GENERALIZED SASAKIAN SPACE FORMS AND TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. In this paper we study the generalized Sasakian space forms and trans-Sasakian manifolds. We present results on generalized recurrent, generalized $\phi$-recurrent, $\phi$-concircular and $\phi$-conharmonically recurrent trans-Sasakian manifolds and generalized Sasakian space forms.

1. INTRODUCTION

P. Alegre, D. E. Blair and A. Carriazo [1] initiated the study of generalized Sasakian space forms and presented some examples. In [2], the authors studied the structure of generalized Sasakian space forms and proved that a K-contact generalized Sasakian space form is Sasakian and if its dimension is greater than 5 then it is a Sasakian space form. Further the authors proved that any three dimensional trans-Sasakian manifold with $\alpha$ and $\beta$ depending only on the direction of $\xi$ is a generalized Sasakian space form. The authors U. C. De and Avijit Sarkar [7], [8] studied curvature properties of generalized Sasakian space forms and obtained important results. Motivated by the above studies, in this paper we study the trans-Sasakian manifolds and the generalized Sasakian space forms. The paper is organized as follows. After preliminaries in Section 2, we study generalized Sasakian space forms and generalized recurrent trans-Sasakian manifolds in Section 3. Generalized $\phi$-recurrent trans-Sasakian manifolds are studied in Section 4. The sections 5 and 6 contain results on $\phi$-concircular recurrent and $\phi$-conharmonically recurrent manifolds.

2. PRELIMINARIES

An odd dimensional Riemannian manifold $(M, g)$ is said to be an almost contact metric manifold if there exist on $M$ a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ such that:

1. $\phi^2 = -I + \eta \otimes \xi$,
2. $\eta(\phi X) = 0$ for any vector field $X$ on $M$,
3. $\phi \eta = 0$,
4. $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields $X$ and $Y$ on $M$.

The tensor field $\phi$ is called the structure tensor field of the almost contact metric manifold. The vector field $\xi$ is called the Reeb vector field. The 1-form $\eta$ is called the Reeb form. The Riemannian metric $g$ is called the almost contact metric metric. A Riemannian manifold $(M, g)$ is a Sasakian manifold if it is an almost contact metric manifold and the $(1,1)$ tensor field $\phi$ is an almost complex structure. A Sasakian manifold is a K-contact manifold if the 1-form $\eta$ is a Killing 1-form. A Sasakian manifold is a K-contact manifold if and only if the $(1,1)$ tensor field $\phi$ is an almost complex structure. A Sasakian manifold is a Sasakian manifold if and only if the $(1,1)$ tensor field $\phi$ is an almost complex structure. A Sasakian manifold is a Sasakian manifold if and only if the $(1,1)$ tensor field $\phi$ is an almost complex structure. A Sasakian manifold is a Sasakian manifold if and only if the $(1,1)$ tensor field $\phi$ is an almost complex structure. A Sasakian manifold is a Sasakian manifold if and only if the $(1,1)$ tensor field $\phi$ is an almost complex structure.
field $\xi$ and a 1-form $\eta$ such that

$$\phi^2 X = -X + \eta(X)\xi,$$

(2.1)

$$\eta(\xi) = 1,$$

(i) $g(X, \xi) = \eta(X),$

(2.2)

$$\eta(\phi X) = 0,$$

(ii) $\phi \xi = 0,$

(2.3)

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.4)

$$g(\phi X, Y) = -g(X, \phi Y),$$

(2.5)

$$(\nabla X \eta)(Y) = g(\nabla X \xi, Y),$$

(2.6)

for any vector fields $X, Y$ on $M$.

An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is called a trans-Sasakian manifold [11] if there exist two functions $\alpha$ and $\beta$ on $M$ such that

$$\nabla X \phi(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

(2.7)

for any vector fields $X, Y$ on $M$. From (2.7), it follows that

$$\nabla X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi),$$

(2.8)

$$\nabla X \eta Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

(2.9)

It is well known that $\alpha$-Sasakian ($\beta = 0$), $\beta$-Kenmotsu ($\alpha = 0$) and co-symplectic ($\alpha = \beta = 0$) manifolds are special cases of trans-Sasakian manifolds.

A $\phi$-section of an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ at $p \in M$ is a section $\Pi \subseteq T_p M$ spanned by a unit vector $X_p$ orthogonal to $\xi$ and $\phi X_p$. The $\phi$-sectional curvature is defined by

$$B(X, \phi X) = R(X, \phi X, \phi X, X).$$

(2.10)

A Sasakian manifold with constant $\phi$-sectional curvature $c$ is called a Sasakian space form and is denoted by $M(c)$. The Riemannian curvature tensor $R$ in $M(c)$ is given by

$$R(X, Y)Z = \frac{c+3}{4} R_1(X, Y)Z + \frac{c-1}{4} R_2(X, Y)Z + \frac{c-1}{4} R_3(X, Y)Z,$$

(2.11)

for any vector fields $X, Y$ and $Z$ on $M$, where

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$

(2.12)

$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$

(2.13)

and

$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$

(2.14)
In [1], the authors introduced the notion of generalized Sasakian space form as an almost contact metric manifold \((M, \phi, \xi, \eta, g)\) whose Riemannian curvature tensor satisfies

\[
R(X, Y)Z = f_1R_1(X, Y)Z + f_2R_2(X, Y)Z + f_3R_3(X, Y)Z,
\]

where \(f_1, f_2\) and \(f_3\) are differentiable functions on \(M\).

Throughout the paper \(M(f_1, f_2, f_3)\) denote a generalized Sasakian space form, where \(M\) is a trans-Sasakian manifold.

In a generalized Sasakian space form the following hold:

\[
\begin{align*}
S(Y, Z) &= [(n - 1)f_1 + 3f_2 - f_3]g(Y, Z) - [3f_2 + (n - 2)f_3]\eta(Y)\eta(Z), \\
QY &= [(n - 1)f_1 + 3f_2 - f_3]Y - [3f_2 + (n - 2)f_3]\eta(Y)\xi, \\
S(Y, \xi) &= (n - 1)(f_1 - f_3)\eta(Y), \\
Q\xi &= (n - 1)(f_1 - f_3)\xi, \\
r &= n(n - 1)f_1 + 3(n - 1)f_2 - 2(n - 1)f_3, \\
R(X, Y)\xi &= (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \\
R(\xi, X)Y &= (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \\
\eta(R(X, Y)Z) &= (f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\end{align*}
\]

3. Generalized recurrent manifolds

A generalized Sasakian space form \(M(f_1, f_2, f_3)\) is called generalized recurrent [5], if its curvature tensor \(R\) satisfies the condition

\[
(\nabla_X R)(Y, Z)(W) = A(X)R(Y, Z)W + B(X)(g(Z, W)Y - g(Y, W)Z),
\]

where \(A\) and \(B\) are two 1-forms and \(B\) is non zero.

Taking \(Y = W = \xi\) in (3.1), we obtain

\[
(\nabla_X R)(\xi, Z)(\xi) = A(X)R(\xi, Z)\xi + B(X)(\eta(Z)\xi - Z).
\]

By definition of covariant derivative, we have

\[
(\nabla_X R)(\xi, Z)\xi
= \nabla_X R(\xi, Z)\xi - R(\nabla_X \xi, Z)\xi - R(\xi, \nabla_X Z)\xi - R(\xi, Z)\nabla_X \xi.
\]

Using (2.21), (2.22) and (2.8) in (3.3), we obtain

\[
(\nabla_X R)(\xi, Z)\xi
= \nabla_X [(f_1 - f_3)(\eta(Z)\xi - Z)] - (f_1 - f_3)[\eta(\nabla_X Z)\xi - \nabla_X Z]
- (f_1 - f_3)[-\alpha\eta(Z)\phi X + \beta\eta(Z)X - \beta\eta(X)\eta(Z)\xi]
- (f_1 - f_3)[-\alpha g(Z, \phi X)\xi + \beta(g(Z, X)\xi - \eta(X)\eta(Z)\xi)].
\]

Using (2.8) and (2.9), equation (3.4) reduces to

\[
(\nabla_X R)(\xi, Z)\xi = d((f_1 - f_3))(X)(\eta(Z)\xi - Z) + (f_1 - f_3)\nabla_X Z.
\]
From (3.2) and (3.5), we have
\[(3.6) \quad [d(f_1 - f_3)(X) - A(X)(f_1 - f_3) - B(X)](\eta(Z)\xi - Z) + (f_1 - f_3)\nabla_X Z = 0.\]
Taking \(Z = \xi\) in (3.6), we obtain
\[(3.7) \quad (f_1 - f_3)\nabla_X \xi = 0.\]
From (3.7), we get that \(M\) is co-symplectic provided \(f_1 = f_3\).
Thus we have

**Proposition 3.1.** A generalized recurrent \(M(f_1, f_2, f_3)\) is co-symplectic provided \(f_1 \neq f_3\).

As it is well known that \(M\) is generalized Ricci-recurrent [6], if its Ricci tensor \(S\) satisfies the condition
\[(3.8) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + (n - 1)B(X)g(Y, Z),\]
where \(A\) and \(B\) are two non-zero 1-forms. By definition of covariant derivative, we have
\[(3.9) \quad (\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi).\]
Using (2.18) and (2.8) in (3.9), we have
\[(3.10) \quad (\nabla_X S)(Y, \xi) = (n - 1)[d(f_1 - f_3)(X)\eta(Y) + (f_1 - f_3)(\alpha g(\phi X, Y) - \beta g(X, Y))] + \alpha S(Y, \phi X) - \beta S(Y, X).\]
Taking \(Z = \xi\) in (3.8) and using (2.18), we obtain
\[(3.11) \quad (\nabla_X S)(Y, \xi) = (n - 1)[(f_1 - f_3)A(X) + B(X)]\eta(Y).\]
From (3.10) and (3.11), we have
\[(3.12) \quad (n - 1)(f_1 - f_3)[A(X)\eta(Y) + \alpha g(\phi X, Y) - \beta g(X, Y)] + (n - 1)B(X)\eta(Y) - (n - 1)d(f_1 - f_3)(X)\eta(Y) - \alpha S(Y, \phi X) + \beta S(Y, X) = 0.\]
Taking \(Y = \xi\) in (3.12), we obtain
\[(3.13) \quad (f_1 - f_3)A(X) + B(X) = d(f_1 - f_3)(X).\]
If \(f_3 - f_1 = c\), a constant then (3.13) reduces to
\[(3.14) \quad B(X) = cA(X).\]
Since \(B(X)\) is not zero, we have \(f_1 \neq f_3\). Thus we can state that

**Proposition 3.2.** In a generalized recurrent \(M(f_1, f_2, f_3)\), \(f_1 \neq f_3\) holds. Further the 1-forms \(A(X)\) and \(B(X)\) are related by (3.14).

It is clear from (3.1) and (3.8) that, a generalized recurrent trans-Sasakian manifold is generalized Ricci-recurrent. Hence from Proposition 3.1 and Proposition 3.2 we have
Theorem 3.3. A generalized recurrent $M(f_1, f_2, f_3)$ is always a co-symplectic manifold.

4. Generalized $\phi$-recurrent manifolds

A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called generalized $\phi$-Ricci recurrent [3], [6] if

\[ \phi^2(\nabla_X Q)(Y) = A(X)QY + (n-1)B(X)Y, \]

where $Q$ is the Ricci operator, $A(X)$ and $B(X)$ are non-zero 1-forms. Using (2.1) and (4.1), we have

\[ -\nabla_X QY - Q(\nabla_X Y) + \eta(\nabla_X Q)(\xi) = A(X)QY + (n-1)B(X)Y. \]

Taking $Y = \xi$ in (4.2) and contracting with respect to $Z$, we obtain

\[ -g(\nabla_X Q\xi, Z) - g(Q(\nabla_X \xi), Z) + \eta(\nabla_X Q)(\xi)\eta(Z) = A(X)g(Q\xi, Z) + (n-1)B(X)\eta(Z). \]

Using (2.8) and (2.19) in (4.3), we have

\[ (n-1)(f_1 - f_3)A(X) + B(X) + d(f_1 - f_3)(X)\eta(Z) = (n-1)(f_1 - f_3)(\alpha g(\phi X, Z) - \beta g(X, Z)) - \alpha S(\phi X, Z) + \beta S(X, Z). \]

Changing $Z$ to $\phi Z$ and taking $\beta = 0$ in (4.4), we obtain

\[ S(X, Z) = (n-1)(f_1 - f_3)g(X, Z). \]

Comparing (4.5) and (2.16), we have $(n - 2)f_3 + 3f_2 = 0$. Changing $Z$ to $\phi Z$ and taking $\alpha = 0$ in (4.4), we obtain

\[ (n-1)(f_1 - f_3)g(X, \phi Z) - S(X, \phi Z) = 0. \]

Using (2.16) in (4.6), we have $(n - 2)f_3 + 3f_2 = 0$.

Thus we have

Theorem 4.1. In an $\alpha$-Sasakian (or a $\beta$-Kenmotsu) generalized Sasakian space form which is $\phi$-Ricci recurrent the relation $(n - 2)f_3 + 3f_2 = 0$ holds.

If $A(X)$ and $B(X)$ are zero in (4.1), then $M(f_1, f_2, f_3)$ is called $\phi$-Ricci symmetric [6].

It is easy to see that the relation (4.6) holds in $\phi$-Ricci-symmetric $\alpha$-Sasakian (or $\beta$-Kenmotsu) generalized Sasakian space form.

Conversely, suppose $(n - 2)f_3 + 3f_2 = 0$ holds in $\phi$-symmetric $M(f_1, f_2, f_3)$. Then from (2.17), we get

\[ QY = (n-1)(f_1 - f_3)Y. \]

Differentiating covariantly with respect to $X$, we obtain

\[ (\nabla_X Q)Y = (n-1)\nabla_X((f_1 - f_3)Y). \]
Applying $\phi^2$ on both sides, we obtain

\begin{equation}
\phi^2((\nabla_X Q)Y) = (n - 1)d(f_1 - f_3)(X)\phi^2 Y.
\end{equation}

i.e $M(f_1, f_2, f_3)$ is $\phi$-Ricci symmetric if and only if $f_1 - f_3$ is a constant.

It follows from (2.15) that in a generalized Sasakian Space form, the $\xi$-sectional curvature $K(X, \xi)$ is given by $K(X, \xi) = R(X, \xi, X, \xi) = f_3 - f_1$.

Thus we can state that

**Theorem 4.2.** An $\alpha$-Sasakian (or a $\beta$-Kenmotsu) generalized Sasakian space form with constant $\xi$-sectional curvature is $\phi$-Ricci symmetric if and only if $(n - 2)f_3 + 3f_2 = 0$ holds.

5. **Concircular curvature tensor of $M(f_1, f_2, f_3)$**

The concircular curvature tensor of $M(f_1, f_2, f_3)$ is given by

\begin{equation}
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n - 1)}[g(Y, Z)X - g(X, Z)Y].
\end{equation}

**Definition 5.1.** $M(f_1, f_2, f_3)$ is said to be $\phi$-concircular recurrent\cite{15} if

\begin{equation}
\phi^2((\nabla_W \tilde{C})(X, Y)Z) = A(W)\tilde{C}(X, Y)Z,
\end{equation}

where $A(W)$ is a nonzero 1-form.

**Definition 5.2.** $M(f_1, f_2, f_3)$ is said to be $\phi$-concircular symmetric if

\begin{equation}
\phi^2((\nabla_W \tilde{C})(X, Y)Z) = 0.
\end{equation}

Taking Covariant differentiation of (5.1), we get

\begin{equation}
(\nabla_W \tilde{C})(X, Y)Z = (\nabla_W R)(X, Y)Z - \frac{dr(W)}{n(n - 1)}[g(Y, Z)X - g(X, Z)Y].
\end{equation}

Applying $\phi^2$ on both the sides, we get

\begin{equation}
\phi^2((\nabla_W \tilde{C})(X, Y)Z
\end{equation}

\begin{equation}
= \phi^2((\nabla_W R)(X, Y)Z - \frac{dr(W)}{n(n - 1)}[g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y].
\end{equation}
Suppose $M(f_1, f_2, f_3)$ is $\phi$-concircular recurrent. Then from (2.15) (2.1) and (5.2) in (5.5), we get

\begin{align}
A(W)\check{C}(X, Y)Z &= df_1(W)[g(X, Z)Y - g(Y, Z)X] \\
+ df_2[g(Y, \phi Z)\phi X - 2g(X, \phi Y)\phi Z] \\
- g(X, \phi Z)\phi Y + f_2[g(X, \phi Z)\phi^2((\nabla_W \phi)Y) \\
- g(X, (\nabla_W \phi)\phi Z)\phi Y - g(Y, \phi Z)\phi^2((\nabla_W \phi)X) \\
+ g(Y, (\nabla_W \phi)Z)\phi X + 2g(X, \phi Y)\phi^2((\nabla_W \phi)Z) \\
- 2g(X, (\nabla_W \phi)Y)\phi Z] \\
- \frac{dr(W)}{n(n-1)}[g(Y, Z)\phi^2X - g(X, Z)\phi^2Y].
\end{align}

Taking $X = \xi$ in (5.6), we get

\begin{align}
A(W)\check{C}(\xi, Y)Z &= -df_1(W)(g(Y, Z)\xi - \eta(Z)Y) \\
- f_2(\eta(\nabla_W \phi)\phi Z)\phi Y - f_2g(Y, \phi Z)\phi^2((\nabla_W \phi)\xi) \\
- 2f_2\eta((\nabla_W \phi)(Y))\phi Z + \frac{dr(W)}{n(n-1)}\eta(Z)\phi^2Y.
\end{align}

Then taking $X = \xi$ in (5.1) and using (2.15), and (2.20), we obtain

\begin{align}
\check{C}(\xi, Y)Z &= \left[\frac{(n-2)f_3 + 3f_2}{n}\right](g(Y, Z)\xi - \eta(Z)Y).
\end{align}

From (5.7) and (5.8), we obtain

\begin{align}
- A(W) \left[\frac{(n-2)f_3 + 3f_2}{n}\right](g(Y, Z)\xi - \eta(Z)Y) \\
= -df_1(W)(g(Y, Z)\xi - \eta(Z)Y) \\
- f_2[\alpha g(W, \phi Z) + \beta g(\phi W, \phi Z)]\phi Y \\
- f_2g(Y, \phi Z)[-\alpha \phi^2W - \beta \phi^3W] + \frac{dr(W)}{n(n-1)}\eta(Z)\phi^2Y \\
- 2f_2[\alpha(g(W, Y) - \eta(Y)\eta(W)) + \beta g(\phi W, Y)]\phi Z.
\end{align}

Taking $Z = \xi$ in (5.9), we obtain

\begin{align}
 dr(W)\phi^2Y \\
= -n(n-1) \left[A(W) \left[\frac{(n-2)f_3 + 3f_2}{n}\right] - df_1(W)\right](\eta(Y)\xi - Y).
\end{align}

For constant $r$, (5.10) yields

\begin{align}
((n-2)f_3 + 3f_2)A(W) = n(df_1(W)).
\end{align}

i.e. $df_1(W) = \left[\frac{3f_2 - 2f_3}{n}\right]A(W)$ if and only if $r$ is a constant.

Thus we have
**Theorem 5.3.** A \(\phi\)-concircular recurrent \(M(f_1, f_2, f_3)\) is of constant curvature if and only if 
\[
d f_1(W) = \frac{(n-2)f_2+3f_3}{n} A(W).
\]

If \(M(f_1, f_2, f_3)\) is \(\phi\)-concircular symmetric then \(A(W) = 0\). From (5.10) it follows that, in a \(\phi\)-concircular symmetric \(M(f_1, f_2, f_3)\), \(f_1\) is constant if and only if \(r\) is a constant. Thus we can state that

**Theorem 5.4.** A \(\phi\)-concircular symmetric \(M(f_1, f_2, f_3)\) is of constant curvature if and only if \(f_1\) is a constant.

### 6. \(\phi\)-Conharmonically Recurrent \(M(f_1, f_2, f_3)\)

The conharmonic curvature tensor of \(M(f_1, f_2, f_3)\) is given by [14]

\[
(6.1) \quad N(X, Y)Z = R(X, Y)Z
\]

\[
- \frac{1}{(n-2)} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].
\]

We know that \(M(f_1, f_2, f_3)\) is said to be \(\phi\)-conharmonically recurrent if

\[
(6.2) \quad \phi^2 ((\nabla_W N)(X, Y)Z) = A(W)N(X, Y)Z.
\]

Suppose the vector fields \(X, Y\) and \(Z\) are orthogonal to \(\xi\). Then from (2.16) (2.17) and (2.15), we have

\[
(6.3) \quad S(Y, Z) = ((n-1)f_1 + 3f_2 - f_3)g(Y, Z),
\]

\[
(6.4) \quad QY = ((n-1)f_1 + 3f_2 - f_3)Y,
\]

\[
(6.5) \quad R(\xi, Y)Z = (f_1 - f_3)g(Y, Z)\xi.
\]

From the equations (6.1) (6.3) and (6.4), we have

\[
(6.6) \quad N(X, Y)Z = R(X, Y)Z
\]

\[
- \frac{2}{(n-2)}[(n-1)f_1 + 3f_2 - f_3][g(Y, Z)X - g(X, Z)Y].
\]

Taking the covariant derivative of (6.6) with respect to \(W\), we get

\[
(6.7) \quad (\nabla_W N)(X, Y)Z = (\nabla_W R)(X, Y)Z
\]

\[
- \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W)(g(Y, Z)X - g(X, Z)Y).
\]
Applying $\phi^2$ on both the sides of (6.7), and using (2.15) and (6.2), we obtain

\begin{align}
A(W)N(X,Y)Z &= df_1(W)[g(X,Z)Y - g(Y,Z)X] \\
&+ df_2[g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z] \\
&- g(X,\phi Z)\phi Y + f_2[g(X,\phi Z)\phi^2((\nabla_W \phi)Y) - g(X,(\nabla_W \phi)\phi Z)\phi Y] \\
&- g(Y,\phi Z)\phi^2((\nabla_W \phi)X) + g(Y,(\nabla_W \phi)Z)\phi X - 2g(X,(\nabla_W \phi)Y)\phi Z \\
&+ 2g(X,\phi Y)\phi^2((\nabla_W \phi)Z]\frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W) \\
&\times [g(Y,Z)\phi^2X - g(X,Z)\phi^2Y].
\end{align}

Taking $X = \xi$ in (6.8), we get

\begin{align}
A(W)N(\xi,Y)Z &= -df_1(W)g(Y,Z)\xi \\
&- f_2\eta((\nabla_W \phi)\phi Z)\phi Y - f_2g(Y,\phi Z)\phi^2((\nabla_W \phi)\xi) - 2\eta((\nabla_W \phi)Y)\phi Z \\
&- \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W)g(Y,Z)\xi.
\end{align}

Taking $X = \xi$ in (6.1), and using (6.3), (6.4) and (6.5), we obtain

\begin{align}
N(\xi,Y)Z &= \left[-nf_1 - 6f_2 + 2f_3 \right]g(Y,Z)\xi.
\end{align}

Using (2.7) and (6.10) in (6.9), we obtain

\begin{align}
A(W) \left[-nf_1 - 6f_2 + 2f_3 \right]g(Y,Z)\xi &= df_1g(Y,Z)\xi \\
&- f_2[\alpha g(W,\phi Z) + \beta g(\phi W,\phi Z)]\phi Y \\
&- f_2g(Y,\phi Z)[\phi^2W - \beta \phi^3W] \\
&- 2[\alpha g(W,Y) - \eta(Y)g(W)] + \beta g(\phi W,Y)\phi Z \\
&- \frac{2}{n-2}d[(n-1)f_1 + 3f_2 - f_3](W)g(Y,Z)\xi.
\end{align}

Contracting (6.11) with respect to $\xi$, we get

\begin{align}
d[(3n - 4)f_1 + 6f_2 - 2f_3](W) &= A(W)[nf_1 + 6f_2 - 2f_3].
\end{align}

Thus we can state that

**Theorem 6.1.** In a $\phi$-conharmonically recurrent $M(f_1, f_2, f_3)$, $nf_1 + 6f_2 - 2f_3 = 0$ holds if and only if $f_1$ and $3f_2 - f_3$ are constants.

**References**


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