ON THE CONVERGENCE ALMOST EVERYWHERE OF DOUBLE SERIES WITH RESPECT TO DIAGONAL BLOCK-ORTONORMAL SYSTEMS

GIVI NADIBAIDZE

Abstract. The diagonal double block-orthonormal system is introduced. The two-dimensional generalization of Menshov-Rademacher’s and V.F. Gaposhkin’s theorems on the almost everywhere convergence of series with respect to block-orthonormal systems is proved.

Block-orthonormal systems were introduced by Gaposhkin [2]. He proved, that the Menshov-Rademacher’s theorem [1] and the strong law of large numbers are valid for such systems in certain conditions. In [3] were obtained some results on convergence and summability of series with respect to block-orthonormal systems. In particular, Menshov-Rademacher’s and Gaposhkin’s theorems were generalized and the exact Weyl multipliers for the convergence and summability almost everywhere of series with respect to block-orthogonal systems were established in the cases, when Menshov-Rademacher’s and Gaposhkin’s theorems are not true.

The two-dimensional analog of Menshov-Rademacher’s theorem was obtained in [4]. In [5] was considered the almost everywhere convergence of multiple orthogonal series.

In the present paper it will be introduced a diagonal block-orthonormal systems and it will be considered the almost everywhere convergence of double series with respect to diagonal block-orthonormal systems.

Definition 1. Let \( \{M_k\} \) and \( \{N_k\} \) be the increasing sequences of natural numbers and \( \Delta_k = ([1, M_{k+1}] \times [1, N_{k+1}]) \setminus ([1, M_k] \times [1, N_k]), (k \geq 1) \). Let \( \{\varphi_{mn}\} \) be a system of functions from \( L^2((0, 1)^2) \). The system \( \{\varphi_{mn}\} \) will be called a diagonal \( \Delta_k \)-orthonormal system (\( \Delta_k \)-ONS) if:
1. \( \|\varphi_{mn}\|_2 = 1, m = 1, 2, \ldots, n = 1, 2, \ldots; \)
2. \( \langle \varphi_{ij}, \varphi_{pq} \rangle = 0, \) for \( (i, j), (p, q) \in \Delta_k, (i, j) \neq (p, q), (k \geq 1). \)

2010 Mathematics Subject Classification. 42C20.

Key words and phrases. block-orthonormal systems, diagonal block-orthonormal systems.

The designated research has been fulfilled by financial support of the Georgian National Science Foundation, Grant GNSF/ST08/3-393.
Let the sequences \( \{M_k\}, \{N_k\} \) be fixed and \( \{\varphi_{mn}\} \) be a diagonal \( \Delta_k \)-ONS. Let the double series
\[
\sum_{m,n=1}^{\infty} a_{mn}\varphi_{mn}(x,y)
\]
is given, where \( \sum_{m,n=1}^{\infty} a_{mn}^2 < \infty \).

Under the convergence of the series (1) it is understood the convergence in Pringsheim’s sense, that is the existence of the limit
\[
\lim_{M,N\to\infty} \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn}\varphi_{mn}(x,y),
\]
as \( M \) and \( N \) independently approaches infinity.

**Definition 2.** Let \( \{\omega(m,n)\} \) be a sequence of positive numbers, for which \( \omega(m,n) \leq \omega(m,n+1) \) and \( \omega(m,n) \leq \omega(m+1,n) \) \((m,n=1,2,\ldots)\). The sequence \( \{\omega(m,n)\} \) will be called the *Weyl multiplier* for the convergence almost everywhere \((a.e.)\) of series (1) with respect to diagonal \( \Delta_k \)-ONS \( \{\varphi_{mn}\} \) if the convergence of the series \( \sum_{m,n=1}^{\infty} a_{mn}^2 \omega(m,n) < \infty \) guarantees the existence of the limit (2) \(a.e.\) on \((0,1)^2\).

In this paper, the logarithms are to the base 2.

**Theorem.** Let the sequences \( \{M_k\}, \{N_k\} \) be fixed and \( \{\omega_1(m)\}, \{\omega_2(n)\} \) be the nondecreasing sequences of positive numbers. In order that a double sequence \( \{\omega_1(m)\omega_2(n)\} \) be the Weyl multiplier for the convergence \(a.e.\) of series (1) with respect to all diagonal \( \Delta_k \)-ONS \( \{\varphi_{mn}\} \), it is necessary and sufficient that the following two conditions be fulfilled:
\[
\sum_{p,q=1}^{\infty} \frac{1}{\omega_1(M_p)\omega_2(N_q)} < \infty,
\]
\[
\log^2 m = O(\omega_i(m)), \quad i = 1, 2, \quad (m \to \infty).
\]

Below we shall use the following lemma, which is the two-dimensional analog of well-known lemma: (see [1, Lemma 2.3.1], [4, lemma 1]).

**Lemma 1.** Let \( \{\varphi_{mn}\} \) be an orthonormal system from \( L^2((0,1)^2) \). Then for all numbers \( \{a_{mn}\}_{0 \leq m \leq M, 0 \leq n \leq N} \) are fulfilled:
\[
\left( \int_0^1 \int_0^1 \max_{0 \leq m \leq M} \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \varphi_{ij}(x,y) \right)^2 \leq c \log^2(M+2) \log^2(N+2) \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij}^2
\]
For (5), (6) and (7) we have generalizations of Kantorovich ([1, p. 89]):

\[
0 \leq m \leq M; 0 \leq n \leq N
\]

\[
\sum_{i=0}^{m} \sum_{j=0}^{N} a_{ij} \varphi_{ij}(x, y) \leq c \log^{2}(M + 2) \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij}^{2},
\]

\[
0 \leq n \leq N
\]

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \varphi_{ij}(x, y) \leq c \log^{2}(N + 2) \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij}^{2}.
\]

Proof of Theorem. Sufficiency. Let for sequence \(\{\omega_1(m)\omega_2(n)\}\) the conditions (3), (4) are fulfilled and let for sequence \(\{a_{mn}\}\) have:

\[
\sum_{m, n=1}^{\infty} a_{mn}^{2} \omega_1(m)\omega_2(n) < \infty.
\]

Let \(\{\varphi_{mn}\}\) be arbitrary diagonal \(\Delta_{k}\)-ONS. In first we shall prove that the limit

\[
\lim_{p, q \to \infty} S_{M_{p}, N_{q}}(x, y) = \sum_{m=1}^{M_{p}} \sum_{n=1}^{N_{q}} a_{mn} \varphi_{mn}(x, y)
\]

exists almost everywhere on \((0, 1)^{2}\).

Without loss of generality it can be assumed that \(M_{0} = N_{0} = 0\) and \(\omega_{1}(0) = \omega_{2}(0) = 1\). Then

\[
\left| S_{M_{p}+p, N_{q}+r}(x, y) - S_{M_{p}, N_{q}}(x, y) \right| = \left| \sum_{m=M_{p}+1}^{M_{p}+p} \sum_{n=1}^{N_{q}+r} a_{mn} \varphi_{mn}(x, y) + \sum_{m=1}^{M_{p}} \sum_{n=N_{q}+1}^{N_{q}+r} a_{mn} \varphi_{mn}(x, y) \right|
\]
We shall prove that the double series
\[ \sum_{i,j=0}^{\infty} \sum_{m=M_i+1}^{M_i+1} \sum_{n=N_j+1}^{N_j+1} a_{mn} \varphi_{mn}(x,y) \]
converges a. e. on \((0,1)^2\). Indeed, we have
\[
\sum_{i,j=0}^{\infty} \int_0^1 \int_0^1 \left| \sum_{m=M_i+1}^{M_i+1} \sum_{n=N_j+1}^{N_j+1} a_{mn} \varphi_{mn}(x,y) \right| \, dx \, dy
\]
\[
\leq \sum_{i,j=0}^{\infty} \left( \int_0^1 \int_0^1 \left| \sum_{m=M_i+1}^{M_i+1} \sum_{n=N_j+1}^{N_j+1} a_{mn} \varphi_{mn}(x,y) \right|^2 \, dx \, dy \right)^{1/2}
\]
\[
= \sum_{i,j=0}^{\infty} \left( \sum_{m=M_i+1}^{M_i+1} \sum_{n=N_j+1}^{N_j+1} a_{mn}^2 \right)^{1/2} \leq c \left( \sum_{i,j=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) \right)^{1/2} < \infty
\]
Hence Levi’s theorem implies, that series (12) converges a. e. on \((0,1)^2\). Then almost everywhere on \((0,1)^2\) we have
\[
\lim_{p \to \infty} I_p(x,y) = 0 \quad \text{and} \quad \lim_{q \to \infty} J_q(x,y) = 0.
\]
Therefore the limit (11) exists almost everywhere on \((0,1)^2\).

Let \(k, l\) be the natural numbers, for which
\[ M_p < k \leq M_{p+1}, \quad N_q < l \leq N_{q+1}. \]
We have
\[
\max_{M_p < k \leq M_{p+1}} \max_{N_q < l \leq N_{q+1}} \left| S_{k,l}(x,y) - S_{M_p,N_q}(x,y) \right|
\]
\[
\leq \max_{N_q < l \leq N_{q+1}} \left| \sum_{m=1}^{M_p} \sum_{n=N_q+1}^{l} a_{mn} \varphi_{mn}(x,y) \right| + \max_{M_p < k \leq M_{p+1}} \left| \sum_{m=M_p+1}^{k} \sum_{n=1}^{N_q} a_{mn} \varphi_{mn}(x,y) \right|
\]
\[
+ \max_{M_p < k \leq M_{p+1}} \max_{N_q < l \leq N_{q+1}} \left| \sum_{m=M_p+1}^{k} \sum_{n=N_q+1}^{l} a_{mn} \varphi_{mn}(x,y) \right|
\]
ON THE CONVERGENCE ALMOST EVERYWHERE OF DOUBLE SERIES 261

\[ \leq \sum_{i=0}^{\infty} \sup_{N_q \leq t \leq l < \infty} \left| \sum_{m=M_{i+1}+1}^{M_i+1} \sum_{n=N_q+1}^{l_2} a_{mn} \varphi_{mn}(x, y) \right| \\
+ \sum_{j=0}^{\infty} \sup_{M_p \leq k_1 < k_2 < \infty} \left| \sum_{m=k_1+1}^{N_{j+1}} \sum_{n=N_{j+1}}^{l_2} a_{mn} \varphi_{mn}(x, y) \right| \\
+ \max_{M_p \leq k \leq M_{p+1}} \sum_{m=M_p+1}^{k} \sum_{n=N_{q+1}+1}^{N_j} a_{mn} \varphi_{mn}(x, y) \right| \\
= \sum_{i=0}^{\infty} \alpha_i^q(x, y) + \sum_{j=0}^{\infty} \beta_j^p(x, y) + \delta_{p,q}(x, y).

It’s clear, that the sequences
\[ \alpha_q(x, y) = \sum_{i=0}^{\infty} \alpha_i^q(x, y) \quad \text{and} \quad \beta_p(x, y) = \sum_{j=0}^{\infty} \beta_j^p(x, y) \]
are increasing sequences. Show that a. e. on \((0,1)^2\)
\[ \lim_{q \to \infty} \alpha_q(x, y) = 0 \quad \text{and} \quad \lim_{p \to \infty} \beta_p(x, y) = 0. \]
Indeed, using lemma we have:
\[ \sum_{i=0}^{\infty} \int_0^1 \int_0^1 \alpha_i^q(x, y) \, dx \, dy \leq \sum_{i=0}^{\infty} \left( \int_0^1 \int_0^1 [\alpha_i^q(x, y)]^2 \, dx \, dy \right)^{\frac{1}{2}} \]
\[ \leq \sum_{i=0}^{\infty} \left( \int_0^1 \int_0^1 \sup_{N_q \leq t \leq l < \infty} \left| \sum_{m=M_{i+1}+1}^{M_i+1} \sum_{n=N_q+1}^{l_2} a_{mn} \varphi_{mn}(x, y) \right| \right)^{\frac{1}{2}} \]
\[ - \sum_{m=M_{i+1}+1}^{M_i+1} \sum_{n=N_q+1}^{l_1} \left| a_{mn} \varphi_{mn}(x, y) \right|^2 dx \, dy \right)^{\frac{1}{2}} \]
\[ \leq c \sum_{i=0}^{\infty} \left( \int_0^1 \int_0^1 \sup_{N_q \leq t \leq l < \infty} \left| \sum_{m=M_{i+1}+1}^{M_i+1} \sum_{n=N_q+1}^{l} a_{mn} \varphi_{mn}(x, y) \right|^2 dx \, dy \right)^{\frac{1}{2}} \]
\[ \leq c \sum_{i=0}^{\infty} \left[ \int_0^1 \int_0^1 \left( \sum_{j=0}^{\infty} \sum_{m=M_{i+1}+1}^{M_i+1} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right)^2 dx \, dy \right]^{\frac{1}{2}} \]
\[ + c \sum_{i=0}^{\infty} \left[ \int_0^1 \int_0^1 \max_{N_j < t \leq N_{j+1}} \sum_{m=M_{i+1}+1}^{M_i+1} \sum_{n=N_j+1}^{l} a_{mn} \varphi_{mn}(x, y) \right]^2 dx \, dy \right]^{\frac{1}{2}} \]
\[
\begin{align*}
&\leq c \sum_{i=0}^{\infty} \left[ \left( \sum_{j=q}^{\infty} \int_{0}^{1} \int_{0}^{1} a_{mn} \varphi_{mn}(x, y) \, dx \, dy \right)^{2} \right]^{\frac{1}{2}} \\
&+ \left( \sum_{j=q}^{\infty} \int_{0}^{1} \int_{0}^{1} \max_{N_j < l \leq N_{j+1}} \left( \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} a_{mn} \varphi_{mn}(x, y) \right)^{2} \, dx \, dy \right)^{\frac{1}{2}} \\
&\leq c \sum_{i=0}^{\infty} \left[ \sum_{i=q}^{\infty} \left( \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^{2} \right)^{\frac{1}{2}} + \left( \sum_{i=q}^{\infty} \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^{2} \log^{2}(n + 2) \right)^{\frac{1}{2}} \right] \\
&\leq c \left( \sum_{i=0}^{\infty} \sum_{i=q}^{\infty} \left( \sum_{m=M_i+1}^{M_{i+1}} \sum_{n=N_j+1}^{N_{j+1}} \alpha_{mn}^{2} \right) \omega_{1}(M_{i}) \omega_{2}(N_{j}) \right)^{\frac{1}{2}} \cdot \left( \sum_{i=0}^{\infty} \sum_{i=q}^{\infty} \frac{1}{\omega_{1}(M_{i}) \omega_{2}(N_{j})} \right)^{\frac{1}{2}} \\
&\leq c \left( \sum_{m=1}^{\infty} \sum_{n=N_{q+1}+1}^{\infty} \alpha_{mn}^{2} \omega_{1}(m) \omega_{2}(n) \right)^{\frac{1}{2}},
\end{align*}
\]

hence

\[
\lim_{q \to \infty} \int_{0}^{1} \int_{0}^{1} \alpha_{q}(x, y) \, dx \, dy = 0.
\]

Then by Fatou’s theorem

\[
\begin{align*}
&\lim_{q \to \infty} \alpha_{q}(x, y) = 0 \text{ a. e. on } (0, 1)^{2}.
\end{align*}
\]

Similarly we obtain

\[
\begin{align*}
&\lim_{p \to \infty} \beta_{p}(x, y) = 0 \text{ a. e. on } (0, 1)^{2}.
\end{align*}
\]

Now we prove that

\[
\begin{align*}
&\lim_{p, q \to \infty} \delta_{p,q}(x, y) = 0 \text{ a. e. on } (0, 1)^{2}.
\end{align*}
\]

Indeed, using inequality (8) we get

\[
\begin{align*}
&\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \int_{0}^{1} \int_{0}^{1} \delta_{p,q}^{2}(x, y) \, dx \, dy \\
&\leq c \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=M_{p+1}+1}^{M_{p+1}} \sum_{n=N_{q+1}+1}^{N_{q+1}} a_{mn}^{2} \log^{2}(m + 2) \log^{2}(n + 2)
\end{align*}
\]
\[
\leq c \sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) < \infty,
\]

hence \( \sum_{p,q=0}^{\infty} \delta_{p,q}^2(x,y) < \infty \) almost everywhere on \((0,1)^2\). Then we obtain (15). Therefore taking into account (13), (14) and (15) we get

\[
\max_{M_p < k \leq M_{p+1}, N_q < l \leq N_{q+1}} |S_{k,l}(x,y) - S_{M_p, N_q}(x,y)| = 0
\]

almost everywhere on \((0,1)^2\). Finally taking into account (11) we finished proof of sufficiency.

**Necessity.** a) Let

\[
\sum_{p,q=1}^{\infty} \frac{1}{\omega_1(M_p) \omega_2(N_q)} = \infty.
\]

Without loss of generality it can be assumed that

\[
\sum_{p=1}^{\infty} \frac{1}{\omega_1(M_p)} = \infty.
\]

Then there exist numbers \(c_p > 0\) such that

\[
\sum_{p,q=1}^{\infty} c_p^2 \omega_1(M_p) < \infty \text{ and } \sum_{p=1}^{\infty} c_p = \infty.
\]

Take \(a_{M_p, N_1} = c_p\), \((p = 1, 2, \ldots)\), \(a_{mn} = 0\), \(((m, n) \neq (M_p, N_1), m \in \mathbb{N}, n \in \mathbb{N}, p \in \mathbb{N})\). Let \(\varphi_{M_p, N_1}(x, y) = 1\), \((p = 1, 2, \ldots), (x, y) \in (0,1)^2\) and choose as other functions an arbitrary ONS orthogonal to 1. The system \(\{\varphi_{mn}\}\) is diagonal \(\Delta_k\)-ONS, for which

\[
\sum_{m,n=1}^{\infty} a_{mn} \varphi_{mn}(x,y) = \sum_{p=1}^{\infty} c_p = \infty \text{ (x,y) } \in (0,1)^2
\]

Though

\[
\sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) = \sum_{p=1}^{\infty} c_p^2 \omega_1(M_p) \omega_2(N_1) < \infty.
\]

b) Let condition (4) is not fulfilled. Without loss of generality it can be assumed that the condition \( \log^2 m = O(\omega_1(m))\), \((m \rightarrow \infty)\) is not fulfilled. Then there exist (see [3, Theorem 1.]) numbers \(b_m\) and \((M_p, M_{p+1})\)-ONS \(\{\varphi_m\}\) such that

\[
\sum_{m=1}^{\infty} b_m^2 \omega_1(m) < \infty,
\]

though

\[
\sum_{m=1}^{\infty} b_m \varphi_m(x)
\]
diverges a.e. on \((0, 1)\).

Take \(a_{m,1} = b_m, (m = 1, 2, \ldots), a_{mn} = 0, (m \in \mathbb{N}, n \geq 2)\). Let \(\{\psi_n\}\) be an ONS from \(L^2(0, 1)\) such that \(\psi_1(y) = 1, y \in (0, 1)\). The system \(\varphi_{mn}(x,y) = \varphi_m(x)\psi_n(y)\) is a diagonal \(\Delta_k\)-orthonormal system. Then taking into account (16), (17) we have

\[
\sum_{m,n=1}^{\infty} a_{mn}^2 \omega_1(m) \omega_2(n) = \omega_2(1) \sum_{m=1}^{\infty} b_m^2 \omega_1(m) < \infty,
\]

though the series

\[
\sum_{m,n=1}^{\infty} a_{mn} \varphi_{mn}(x,y) = \sum_{m=1}^{\infty} b_m \varphi_m(x) \psi_1(y)
\]
diverges a.e. on \((0, 1)^2\).

**Corollary.** If we take \(\omega_1(m) = \omega_2(m) = \log^2 m\) then we obtain the following theorem:

a) If

\[
\sum_{p,q=1}^{\infty} \frac{1}{\log^2(M_p) \log^2(N_q)} < \infty,
\]

then for every diagonal \(\Delta_k\)-ONS \(\{\varphi_{mn}\}\) the condition

\[
\sum_{m,n=1}^{\infty} a_{mn}^2 \log^2 m \log^2 n < \infty
\]

guarantees the convergence a.e. on \((0, 1)^2\) of the series (1).

b) If however

\[
\sum_{p,q=1}^{\infty} \frac{1}{\log^2 M_p \log^2 N_q} = \infty,
\]

then there exist numbers \(b_{mn}\) and diagonal \(\Delta_k\)-ONS \(\{\psi_{mn}\}\) such that the series

\[
\sum_{m,n=1}^{\infty} b_{mn} \psi_{mn}(x,y)
\]
diverges a.e. on \((0, 1)^2\) though

\[
\sum_{m,n=1}^{\infty} b_{mn}^2 \log^2 m \log^2 n < \infty.
\]

**Remark 1.** For example if we take \(M_p = [2^p]\), \(N_q = [2^q]\), \(\alpha > 1/2\), then the condition (18) is fulfilled. Therefore the two-dimensional analog of Menshov-Rademacher’s Theorem (see [4], theorem 1) is fulfilled for any \(\Delta_k\)-ONS \(\{\varphi_{mn}\}\).

If \(M_p = [2^p]\), \(N_q = [2^q]\), \(0 < \alpha \leq 1/2\), then \(\{\log^2 m \log^2 n\}\) will be the Weyl multiplier for the convergence a.e. not for each \(\Delta_k\)-ONS. From proved
Theorem it follows that in that case \( \{ \log^{1+\varepsilon} m \log^{1+\varepsilon} n \} \) \((\varepsilon > 0)\) is the Weyl multiplier.

REFERENCES


Received April 03, 2011.

Department of Mathematics, Tbilisi State University, Chavchavadze av. 1, 0128, Tbilisi, Georgia
E-mail address: g.nadibaidze@gmail.com