completely prime ideal ring (CPI-ring) if every prime ideal of $R$ is completely prime. We say that a ring $R$ is a near completely prime ideal ring (NCPI-ring) if every minimal prime ideal of $R$ is completely prime. We say that a ring $R$ is an almost completely prime ideal ring (ACPI-ring) if every associated prime ideal of $R$ ($R$ viewed as a right module over itself) is completely prime.

Let now $R$ be a Noetherian ring which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers) and $\delta$ a derivation of $R$. Then we prove the following:

1. $R$ is a near completely prime ideal ring if and only if $R[x; \delta]$ is a near completely prime ideal ring.
2. $R$ is an almost completely prime ideal ring if and only if $R[x; \delta]$ is an almost completely prime ideal ring.

1. Introduction

We follow notation as in Bhat [3] but to make the paper self contained, we have the following:

Notation. A ring $R$ means an associative ring with identity $1 \neq 0$, and any $R$-module unitary. $\mathbb{R}$ denotes the field of real numbers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless other wise stated. Let $R$ be a ring. The set of prime ideals of $R$ is denoted by $\text{Spec}(R)$, the set of associated prime ideals of $R$ (where $R$ is viewed as a right module over itself) is denoted by $\text{Ass}(R_R)$, the

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set of minimal prime ideals of $R$ is denoted by $\text{Min}\text{-Spec}(R)$ and the set of completely prime ideals of $R$ is denoted by $\text{C-Spec}(R)$. Let $K$ be an ideal of a ring $R$ such that $\sigma^m(K) = K$ for some integer $m \geq 1$, we denote $\cap_{i=1}^m \sigma^i(K)$ by $K^0$.

Let $R$ be a ring, $\sigma$ an automorphisms of $R$ and $\delta$ a $\sigma$-derivation of $R$; i.e. $\delta: R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example for any endomorphism $\sigma$ of a ring $R$ and for any $a \in R$, $\varrho: R \to R$ defined as $\varrho(r) = ra - a\sigma(r)$ is a $\sigma$-derivation of $R$.

By a $\sigma$-derivation we mean a right $\sigma$-derivation. We note that for a left $\sigma$-derivation of $\delta$ of $R$, $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$.

We recall that the Ore extension

$$R[x; \sigma, \delta] = \{ f = \sum x^i a_i, \quad a_i \in R, \quad 0 \leq i \leq n \}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We would like to mention that we take coefficients of the polynomials on the right as in McConnell and Robson [12]. We denote $R[x; \sigma, \delta]$ by $O(R)$. If $I$ is an ideal of $R$ such that $I$ is $\sigma$-stable (i.e. $\sigma(I) = I$) and is also $\delta$-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x; \sigma, \delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$.

In case $\sigma$ is the identity map, we denote the ring of differential operators $R[x; \delta]$ by $D(R)$. If $J$ is an ideal of $R$ such that $J$ is $\delta$-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x; \delta]$ is an ideal of $D(R)$, and we denote it as usual by $D(J)$.

In case $\delta$ is the zero map, we denote $R[x; \sigma]$ by $S(R)$. If $K$ is an ideal of $R$ such that $K$ is $\sigma$-stable (i.e. $\sigma(K) = K$), then clearly $K[x; \sigma]$ is an ideal of $S(R)$, and we denote it as usual by $S(K)$.

**Completely prime ideals.** Study of prime ideals in Ore extensions has been an area of active research in recent past. For more details the reader is referred to S. Annin [1], Carl Faith [6], Gabriel [8], Goodearl and Warfield [9], Leroy and Matczuk [11], H. Nordstrom [14], Bhat [3].

We shall now discuss some more types of prime ideals; i.e. completely prime ideals and minimal prime ideals.

Recall that an ideal $P$ of a ring $R$ is completely prime if $R/P$ is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$ (McCoy [13]). In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring $R$ is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

**Example 1.1** (Bhat [3]). Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If $p$ is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of $R$, but is not completely prime,
since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

A relation between the completely prime ideals of a ring $R$ and those of $O(\mathbb{R})$ has been given in [3, Theorem 2.4.] as follows.

**Theorem** (Bhat [3]). Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then

1. For any completely prime ideal $P$ of $R$ with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P) = P[x; \sigma, \delta]$ is a completely prime ideal of $O(\mathbb{R})$.
2. For any completely prime ideal $U$ of $O(R)$, $U \cap R$ is a completely prime ideal of $R$.

**Minimal prime ideals.** Towards minimal prime ideals and completely prime ideals of a ring, J. Krempa [10, Theorem 2.2.] has proved the following:

**Theorem** (Krempa [10]). For a ring $R$ the following conditions are equivalent:

1. $R$ is reduced.
2. $R$ is semiprime and all minimal prime ideals of $R$ are completely prime.
3. $R$ is a subdirect product of domains.

Towards the minimal prime ideals of $R[x; \delta]$, the following has been proved by Krempa [10, Theorem 3.1.]:

**Theorem** (Krempa [10]). Let $R$ be a reduced ring and let $\delta$ be a derivation of $R$. Then

1. The differential operator ring $R[x; \delta]$ is reduced.
2. Any annihilator and any minimal prime ideal of $R$ is $\delta$-invariant.
3. Any minimal prime ideal in $R[x; \delta]$ is of the form $P[x; \delta]$ where $P$ is a minimal prime ideal in $R$.

**Completely Prime Ideal Rings (CPI-rings).** In this paper we introduce a new class of rings called completely prime ideal rings (CPI-rings) as follows:

**Definition 1.2.** Let $R$ be a ring. We say that $R$ is a completely prime ideal ring (CPI-ring) if every prime ideal of $R$ is completely prime.

**Definition 1.3.** Let $R$ be a ring. We say that $R$ is a near completely prime ideal ring (NCPI-ring) if every minimal prime ideal of $R$ is completely prime.

For example a reduced ring is a near completely primal ring.

**Definition 1.4.** Let $R$ be a ring. We say that $R$ is an almost completely prime ideal ring (ACPI-ring) if every associated prime ideal of $R$ ($R$ viewed as a right module over itself) is completely prime.
Our aim is to find the relation between completely prime ideal rings (CPI-rings) (near completely prime ideal rings (NCPI-rings), almost completely prime ideal rings (ACPI-rings)) and their extensions. It is known that if \( P \) is a prime ideal of a ring \( R \), then \( P[x] \) is a prime ideal of \( R[x] \) (Brewer and Heinzer [5]).

It is known (Lemma 1.6 of Ferrero [7]) that for any ring \( R \), an ideal \( P \) of \( R[x] \) is prime if and only if \( P \cap R \) is a prime ideal of \( R \) and

(1) either \( P = (P \cap R)[x] \)

(2) or \( P \) is maximal amongst ideals \( I \) of \( R[x] \) such that \( I \cap R = P \cap R \).

Let \( R \) be ring satisfying (1) above. Then, in Theorem (3.1), we prove the following:

\( R \) is a CPI-ring if and only if \( R[x] \) is a CPI-ring.

Let \( R \) be a Noetherian \( \mathbb{Q} \)-algebra and \( \delta \) a derivation of \( R \). It is known that if \( U \) is a minimal prime ideal (associated prime ideal) of a ring \( R \), then \( U[x; \delta] \) is a minimal prime ideal (associated prime ideal) of \( R[x; \delta] \). Conversely for any minimal prime ideal (associated prime ideal) \( P \) of \( R[x; \delta] \), there exists a minimal prime ideal (associated prime ideal) \( V \) of \( R \) such that \( P = V[x; \delta] \).

In case of associated prime ideals a ring is viewed as a right module over itself (Bhat [4, Theorem 3.7]).

Let \( R \) be a Noetherian ring which is also an algebra over \( \mathbb{Q} \) and \( \delta \) a derivation of \( R \). Using the above facts, in Theorem (3.3), we prove the following concerning near completely primal rings and almost completely primal rings:

\( R \) is an NCPI-ring if and only if \( R[x; \delta] \) is an NCPI-ring. Moreover, in Theorem (3.5), we show that \( R \) is an ACPI-ring if and only if \( R[x; \delta] \) is an ACPI-ring.

2. Preliminaries

We begin with the following known results:

**Lemma 2.1.** Let \( R \) be a ring and \( \sigma \) an automorphism of \( R 

(1) If \( P \) is a prime ideal of \( S(R) \) such that \( x \notin P \), then \( P \cap R \) is a prime ideal of \( R \) and \( \sigma(P \cap R) = P \cap R \).

(2) If \( U \) is a prime ideal of \( R \) such that \( \sigma(U) = U \), then \( S(U) \) is a prime ideal of \( S(R) \) and \( S(U) \cap R = U \).

**Proof.** The proof follows on the same lines as in the lemma of McConnell and Robson [10, Lemma (10.6.4)]. \( \square \)

**Lemma 2.2.** Let \( R \) be a commutative Noetherian \( \mathbb{Q} \)-algebra. Let \( \delta \) be a derivation of \( R 

(1) If \( P \) is a prime ideal of \( D(R) \), then \( P \cap R \) is a prime ideal of \( R \) and \( \delta(P \cap R) \subseteq P \cap R \).

(2) If \( U \) is a prime ideal of \( R \) such that \( \delta(U) \subseteq U \), then \( D(U) \) is a prime ideal of \( D(R) \) and \( D(U) \cap R = U \).

**Proof.** See the theorem of Goodearl and Warfield [9, Theorem (2.22)]. \( \square \)
Theorem 2.3 (Hilbert Basis Theorem). Let $R$ be a right/left Noetherian ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian. Also $R[x, x^{-1}, \sigma]$ is right/left Noetherian.

Proof. See the theorems of Goodearl and Warfield [9, Theorem (1.12) and (1.17)].

Let $R$ be a right Noetherian ring. Then we know that $\text{Min.} \text{Spec}(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [9] and for any automorphism $\sigma$ of $R$, $U \in \text{Min.} \text{Spec}(R)$ implies that $\sigma^j(U) \in \text{Min.} \text{Spec}(R)$ for all positive integers $j$. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(U) = U$ for all $U \in \text{Min.} \text{Spec}(R)$. We denote $\cap_{i=1}^m \sigma^i(U)$ by $U^0$ as mentioned in introduction. We have a similar statement and notation for associated prime ideals of a right Noetherian ring $R$ (where $R$ is viewed as a right module over itself).

Theorem 2.4. Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$. Then:

1. $P \in \text{Ass}(S(R)S(R))$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $S(P \cap R) = P$ and $P \cap R = U^0$.
2. $P \in \text{Min.} \text{Spec}(S(R))$ if and only if there exists $U \in \text{Min.} \text{Spec}(R)$ such that $S(P \cap R) = P$ and $P \cap R = U^0$.

Proof. See the theorem of Bhat [2, Theorem (2.4)].

Theorem 2.5. Let $R$ be a Noetherian $\mathbb{Q}$-algebra and $\delta$ a derivation of $R$. Then:

1. $P \in \text{Ass}(D(R)D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Ass}(R_R)$.
2. $P \in \text{Min.} \text{Spec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Min.} \text{Spec}(R)$.

Proof. See the theorem of Bhat [2, Theorem (3.7)].

3. Completely prime ideals of Polynomial rings

Theorem 3.1. Let $R$ be a ring such that for any prime ideal $P$ of $R[x]$, $P = (P \cap R)[x]$. Then $R$ is a CPI-ring if and only if $R[x]$ is a CPI-ring.

Proof. Let $R$ be a CPI-ring. Let $P$ be a prime ideal of $R[x]$. Now, Lemma (1.6) of Ferrero [7] implies that $P$ is a prime ideal of $R[x]$ if and only if $P \cap R = V$ (say) is a prime ideal of $R$. Now, by hypothesis $P = V[x]$. Now, $R$ is a CPI-ring implies that $V$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $V[x]$ is completely prime. Therefore $R[x]$ is a CPI-ring.

Conversely, let $R[x]$ be a CPI-ring. Let $U$ be a prime ideal of $R$. Now, by hypothesis $U[x] \in \text{Spec}(R)$. Now, $R[x]$ is a CPI-ring implies that $U[x]$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $U[x] \cap R = U$ is completely prime. Therefore, $R$ is a CPI-ring.
Example 3.2. Let $R = \mathbb{Z}(2) = \{ p/q : p, q \in \mathbb{Z}, q \text{ odd } \}$. This is a PID and the field of fractions of $\mathbb{Z}(2)$ is $\mathbb{Q}$. Now it can be seen that the principal ideal generated by 2 is the unique non zero prime ideal (indeed it is unique maximal ideal) of $\mathbb{Z}(2)$. Let $P$ be any prime ideal of $R[x]$. Then $P \cap R$ is a prime ideal of $R$; i.e. $P \cap R = (2)$.

Theorem 3.3. Let $R$ be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and $\delta$ a derivation of $R$. Then $R$ is an NCPI-ring if and only if $R[x; \delta]$ is an NCPI-ring.

Proof. Let $R$ be an NCPI-ring. Let $P$ be a minimal prime ideal of $R[x; \delta]$. Now, Theorem (3.7) of Bhat [2] implies that $P \cap R \in \text{Min} \text{Spec}(R)$ and $\delta(P \cap R) \subseteq P \cap R$ and $(P \cap R)[x; \delta] = P$. Now, $R$ is an NCPI-ring implies that $P \cap R$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $(P \cap R)[x; \delta] = P$ is completely prime. Therefore, $R[x; \delta]$ is an NCPI-ring.

Conversely, let $R[x; \delta]$ be an NCPI-ring. Let $U$ be a minimal prime ideal of $R$. Now, Theorem (3.7) of Bhat [2] implies that $U[x; \delta] \in \text{Min} \text{Spec}(R[x; \delta])$. Now, $R[x; \delta]$ is an NCPI-ring implies that $U[x; \delta]$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $U[x; \delta] \cap R = U$ is completely prime. Therefore $R$ is an NCPI-ring.

Taking $\delta = 0$ in above theorem, we get the following Corollary:

Corollary 3.4. Let $R$ be a Noetherian ring. Then $R$ is an NCPI-ring if and only if $R[x]$ is NCPI-ring.

Theorem 3.5. Let $R$ be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and $\delta$ a derivation of $R$. Then $R$ is an ACPI-ring if and only if $R[x; \delta]$ is an ACPI-ring.

Proof. Let $R$ be an ACPI-ring. Let $P \in \text{Ass}(D(R)D(R))$. Now Theorem (3.7) of Bhat [2] implies that $P \cap R \in \text{Ass}(R_R)$ and $\delta(P \cap R) \subseteq P \cap R$ and $(P \cap R)[x; \delta] = P$. Now $R$ is an ACPI-ring implies that $P \cap R$ is completely prime.

Rest is on the same lines as in Theorem (3.3) above.

Corollary 3.6. Let $R$ be a Noetherian ring. Then $R$ is an ACPI-ring if and only if $R[x]$ is an ACPI-ring.

Remark 3.7. Let $R$ be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and $\delta$ a derivation of $R$. Let $R$ be a CPI-ring. Then $R[x]$ need not be a CPI-ring.

Example 3.8. Let $R = \mathbb{H}$, the ring of Quaternians. This is a CPI-ring. Now, $\mathbb{H}[x]/(x^2 + 1) \cong H \otimes \mathbb{C} \cong M_2(\mathbb{C})$. Therefore, the maximal ideal $(x^2 + 1)$ is not completely prime.

References


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