THE MAXIMAL OPERATORS OF LOGARITHMIC MEANS OF ONE-DIMENSIONAL VILENKIN-FOURIER SERIES

GEORGE TEPHNADZE

Abstract. The main aim of this paper is to investigate \((H_p, L_p)\)-type inequalities for maximal operators of logarithmic means of one-dimensional bounded Vilenkin-Fourier series.

1. Introduction

In one-dimensional case the weak type inequality
\[
\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0)
\]
can be found in Zygmund [20] for the trigonometric series, in Schipp [11] for Walsh series and in Pálfy and Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujii [3] and Simon [12] verified that \(\sigma^*\) is bounded from \(H_1\) to \(L_1\). Weisz [17] generalized this result and proved the boundedness of \(\sigma^*\) from the martingale space \(H_p\) to the space \(L_p\) for \(p > 1/2\). Simon [13] gave a counterexample, which shows that boundedness does not hold for \(0 < p < 1/2\).

The counterexample for \(p = 1/2\) due to Goginava ([7], see also [2]).

Riesz’s logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mention, for instance, the paper by Szász [14] and Yabuta [19]. This means with respect to the Walsh and Vilenkin systems was discussed by Simon[13] and Gát[4].

Móricz and Siddiqi[9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of \(L_p\) function in norm. The case when \(q_k = 1/k\) is excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [5] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space \(L_1\). Among there, they gave a negative answer to the question of Móricz and

2010 Mathematics Subject Classification. 42C10.

Key words and phrases. Vilenkin system, Logarithmic means, martingale Hardy space.
Siddiqi [9], Gát and Goginava [6] proved that for each measurable function 
\( \phi(u) = o(u^{\log u}) \) there exists an integrable function \( f \), such that 
\[
\int_{G_m} \phi(|f(x)|) \, d\mu(x) < \infty
\]
and there exist a set with positive measure, such that the Walsh-logarithmic 
means of the function diverge on this set.

The main aim of this paper is to investigate \((H_p, L_p)\)-type inequalities for the
maximal operators of Riesz and Nörlund logarithmic means of one-dimensional
Vilenkin-Fourier series. We prove that the maximal operator \( R^* \) is bounded
from the Hardy space \( H_p \) to the space \( L_p \) when \( p > 1/2 \). We also shows that
when \( 0 < p \leq 1/2 \) there exists a martingale \( f \in H_p \) for which
\[
\|R^*f\|_{L_p} = +\infty.
\]

For the Nörlund logarithmic means we prove that when \( 0 < p \leq 1 \) there
exists a martingale \( f \in H_p \) for which
\[
\|L^*f\|_{L_p} = +\infty.
\]

Analogical theorems for Walsh-Paley system is proved in [8].

2. Definitions and notation

Let \( N_+ \) denote the set of the positive integers, \( N := N_+ \cup \{0\} \). Let \( m := (m_0, m_1, \ldots) \) denote a sequence of positive integers not less than 2. Denote by
\( Z_{m_k} := \{0, 1, \ldots, m_k - 1\} \) the addition group of integers modulo \( m_k \).

Define the group \( G_m \) as the complete direct product of the groups \( Z_{m_j} \), with
the product of the discrete topologies of the groups \( Z_{m_j} \).

The direct product \( \mu \) of the measures
\[
\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})
\]
is the Haar measure on \( G_{m_k} \), with \( \mu(G_m) = 1 \).

If the sequence \( m \) is bounded then \( G_m \) is called a bounded Vilenkin group,
else it is called an unbounded one. In this paper we discuss bounded Vilenkin
groups only. The elements of \( G_m \) are represented by sequences
\[
x := (x_0, x_1, \ldots, x_j, \ldots) \quad (x_i \in Z_{m_j}).
\]

It is easy to give a base for the neighborhood of \( G_m \)
\[
I_0(x) := G_m \quad \text{and} \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in N)
\]

Denote \( I_n := I_n(0) \) for \( n \in N_+ \).

If we define the so-called generalized number system based on \( m \) in the
following way:
\[
M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in N),
\]
then every \( n \in N \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} n_j M_j \) where \( n_j \in Z_{m_j} \) \( (j \in N_+) \) and only a finite number of \( n_j \)s differ from zero.

Next, we introduce on \( G_m \) an orthonormal system which is called the Vilenkin system. At first, define the complex valued function \( r_k(x) : G_m \to C \), the generalized Rademacher functions as

\[
    r_k(x) := \exp \left( \frac{2\pi i x_k}{m_k} \right) \quad (i^2 = -1, x \in G_m, k \in N).
\]

Now, define the Vilenkin system \( \psi := (\psi_n : n \in N) \) on \( G_m \) as:

\[
    \psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in N).
\]

Specifically, we call this system the Walsh-Paley one if \( m \equiv 2 \). The Vilenkin system is orthonormal and complete in \( L^2(G_m) \) \cite{1, 15}.

Now, we introduce analogues of the usual definitions in Fourier-analysis. If \( f \in L^1(G_m) \) we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system \( \psi \) in the usual manner:

\[
    \hat{f}(k) := \int_{G_m} f \psi_k d\mu \quad (k \in N),
\]

\[
    S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in N_+, S_0 f := 0),
\]

\[
    \sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in N_+),
\]

\[
    D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in N_+).
\]

Recall that

\[
    D_M_n(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}
\]

The norm (or quasinorm) of the space \( L^p(G_m) \) is defined by

\[
    \|f\|_p := \left( \int_{G_m} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \quad (0 < p < \infty).
\]

The \( \sigma \)-algebra generated by the intervals \( \{I_n(x) : x \in G_m\} \) is denoted by \( F_n (n \in N) \). Denote by \( f = (f^{(n)}, n \in N) \) a martingale with respect to \( F_n \ (n \in N) \) (for details see e.g. \cite{16}).

The maximal function of a martingale \( f \) is defined by

\[
    f^* = \sup_{n \in N} |f^{(n)}|.
\]
In case \( f \in L_1(G_m) \), the maximal functions are also be given by

\[
\max^* (x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \int_{\mu_n(x)} f(u) \, d\mu(u).
\]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H_p(G_m) \) consist of all martingale for which

\[
\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.
\]

If \( f \in L_1(G_m) \), then it is easy to show that the sequence \( (S_{M_n}(f) : n \in \mathbb{N}) \) is a martingale.

If \( f = \left(f^{(n)}, n \in \mathbb{N}\right) \) is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

\[
\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \overline{\varphi}_i(x) \, d\mu(x).
\]

The Vilenkin-Fourier coefficients of \( f \in L_1(G_m) \) are the same as those of the martingale \( (S_{M_n}(f) : n \in \mathbb{N}) \) obtained from \( f \).

In the literature, there is the notion of Riesz’s logarithmic means of the Fourier series. The \( n \)-th Riesz’s logarithmic means of the Fourier series of an integrable function \( f \) is defined by

\[
R_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} S_k f(x) \cdot \frac{k}{n},
\]

where

\[
l_n := \sum_{k=1}^{n} (1/k).
\]

Let \( \{q_k : k > 0\} \) be a sequence of nonnegative numbers. The \( n \)-th Nörlund means for the Fourier series of \( f \) is defined by

\[
\frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,
\]

where

\[
Q_n := \sum_{k=1}^{n} q_k.
\]

If \( q_k = 1/k \), then we get Nörlund logarithmic means

\[
L_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} S_k f(x) \cdot \frac{n}{n-k}
\]

It is a kind of ”reverse” Riesz’s logarithmic mean. In this paper we call these means logarithmic means.
For the martingale $f$ we consider the following maximal operators of

\[ R^* f(x) := \sup_{n \in \mathbb{N}} |R_n f(x)|, \]
\[ L^* f(x) := \sup_{n \in \mathbb{N}} |L_n f(x)|, \]
\[ \sigma^* f(x) := \sup_{n \in \mathbb{N}} |\sigma_n f(x)|. \]

A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic interval $I$, such that

\begin{enumerate}[a)]  
\item $\int_I a \, d\mu = 0$,  
\item $\|a\|_\infty \leq \mu(I)^{-1/p}$,  
\item $\text{supp}(a) \subset I$.  
\end{enumerate}

3. FORMULATION OF MAIN RESULTS

**Theorem 1.** Let $p > 1/2$. Then the maximal operator $R^*$ is bounded from the Hardy space $H_p$ to the space $L_p$.

**Theorem 2.** Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that

\[ \|R^* f\|_p = +\infty. \]

**Corollary 1.** Let $0 < p \leq 1/2$. Then there exists a martingale $f \in H_p$ such that

\[ \|\sigma^* f\|_p = +\infty. \]

**Theorem 3.** Let $0 < p \leq 1$. Then there exists a martingale $f \in L_p$ such that

\[ \|L^* f\|_p = +\infty. \]

4. AUXILIARY PROPOSITIONS

**Lemma 1.** [18] A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p (0 < p \leq 1)$ if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers such that for every $n \in \mathbb{N}$

\[ \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \]
\[ \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \]

Moreover,

\[ \|f\|_{H_p} \sim \inf \left( \sum_{K=0}^{\infty} |\mu_k|^p \right)^{1/p}, \]
where the infimum is taken over all decomposition of $f$ of the form (1).

5. PROOF OF THE THEOREMS

Proof of Theorem 1: Using Abel transformation we obtain

$$R_n f(x) = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\sigma_j f(x)}{j+1} + \frac{\sigma_n f(x)}{l_n},$$

Consequently,

(2) \quad L^* f \leq c \sigma^* f.

On the other hand Weisz[17] proved that $\sigma^*$ is bounded from the Hardy space $H_p$ to the space $L_p$ when $p > 1/2$. Hence, from (2) we conclude that $R^*$ is bounded from the martingale Hardy space $H_p$ to the space $L_p$ when $p > 1/2$.

Proof of Theorem 2: Let $\{\alpha_k : k \in \mathbb{N}\}$ be an increasing sequence of the positive integers such that

(3) \quad \sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty,

(4) \quad \sum_{q=0}^{k-1} \frac{(M_{2\alpha_q})^{1/p}}{\sqrt{\alpha_q}} < \frac{(M_{2\alpha_k})^{1/p}}{\sqrt{\alpha_k}},

(5) \quad \frac{(M_{2\alpha_{k-1}})^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{M_{\alpha_k}}{\alpha_k^{3/2}}.

We note that such an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$ which satisfies conditions (3)-(5) can be constructed.

Let

$$f^{(A)}(x) = \sum_{\{k : 2\alpha_k < A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{m_{2\alpha_k}}{\sqrt{\alpha_k}}$$

and

$$a_k(x) = \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} \left( D_{M(2\alpha_k+1)}(x) - D_{M_{2\alpha_k}}(x) \right).$$

It is easy to show that

$$\|a_k\|_\infty \leq \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} M_{2\alpha_k+1} \leq (M_{2\alpha_k})^{1/p} = (\mu(\text{supp } a_k))^{-1/p},$$

(6) \quad S_{\lambda_k} a_k(x) = \begin{cases} a_k(x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A. \end{cases}$$
\[ f^{(A)}(x) = \sum_{k: 2^k < A} \lambda_k a_k = \sum_{k=0}^{\infty} \lambda_k S_{M^A} a_k(x) , \]

\[ \text{supp}(a_k) = I_{2^k} , \]

\[ \int_{I_{2^k}} a_k d\mu = 0 . \]

From (3) and Lemma 1 we conclude that \( f = (f^{(n)}, n \in N) \in H_p . \)

Let

\[ q_A^s = M_{2A} + M_{2s} - 1, \quad A > S . \]

Then we can write

\[ R_{q_{n_k}} f(x) = \frac{1}{l_{q_{n_k}}} \sum_{j=1}^{q_{n_k}} S_j f(x) \]

\[ = \frac{1}{l_{q_{n_k}}} \sum_{j=1}^{M_{2n_k}} S_j f(x) + \frac{1}{l_{q_{n_k}}} \sum_{j=M_{2n_k}}^{q_{n_k}} S_j f(x) = I + II . \]

It is easy to show that

\[ \hat{f}(j) = \begin{cases} \frac{M_{1/p-1}}{\sqrt{\alpha_k}}, & \text{if } j \in \{M_{2n_k}, \ldots, M_{2n_k+1} - 1\} \quad k = 0, 1, 2, \ldots, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{2n_k}, \ldots, M_{2n_k+1} - 1\} . \end{cases} \]

Let \( j < M_{2n_k} . \) Then from (4) and (8) we have

\[ |S_j f(x)| \leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\eta}}^{M_{2\eta+1} - 1} \left| \hat{f}(v) \right| \]

\[ \leq \sum_{\eta=0}^{k-1} \sum_{v=M_{2\eta}}^{M_{2\eta+1} - 1} \frac{M_{1/p-1}}{\sqrt{\alpha_\eta}} \leq c \sum_{\eta=0}^{k-1} \frac{M_{1/p}}{\sqrt{\alpha_\eta}} \leq c \frac{M_{1/p}}{\sqrt{\alpha_k-1}} . \]

Consequently,

\[ |I| \leq \frac{1}{l_{q_{n_k}}} \sum_{j=1}^{M_{2n_k} - 1} \frac{|S_j f(x)|}{j} \leq \frac{c}{\alpha_k} \frac{M_{2n_k-1}^{1/p}}{\sqrt{\alpha_k-1}} \sum_{j=1}^{M_{2n_k}-1} \frac{1}{j} \leq c \frac{M_{2n_k-1}^{1/p}}{\sqrt{\alpha_k-1}} . \]
Let $M_{2\alpha k} \leq j \leq q'_{\alpha k}$. Then we have the following

\begin{align*}
S_j f (x) &= \sum_{\eta=0}^{k-1} \sum_{\nu=0}^{M_{2\alpha\eta}+1} \hat{f}(\nu) \psi_\nu (x) + \sum_{\nu=M_{2\alpha k}}^{j-1} \hat{f}(\nu) \psi_\nu (x) \\
&= \sum_{\eta=0}^{k-1} \frac{M_{2\alpha\eta}^{1/p-1}}{\sqrt{\alpha \eta}} \left( D_{M_{2\alpha\eta+1}} (x) - D_{M_{2\alpha\eta}} (x) \right) \\
&\quad + \frac{M_{2\alpha k}^{1/p-1}}{\sqrt{\alpha k}} \left( D_j (x) - D_{M_{2\alpha k}} (x) \right).
\end{align*}

This gives that

\begin{align*}
II &= \frac{1}{l_{q'_{\alpha k}}} \sum_{j=M_{2\alpha k}}^{q'_{\alpha k}} \frac{1}{j} \left( \sum_{\eta=0}^{k-1} \frac{M_{2\alpha\eta}^{1/p-1}}{\sqrt{\alpha \eta}} \left( D_{M_{2\alpha\eta+1}} (x) - D_{M_{2\alpha\eta}} (x) \right) \right) \\
&\quad + \frac{1}{l_{q'_{\alpha k}}} \frac{M_{2\alpha k}^{1/p-1}}{\sqrt{\alpha k}} \sum_{j=M_{2\alpha k}}^{q'_{\alpha k}} \frac{1}{j} \left( D_j (x) - D_{M_{2\alpha k}} (x) \right) = II_1 + II_2.
\end{align*}

To discuss $II_1$, we use (4). Thus, we can write that

\begin{equation}
|II_1| \leq c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha\eta}^{1/p}}{\sqrt{\alpha \eta}} \leq c \frac{M_{2\alpha k}^{1/p-1}}{\sqrt{\alpha k-1}}.
\end{equation}

Since,

\begin{equation}
D_{j+M_{2\alpha k}} (x) = D_{M_{2\alpha k}} (x) + \psi_{M_{2\alpha k}} (x) D_j (x), \quad \text{when } j < M_{2\alpha k},
\end{equation}

for $II_2$ we have,

\begin{align*}
II_2 &= \frac{1}{l_{q'_{\alpha k}}} \frac{M_{2\alpha k}^{1/p-1}}{\sqrt{\alpha k}} \sum_{j=0}^{M_{2\alpha k}} D_{j+M_{2\alpha k}} (x) - D_{M_{2\alpha k}} (x) \\
&= \frac{1}{l_{q'_{\alpha k}}} \frac{M_{2\alpha k}^{1/p-1}}{\sqrt{\alpha k}} \psi_{M_{2\alpha k}} \sum_{j=0}^{M_{2\alpha k}-1} D_j (x) \\
&\geq \frac{c}{\alpha k} \frac{M_{2\alpha k}^{1/p-1}}{\sqrt{\alpha k}} \sum_{j=0}^{M_{2\alpha k}-1} D_j (x).
\end{align*}

We write

\begin{equation}
R_{q'_{\alpha k}} f (x) = I + II_1 + II_2,
\end{equation}

Then by (5), (7), (10) and (12)-(15) we have

\begin{equation}
\left| R_{q'_{\alpha k}} f (x) \right| \geq |II_2| - |I| - |II_1| \geq |II_2| - c \frac{M_{q'_{\alpha k}}}{\alpha k^{3/2}}
\end{equation}
Let $0 < p \leq 1/2$, $x \in I_{2s} \setminus I_{2s+1}$ for $s = [\alpha_k/2], \ldots, \alpha_k$. Then it is evident
\[
\left| \sum_{j=0}^{M_{2s}} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| \geq c \frac{M_{2s}^2}{M_{2\alpha_k}}
\]

Hence, we can write
\[
\left| R_{q^*\alpha_k} f(x) \right| \geq c \frac{M_{2\alpha_k}^{1/p-1}}{\alpha_k} \frac{M_{2s}^2}{M_{2\alpha_k}} - c \frac{M_{\alpha_k}/\alpha_k^{3/2}}{\alpha_k^{3/2}} \geq c \frac{M_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} - c \frac{M_{\alpha_k}/\alpha_k^{3/2}}{\alpha_k^{3/2}} \geq c \frac{M_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}}.
\]

Then we have
\[
\int_{G_m} |R^* f(x)|^p \, d\mu(x) \geq \sum_{s=[\alpha_k/2]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left| R_{q^*\alpha_k} f(x) \right|^p \, d\mu(x)
\]
\[
\geq \sum_{s=[\alpha_k/2]}^{\alpha_k} \int_{I_{2s} \setminus I_{2s+1}} \left( c M_{2\alpha_k}^{1/p-2} M_{2s}^2 \right)^p \, d\mu(x)
\]
\[
\geq c \sum_{s=[\alpha_k/2]}^{\alpha_k} \frac{M_{2\alpha_k}^{1-2p} M_{2s}^{2p-1}}{\alpha_k^{3p/2}}
\]
\[
\geq \left\{ \begin{array}{ll}
\frac{2^{p_0(k-1) - 2p}}{(2p_0)^{2p_0}} & \text{when } 0 < p < 1/2 \\
\frac{2^{p_0(k-1)/4}}{2^{p_0(k-1)/4}} & \text{when } p = 1/2
\end{array} \right.
\]
which completes the proof of the Theorem 2. \hfill \Box

**Proof of Theorem 3:** We write

\[ \text{(16)} \quad L_{q^*\alpha_k} f(x) = \frac{1}{q^*_{\alpha_k}} \sum_{j=1}^{q^*_{\alpha_k}} S_j f(x) \]

\[ = \frac{1}{q^*_{\alpha_k}} \sum_{j=1}^{M_{2\alpha_k}-1} S_j f(x) + \frac{1}{q^*_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q^*_{\alpha_k}} S_j f(x) = III + IV. \]

Since (see 9)
\[
|S_j f(x)| \leq c \frac{M_{2\alpha_k}^{1/p}}{\sqrt{\alpha_k-1}}, \quad j < M_{2\alpha_k}.
\]

For $III$ we can write

\[ \text{(17)} \quad |III| \leq c \frac{M_{2\alpha_k}^{1/p}}{\sqrt{\alpha_k-1}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{1}{q^*_{\alpha_k} - j} \leq c \frac{M_{2\alpha_k}^{1/p}}{\sqrt{\alpha_k-1}} \]

Using (11) we have

\[
IV = \frac{1}{lq_{sk}^{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{sk}^{\alpha_k}} \frac{1}{q_{\alpha_k,s} - j} \left( \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_{\eta}}^{1/p-1}}{\sqrt{\alpha_{\eta}}} \left( D_{M_{2\alpha_{\eta}+1}}(x) - D_{M_{2\alpha_{\eta}}}(x) \right) \right)
+ \frac{1}{lq_{sk}^{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{sk}^{\alpha_k}} \left( D_j(x) - D_{M_{2\alpha_k}}(x) \right) = IV_1 + IV_2.
\]

Applying (4) in \( IV_1 \) we have

\[
|IV_1| \leq c \frac{M_{2\alpha_k-1}^{1/p}}{\sqrt{\alpha_k-1}}.
\]

From (14) we obtain

\[
IV_2 = \frac{1}{lq_{sk}^{\alpha_k}} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \psi_{M_{2\alpha_k}}^{1/p-1} \sum_{j=0}^{M_{2\alpha_k}-1} \frac{D_j(x)}{M_{2\alpha_k} - j}.
\]

Let \( x \in I_{2s} \setminus I_{2s+1} \). Then \( D_j(x) = j, j < M_{2s} \). Consequently,

\[
\sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \frac{j}{M_{2s} - j} = \sum_{j=0}^{M_{2s}-1} \left( \frac{M_{2s}}{M_{2s} - j} - 1 \right) \geq c s M_{2s}.
\]

Then

\[
|IV_2| \geq c \frac{M_{2\alpha_k-1}^{1/p}}{\alpha_k^{3/2}} s M_{2s}, \quad x \in I_{2s} \setminus I_{2s+1}.
\]

Combining (5), (16)-(21) for \( x \in I_{2s} \setminus I_{2s+1}, s = [\alpha_k/2], \ldots, \alpha_k \), and \( 0 < p \leq 1 \) we have

\[
\left| Lq_{sk}^{\alpha_k} f(x) \right| \geq c \frac{M_{2\alpha_k-1}^{1/p}}{\alpha_k^{3/2}} s M_{2s} - c \frac{M_{\alpha_k}}{\alpha_k} \geq c \frac{M_{2\alpha_k-1}^{1/p}}{\alpha_k^{3/2}} s M_{2s}
\]
Then
\[
\int_{G_m} |L^s f (x)|^p \, d\mu (x) \geq \sum_{s=\lceil mk/2 \rceil}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} |L^s f (x)|^p \, d\mu (x)
\]
\[
\geq c \sum_{s=\lceil mk/2 \rceil}^{m_k} \int_{I_{2s} \setminus I_{2s+1}} \left( \frac{M_{2s}^{1/p - 1}}{\alpha_k^{3/2}} \right)^p \, d\mu (x)
\]
\[
\geq c \sum_{s=\lceil mk/2 \rceil}^{m_k} \frac{M_{2s}^{1-p}}{\alpha_k^{p/2}} M_{2s}^{p-1}
\]
\[
\geq \begin{cases} 
\frac{\alpha_k^{3/2}}{\alpha_k^{p/2}}, & \text{when } 0 < p < 1, \\
\frac{1}{c \alpha_k}, & \text{when } p = 1,
\end{cases} \quad \text{when } k \to \infty.
\]

Theorem 3 is proved. \qed

REFERENCES


256 GEORGE TEPHNADZE


Received December 01, 2010.

Institute of Mathematics, Faculty of Exact and Natural Sciences, Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
E-mail address: giorgitephnadze@gmail.com