Abstract. In this paper, we define and study Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces. Further, we consider the finite dimensional Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces and prove some theorems about completeness, compactness and weak convergence in these spaces. As application, we get a stability result in the setting of Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces.

1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by Zadeh in 1965 [44]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [2, 21, 15, 16, 18, 19, 20, 29, 39]. One of the problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept fuzzy normed spaces. In 1984, Katsaras [26] defined a fuzzy norm on a linear space and at the same year Wu and Fang [42] also introduced fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. Some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [8, 9, 14, 28, 40, 43]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [27]. In 2003, Bag and Samanta [6] modified the definition of Cheng and Mordeson [10] by removing a regular condition.

In this paper, we define the notion of Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces using [37]. Further, we consider finite dimensional Cheng-Mordeson...
\(\mathcal{L}\)-fuzzy normed spaces and prove some theorems about completeness, compactness and weak convergence in these spaces.

In this paper, \(\mathcal{L} = (L, \leq_L)\) is a complete lattice, i.e. a partially ordered set in which every nonempty subset admits supremum and infimum, and \(0_L = \inf L, 1_L = \sup L\).

**Definition 1.1** (see [17]). 1.1 Let \(\mathcal{L} = (L, \leq_L)\) be a complete lattice and let \(U\) be a non-empty set called the universe. An \(\mathcal{L}\)-fuzzy set in \(U\) is defined as a mapping \(A: U \rightarrow L\). For each \(u \in U\), \(A(u)\) represents the degree (in \(L\)) to which \(u\) is an element of \(A\).

**Lemma 1.2** (see [12]). Consider the set \(L^*\) and operation \(\leq_{L^*}\) defined by

\[
L^* = \{ (x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1 \},
\]

\[
(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, \ x_2 \geq y_2
\]

for all \((x_1, x_2), (y_1, y_2) \in L^*\). Then \(L^*, \leq_{L^*}\) is a complete lattice.

**Definition 1.3** (see [4]). An intuitionistic fuzzy set \(\mathcal{A}_{c,n}\) in the universe \(U\) is an object \(\mathcal{A}_{c,n} = \{ (u, \zeta_A(u), \eta_A(u)) : u \in U \}\), where \(\zeta_A(u) \in [0,1]\) and \(\eta_A(u) \in [0,1]\) for all \(u \in U\) are called the membership degree and the non-membership degree, respectively, of \(u\) in \(\mathcal{A}_{c,n}\) and, furthermore, satisfy \(\zeta_A(u) + \eta_A(u) \leq 1\).

We define mapping \(\wedge: L^2 \rightarrow L\) as

\[
\wedge(x, y) = \begin{cases} x, & \text{if } x \leq_L y \\ y, & \text{if } y \leq_L x \end{cases}
\]

For example,

\[
\wedge(x, y) = (\min(x_1, y_1), \max(x_2, y_2)),
\]

in which \(x = (x_1, x_2), y = (y_1, y_2) \in L^*\).

**Definition 1.4.** A negator on \(\mathcal{L}\) is any decreasing mapping \(\mathcal{N}: L \rightarrow L\) satisfying \(\mathcal{N}(0_L) = 1_L\) and \(\mathcal{N}(1_L) = 0_L\). If \(\mathcal{N}(\mathcal{N}(x)) = x\) for all \(x \in L\), then \(\mathcal{N}\) is called an involutive negator.

The negator \(N_s\) on \([0,1], \leq\) defined as \(N_s(x) = 1 - x\) for all \(x \in [0,1]\) is called the standard negator on \([0,1], \leq\). In this paper, the involutive negator \(\mathcal{N}\) is fixed.

**Definition 1.5.** The pair \((V, \mathcal{P})\) is said to be an Cheng-Mordeson \(\mathcal{L}\)-fuzzy normed space (briefly, \(\text{CM}\)-fuzzy normed space) if \(V\) is vector space and \(\mathcal{P}\) is a \(\mathcal{L}\)-fuzzy set on \(V \times [0, +\infty]\) satisfying the following conditions: for all \(x, y \in V\) and \(t, s \in [0, +\infty]\),

(a) \(\mathcal{P}(x, t) = 0_L\) for all \(t \leq 0\);

(b) \(\mathcal{P}(x, t) = 1_L\) if and only if \(x = 0\);

(c) \(\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})\) for each \(\alpha \neq 0\);

(d) \(\wedge(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)\);

(e) \(\mathcal{P}(x, \cdot) : [0, \infty] \rightarrow L\) is continuous;
(f) $\lim_{t \to 0} \mathcal{P}(x, t) = 0 \mathcal{L}$ and $\lim_{t \to \infty} \mathcal{P}(x, t) = 1 \mathcal{L}$.

In this case $\mathcal{P}$ is called an $\mathcal{L}$-fuzzy norm. If $\mathcal{P} = \mathcal{P}_{\mu, \nu}$ is an intuitionistic fuzzy set (see Definition 1.3), then the pair $(V, \mathcal{P}_{\mu, \nu})$ is said to be an 

**Cheng-Mordeson intuitionistic fuzzy normed space.**

**Example 1.6.** Let $(V, \| \cdot \|)$ be a normed space. We define $\wedge(a, b)$ by $\wedge(a, b) := (\min(a_1, b_1), \max(a_2, b_2))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^2$ and let $\mathcal{P}_{\mu, \nu}$ be the intuitionistic fuzzy set on $V \times [0, +\infty[$ defined as follows:

$$\mathcal{P}_{\mu, \nu}(x, t) = \left( \frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $t \in \mathbb{R}^+$. Then $(V, \mathcal{P}_{\mu, \nu})$ is a Cheng-Mordeson intuitionistic fuzzy normed space.

**Definition 1.7.** (1) A sequence $(x_n)_{n \in \mathbb{N}}$ in a $\mathcal{CML}$-fuzzy normed space $(V, \mathcal{P})$ is called a Cauchy sequence if, for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$\mathcal{P}(x_n - x_m, t) > _{\mathcal{L}} \mathcal{N}(\varepsilon),$$

where $\mathcal{N}$ is a negator on $\mathcal{L}$.

(2) A sequence $(x_n)_{n \in \mathbb{N}}$ is said to be convergent to $x \in V$ in the $\mathcal{CML}$-fuzzy normed space $(V, \mathcal{P})$, which is denoted by $x_n \xrightarrow{\mathcal{P}} x$ if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.

(3) A $\mathcal{CML}$-fuzzy normed space $(V, \mathcal{P})$ is said to be complete if and only if every Cauchy sequence in $V$ is convergent.

**Lemma 1.8** (see [37]). Let $\mathcal{P}$ be a $\mathcal{CML}$-fuzzy norm on $V$. Then we have the following:

(i) $\mathcal{P}(x, t)$ is nondecreasing with respect to $t$ for all $x \in V$;

(ii) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in V$ and $t \in [0, +\infty[$.

**Definition 1.9.** Let $(V, \mathcal{P})$ be an $\mathcal{CML}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. For all $t \in [0, +\infty[$, we define the open ball $B(x, r, t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as follows:

$$B(x, r, t) = \{ y \in V \mid \mathcal{P}(x - y, t) > _{\mathcal{L}} \mathcal{N}(r) \}$$

and define the unit ball of $V$ by

$$B(0, r, 1) = \{ x : \mathcal{P}(x, 1) > _{\mathcal{L}} \mathcal{N}(r) \}.$$

A subset $A \subseteq V$ is said to be open if, for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_\mathcal{P}$ denote the family of all open subsets of $V$. Then $\tau_\mathcal{P}$ is called the topology induced by the $\mathcal{CML}$-fuzzy norm $\mathcal{P}$.

**Definition 1.10.** Let $(V, \mathcal{P})$ be a $\mathcal{CML}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. A subset $A$ of $V$ is said to be $\mathcal{LF}$-bounded if there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x, t) > _{\mathcal{L}} \mathcal{N}(r)$ for all $x \in A$. 
Theorem 1.11. In a $\text{CML}$-fuzzy normed space $(V, \mathcal{P})$, every compact set is closed and $L$-bounded.

Lemma 1.12 (see [13]). Let $(V, \mathcal{P})$ be a $\text{CML}$-fuzzy normed space. Let $\mathcal{N}$ be a continuous negator on $\mathcal{L}$. If we define $E_{\lambda, \mathcal{P}} : V \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$$E_{\lambda, \mathcal{P}}(x) = \inf \{ t > 0 : \mathcal{P}(x, t) > L\mathcal{N}(\lambda) \}$$

for all $\lambda \in \mathcal{L} \setminus \{0, 1\}$ and $x \in V$. Then we have the following:

(i) $E_{\lambda, \mathcal{P}}(\alpha x) = |\alpha|E_{\lambda, \mathcal{P}}(x)$ for all $x \in A$ and $\alpha \in \mathbb{R}$.

(ii) $E_{\lambda, \mathcal{P}}(x + y) \leq E_{\lambda, \mathcal{P}}(x) + E_{\lambda, \mathcal{P}}(y)$ for all $x, y \in V$.

(iii) A sequence $(x_n)_{n \in \mathbb{N}}$ is convergent with respect to the $\text{CML}$-fuzzy norm $\mathcal{P}$ if and only if $E_{\lambda, \mathcal{P}}(x_n - x) \to 0$. Also, the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the $\text{CML}$-fuzzy norm $\mathcal{P}$ if and only if it is a Cauchy sequence with respect to $E_{\lambda, \mathcal{P}}$.

Lemma 1.13 (see [13]). A subset $A$ of $\mathbb{R}$ is $L$-bounded in $(\mathbb{R}, \mathcal{P})$ if and only if it is bounded in $\mathbb{R}$.

Corollary 1.14 (see [13]). If the real sequence $(\beta_n)_{n \in \mathbb{N}}$ is $L$-bounded, then it has at least one limit point.

Definition 1.15. Let $V$ be a vector space and let $f$ be a real functional on $V$. We define

$$\tilde{V} = \{ f : \mathcal{P}_0(f(x), t) \geq_L \mathcal{P}(cx, t), c \neq 0 \}$$

for all $t > 0$.

Lemma 1.16 (see [38]). If $(V, \mathcal{P})$ is a $\text{CML}$-fuzzy normed space, then we have

(a) the function $(x, y) \mapsto x + y$ is continuous.

(b) the function $(\alpha, x) \mapsto \alpha x$ is continuous.

By the above lemma, a $\text{CML}$-fuzzy normed space is Hausdorff Topological Vector Space.

2. CML-Fuzzy Finite Dimensional Normed Spaces

Theorem 2.1. Let $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in vector space $V$ and let $(V, \mathcal{P})$ be a $\text{CML}$-fuzzy normed space. Then there exist $c \neq 0$ and a $\text{CML}$-fuzzy normed space $(\mathbb{R}, \mathcal{P}_0)$ such that, for every choice of the $n$ real scalars $\alpha_1, \cdots, \alpha_n$,

$$\mathcal{P}(\alpha_1 x_1 + \cdots + \alpha_n x_n, t) \leq_L \mathcal{P}_0(c \sum_{j=1}^{n} |\alpha_j|, t).$$

Proof. Put $s = |\alpha_1| + \cdots + |\alpha_n|$. If $s = 0$, all $\alpha_j$’s must be zero and so (2.1) holds for any $c$. Let $s > 0$. Then (2.1) is equivalent to the inequality which we
obtain from (2.1) by dividing by $s$ and putting $\beta_j = \frac{a_j}{s}$, that is,

$$(2.2) \quad \mathcal{P}(\beta_1 x_1 + \cdots + \beta_n x_n, t') \leq \mathcal{P}_0(c, t'), \quad (t' = \frac{t}{s} \sum_{j=1}^n |\beta_j| = 1).$$

Hence, it suffices to prove the existence of a $c \neq 0$ and $\mathcal{L}$-fuzzy norm $\mathcal{P}_0$ such that (2.2) holds. Suppose that this is not true. Then there exists a sequence $(y_m)_{m \in \mathbb{N}}$ of vectors,

$$y_m = \beta_{1,m} x_1 + \cdots + \beta_{n,m} x_n, \quad (\sum_{j=1}^n |\beta_{j,m}| = 1)$$

such that $\mathcal{P}(y_m, t) \to 1_\mathcal{L}$ as $m \to \infty$ for all $t > 0$. Since $\sum_{j=1}^n |\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \leq 1$ and so, by Lemma 1.13, the sequence of $(\beta_{j,m})$ is $\mathcal{L}F$-bounded. By Corollary 1.14, $(\beta_{1,m})$ has a convergent subsequence. Let $\beta_1$ denote the limit of that subsequence and let $(y_{1,m})$ denote the corresponding subsequence of $(y_m)$. By the same argument, $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence $(\beta_{2,m})$ of real scalars convergence. Let $\beta_2$ denote the limit. Continuing this process, after $n$ steps, we obtain a subsequence $(y_{n,m})$ of $(y_m)$ such that

$$y_{n,m} = \sum_{j=1}^n \gamma_{j,m} x_j (\sum_{j=1}^n |\gamma_{j,m}| = 1)$$

and $\gamma_{j,m} \to \beta_j$ as $m \to \infty$. By Lemma 1.12 (ii), for any $\mu \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, we have

$$E_{\mu, \mathcal{P}}(y_{n,m} - \sum_{j=1}^n \beta_j x_j) = E_{\mu, \mathcal{P}}(\sum_{j=1}^n (\gamma_{j,m} - \beta_j) x_j) \leq \sum_{j=1}^n |\gamma_{j,m} - \beta_j| E_{\mu, \mathcal{P}}(x_j) \to 0$$

as $m \to \infty$. By Lemma 1.12 (iii), we conclude

$$\lim_{m \to \infty} y_{n,m} = \sum_{j=1}^n \beta_j x_j (\sum_{j=1}^n |\beta_j| = 1),$$

so that not all $\beta_j$ can be zero. Put $y = \sum_{j=1}^n \beta_j x_j$. Since $\{x_1, \cdots, x_n\}$ is a linearly independent set, we have $y \neq 0$. Since $\mathcal{P}(y_m, t) \to 1_\mathcal{L}$ by assumption, we have $\mathcal{P}(y_{n,m}, t) \to 1_\mathcal{L}$. Hence it follows that

$$\mathcal{P}(y, t) = \mathcal{P}((y - y_{n,m}) + y_{n,m}, t) \geq_{\mathcal{L}} \mathcal{P}(y - y_{n,m}, t/2), \mathcal{P}(y_{n,m}, t/2) \to 1_\mathcal{L}$$

and so $y = 0$, which is a contradiction. \qed
Theorem 2.2. Every finite dimensional subspace \( W \) of a \( CML \)-fuzzy normed space \((V, P)\) is complete. In particular, every finite dimensional \( CML \)-fuzzy normed space is complete.

Proof. Let \((y_m)_{m \in \mathbb{N}}\) be a Cauchy sequence in \( W \) such that \( y \) is its limit. Then we show that \( y \in W \). Let \( \dim W = n \) and \( \{x_1, \ldots, x_n\} \) any linearly independent subset for \( Y \). Then each \( y_m \) has a unique representation of the form

\[
y_m = \alpha_1^{(m)} x_1 + \cdots + \alpha_n^{(m)} x_n.
\]

Since \((y_m)_{m \in \mathbb{N}}\) is a Cauchy sequence, for any \( \varepsilon \in L \setminus \{0, 1\} \), there is a positive integer \( n_0 \) such that

\[
\mathcal{N}(\varepsilon) < L \mathcal{P}(y_m - y_k, t),
\]

whenever \( m, k > n_0 \) and \( t > 0 \). From this and the last theorem, we have

\[
\mathcal{N}(\varepsilon) < L \mathcal{P}(y_m - y_k, t) = L \mathcal{P}\left(\sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j^{(k)}) x_j, t\right)
\]

\[
\leq L \mathcal{P}_0\left(\sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(k)}| c, t\right) \leq L \mathcal{P}_0\left(1, \frac{t}{\sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(k)}|}\right)
\]

\[
\leq L \mathcal{P}_0\left(1, \frac{t}{|\alpha_j^{(m)} - \alpha_j^{(k)}|}\right) = \mathcal{P}_0\left(\alpha_j^{(m)} - \alpha_j^{(k)}\right) = c
\]

for some \( c \neq 0 \) and \( \mathcal{P}_0 \). This shows that each of the \( n \) sequences \((\alpha_j^{(m)})_{m \in \mathbb{N}}\) where \( j \in \{1, 2, 3, \ldots, n\} \) is a Cauchy sequence in \( \mathbb{R} \). Hence these sequences converge. Let \( \alpha_j \) denote the limit. Using these \( n \) limits \( \alpha_1, \ldots, \alpha_n \), we define

\[
y = \alpha_1 x_1 + \cdots + \alpha_n x_n.
\]

Clearly, \( y \in W \). Furthermore, by Lemma 1.12 (ii), for any \( \mu \in L \setminus \{0, 1\} \), we have

\[
E_{\mu, \mathcal{P}}(y_m - y) = E_{\mu, \mathcal{P}}\left(\sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j) x_j\right) \leq \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j| E_{\mu, \mathcal{P}}(x_j) \to 0
\]

whenever \( m \to \infty \). This shows that the arbitrary sequence \((y_m)_{m \in \mathbb{N}}\) is convergent in \( W \). Hence \( W \) is complete. \( \square \)

Corollary 2.3. Every finite dimensional subspace \( W \) of a \( CML \)-fuzzy normed space \((V, \mathcal{P})\) is closed in \( V \).

Theorem 2.4. In a finite dimensional \( CML \)-fuzzy normed space \((V, \mathcal{P})\), any subset \( K \subset V \) is compact if and only if \( K \) is closed and \( LF \)-bounded.

Proof. By Theorem 1.11, compactness implies closedness and \( LF \)-boundedness.

Conversely, let \( K \) be closed and \( LF \)-bounded. Let \( \dim V = n \) and \( \{x_1, \ldots, x_n\} \) be a linearly independent set of \( V \). We consider a sequence \((x^{(m)})_{m \in \mathbb{N}}\) in


K. Each $x^{(m)}$ has a representation by

$$x^{(m)} = \alpha_1^{(m)} x_1 + \cdots + \alpha_n^{(m)} x_n.$$  

Since, $K$ is $\mathcal{LF}$-bounded, so is $(x^{(m)})_{m \in \mathbb{N}}$ and so there exist $t > 0$ and $r \in L \setminus \{0, 1\}$ such that $P(x^{(m)}, t) >_L N(r)$ for all $m \in \mathbb{N}$.

On the other hand, by Theorem 2.1, there exist $c \neq 0$ and a $\mathcal{L}$-fuzzy norm $P_0$ such that

$$N(r) <_L P(x^{(m)}, t) = P\left(\sum_{j=1}^{n} \alpha_j^{(m)} x_j, t\right)$$

$$\leq_L P_0\left(c \sum_{j=1}^{n} |\alpha_j^{(m)}|, t\right) \leq_L P_0\left(1, \frac{t}{c \sum_{j=1}^{n} |\alpha_j^{(m)}|}\right)$$

$$\leq_L P_0\left(1, \frac{t}{c |\alpha_j^{(m)}|}\right) = P_0\left(\alpha_j^{(m)}, \frac{t}{c}\right).$$

Hence, the sequence $(\alpha_j^{(m)})_{m \in \mathbb{N}}$ for any fixed $j$ is $\mathcal{LF}$-bounded and, by Corollary 1.14, has a limit point $\alpha_j$, where $1 \leq j \leq n$. We consider that $(x^{(m)})_{m \in \mathbb{N}}$ has a subsequence $(z_m)_{m \in \mathbb{N}}$ which converges to $z = \sum_{j=1}^{n} \alpha_j x_j$. Since $K$ is closed, $z \in K$. This shows that an arbitrary sequence $(x^{(m)})_{m \in \mathbb{N}}$ in $K$ has a subsequence which converges in $K$. Hence, $K$ is compact. □

**Remark 2.5.** In a CML-fuzzy normed space $(V, P)$ whenever $P(x, t) >_L N(r)$ for all $x \in V$, $t > 0$ and $r \in L \setminus \{0, 1\}$, we can find $t_0 \in ]0, t[\]$ such that $P(x, t_0) >_L N(r)$ (see [15]).

**Lemma 2.6.** Let $(V, P)$ be a CML-fuzzy normed space and let $A$ be a subspace of $V$. Define

$$D(x_1 - A, t) = \sup\{P(x_1 - y, t) : y \in A\}$$

for all $x_1 \in V$ and $t > 0$. Then, for any $\varepsilon \in L \setminus \{1\}$ and $x_1 \in V \setminus A$, there exists $y_1 \in A$ such that

$$\wedge(D(x_1 - A, t), \varepsilon) <_L P(x_1 - y_1, t) \leq_L D(x_1 - A, t).$$

The proof is straightforward.

**Lemma 2.7.** Let $(V, P)$ be a CML-fuzzy normed space and let $A$ be a subset of $V$. If we define

$$p_1 = \inf\{t > 0 : D(x_1 - A, t) >_L N(\lambda)\}$$

and

$$p_2 = \inf\{t > 0 : \wedge(D(x_1 - A, t), \varepsilon) >_L N(\lambda)\},$$

in which $\varepsilon \in L \setminus \{1\}$. Then there exists $\delta \in ]0, t[\]$ such that $p_2 \geq p_1 + \delta$.  

Proof. Since \( \land \left( \mathcal{D}(x_1 - A, t), \varepsilon \right) <_L \mathcal{D}(x_1 - A, t) \), by Remark 2.5, there exists \( \delta \in [0, t] \) such that \( \land \left( \mathcal{D}(x_1 - A, t), \varepsilon \right) <_L \mathcal{D}(x_1 - A, t - \delta) \) and so

\[
p_2 = \inf \{ t > 0 : \land \left( \mathcal{D}(x_1 - A, t), \varepsilon \right) >_L \mathcal{N}(\lambda) \}
\geq \inf \{ t > 0 : \mathcal{D}(x_1 - A, t - \delta) >_L \mathcal{N}(\lambda) \}
= \inf \{ t + \delta > 0 : \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda) \} = p_1 + \delta. \]

\[\square\]

Lemma 2.8. Let \((V, \mathcal{P})\) be a CML-fuzzy normed space and let \(A\) be a nonempty closed subspace of \(V\). Then \(x \in A\) if and only if \(\mathcal{D}(x - A, t) = 1_L\) for all \(t > 0\).

Proof. Let \(\mathcal{D}(x - A, t) = 1_L\). By definition, there exists a sequence \((x_n)_{n \in \mathbb{N}}\) in \(A\) such that \(\mathcal{P}(x - x_n, t) \rightarrow 1_L\). Hence \(x - x_n \rightarrow 0\) or equivalently \(x_n \rightarrow x\) and, since \(A\) is closed, \(x \in A\). The converse is trivial. \[\square\]

Theorem 2.9. Let \((V, \mathcal{P})\) be a CML-fuzzy normed space and let \(A\) be a nonempty closed subspace of \(V\). Then, for any \(y \in A\), there exist \(x_0 \in V \setminus A\) and \(\lambda_0 \in L\) such that \(x_0 \in B(0, \lambda_1)\) and \(E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1\) for all \(\lambda_0 \leq \lambda \leq 1_L\).

Proof. Since \(A\) is a nonempty closed subspace of \(V\), by Lemma 2.8, there exists \(x_1 \in V \setminus A\) such that \(\mathcal{D}(x_1 - A, t) <_L 1_L\) for all \(t > 0\). Let

\[
\sup_{t > 0} \mathcal{D}(x_1 - A, t) = \sigma.
\]

Let \(\lambda_0 = \mathcal{N}(\sigma)\). Then, for all \(\lambda_0 \leq \lambda \leq 1_L\), we have

\[
\sup_{t > 0} \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda).
\]

By the property of sup, there exists \(t_0 > 0\) such that \(\mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)\) for all \(t \geq t_0\). By Lemma 2.6, there exists \(y_1 \in A\) such that

\[
\land \left( \mathcal{D}(x_1 - A, t), \varepsilon \right) <_L \mathcal{P}(x_1 - y_1, t)
\]

for all \(\varepsilon \in L \setminus \{1_L\}\) and \(t \geq 0\). Taking \(x_0 = \frac{x_1 - y_1}{p_2}\), by Lemma 2.7, we have

\[
\mathcal{P}(x_0, 1) = \mathcal{P} \left( \frac{x_1 - y_1}{p_2}, 1 \right) = \mathcal{P}(x_1 - y_1, p_2) \leq_L \land \left( \mathcal{D}(x_1 - A, p_2), \varepsilon \right)
\geq_L \land \left( \mathcal{D}(x_1 - A, p_1 + \delta), \varepsilon \right) >_L \land \left( \mathcal{N}(\lambda), \varepsilon \right).
\]

Since, \(\varepsilon \in L \setminus \{1_L\}\) is arbitrary, we have \(\mathcal{P}(x_0, 1) >_L \mathcal{N}(\lambda)\), i.e., \(x_0 \in B(0, \lambda, 1)\) for all \(\lambda_0 \leq \lambda \leq 1_L\). Taking \(\delta_1 = \frac{\delta}{p_2}\), by Lemma 2.7, we have

\[
\land \left( \mathcal{P}(x_0 - y, N_{\delta_1}(\delta_1)), \varepsilon \right) = \land \left( \mathcal{P}(x_1 - (y_1 + p_2 y), p_2 N_{\delta_1}(\delta_1)), \varepsilon \right)
\leq_L \land \left( \mathcal{D}(x_1 - A, p_2 - \delta), \varepsilon \right) \leq_L \mathcal{N}(\lambda).
\]

Letting \(\varepsilon \rightarrow 1_L\) and \(\delta \rightarrow 0\), we have \(\mathcal{P}(x_0 - y, 1) \leq_L \mathcal{N}(\lambda)\) and so

\[
E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1
\]

for all \(y \in A\) and \(x_0 \in B(0, \lambda, 1)\). \[\square\]
Lemma 2.10. Let \{x_1, \ldots, x_n\} be a linearly independent set of vectors in vector space \(V\) and \((V, \mathcal{P})\) be a CML-fuzzy normed space. Then there exists \(k \neq 0\) such that, for every choice of the \(n\) real scalars \(\alpha_1, \ldots, \alpha_n\),

\[
E_{\lambda, \mathcal{P}} \left( \sum_{j=1}^{n} \alpha_j x_j \right) \geq |k| \sum_{j=1}^{n} |\alpha_j|.
\]

Proof. By Theorem 2.1, there exist \(c \neq 0\) and an \(\mathcal{L}\)-fuzzy norm \(\mathcal{P}_0\) such that

\[
\mathcal{P} \left( \sum_{j=1}^{n} \alpha_j x_j, t \right) \leq_L \mathcal{P}_0 \left( c \sum_{j=1}^{n} |\alpha_j|, t \right).
\]

Therefore, we have

\[
E_{\lambda, \mathcal{P}} \left( \sum_{j=1}^{n} \alpha_j x_j \right) \geq E_{\lambda, \mathcal{P}_0} \left( c \sum_{j=1}^{n} |\alpha_j| \right) = |c| \sum_{j=1}^{n} |\alpha_j|E_{\lambda, \mathcal{P}_0}(1).
\]

Taking \(k = cE_{\lambda, \mathcal{P}_0}(1)\), we have

\[
E_{\lambda, \mathcal{P}} \left( \sum_{j=1}^{n} \alpha_j x_j \right) \geq |k| \sum_{j=1}^{n} |\alpha_j|.
\]

\(\square\)

Theorem 2.11. Let \((V, \mathcal{P})\) be a CML-fuzzy normed space. Then \((V, \mathcal{P})\) is finite dimensional if and only if the unit ball \(B(0, \lambda, 1)\) is compact.

Proof. Let \(\dim V = n\) and \(\{x_1, \ldots, x_n\}\) a basis for \(V\). We consider any sequence \((x^{(m)})_{m \in \mathbb{N}}\) in \(B(0, \lambda, 1)\). Each \(x^{(m)}\) has the representation by

\[
x^{(m)} = \sum_{j=1}^{n} \alpha_j^{(m)} x_j.
\]

By Lemmas 2.7 and 2.10, we have

\[
1 \geq E_{\lambda, \mathcal{P}}(x^{(m)}) \geq |k| \sum_{j=1}^{n} |\alpha_j^{(m)}|,
\]

where \(k \neq 0\). Hence the sequence \((\alpha_j^{(m)})_{m \in \mathbb{N}}\) is bounded and has a limit point \(\alpha_j\) (1 \(\leq j \leq n\)). Therefore, \((x^{(m)})_{m \in \mathbb{N}}\) has a subsequence \((x^{(m_k)})_{k \in \mathbb{N}}\) which converges to \(x = \sum_{j=1}^{n} \alpha_j x_j\).

On the other hand, for any \(\varepsilon \neq 0\), there exists \(k_0 \in \mathbb{N}\) such that, for all \(k \geq k_0\),

\[
\mathcal{P}(x, 1 + \delta) \geq_L \mathcal{P}(x^{(m_k)} - x, \delta), \mathcal{P}(x^{(m_k)}, 1)) \geq_L \mathcal{N}(\varepsilon), \mathcal{N}(\lambda))
\]

for all \(\delta > 0\). Since \(\varepsilon \neq 0\) and \(\delta > 0\) are arbitrary, it follows that

\[
\mathcal{P}(x, 1) \geq_L 1_L, \mathcal{N}(\lambda)) = \mathcal{N}(\lambda)
\]

and, consequently, \(x \in B(0, \lambda, 1)\). Hence, \(B(0, \lambda, 1)\) is compact.

Conversely, assume that the unit balls be compact, but \((V, \mathcal{P})\) is not finite dimensional. We choose \(x_1 \neq 0\) in \(V\), for any \(k_1 \in \mathbb{R}\), let \(V_1 = \{k_1 x_1 : x_1 \in \}

\]
V, k_1 \in \mathbb{R}\}. By Theorem 2.9, for all \lambda_{0,i} < \lambda \leq 1, there exist \(x_2 \in V \setminus V_1\) and \(x_2 \in B(0, \lambda, 1)\) such that \(E_{\lambda, \mathcal{P}}(x_2 - x_1) \geq 1\).

In this case, \(x_1\) and \(x_2\) are linear independent. In fact, if \(x_1\) and \(x_2\) are dependent, then there exists \(k_1, k_2 \in \mathbb{R}\) (we might as well assume \(k_2 \neq 0\)) such that \(k_1x_1 + k_2x_2 = 0\) and \(x_2 = \frac{-k_1}{k_2}x_1 \in V_1\), which is a contradiction.

Let \(V_2 = \{k_1x_1 + k_2x_2 : x_1 \in V_1, x_2 \in V \setminus V_1, k_1, k_2 \in \mathbb{R}\}\). By Theorem 2.9, for all \(\lambda_{0,2} < \lambda \leq 1\), there exist \(x_3 \in V \setminus V_2\) and \(x_3 \in B(0, \lambda, 1)\) such that \(E_{\lambda, \mathcal{P}}(x_3 - y) \geq 1\) where \(y \in V_2\). In particular, if we choose \(y = x_1\) and \(y = x_2\), then \(E_{\lambda, \mathcal{P}}(x_3 - x_1) \geq 1\) and \(E_{\lambda, \mathcal{P}}(x_3 - x_2) \geq 1\). By the same way, we can choose \((x_n)_{n \in \mathbb{N}} \subset B(0, \lambda, 1)\) such that \(E_{\lambda, \mathcal{P}}(x_m - x_n) \geq 1\) where \(m \neq n\) for all \(\lambda_{0,n-1} < \lambda \leq 1\). If we put \(\lambda_0 = \bigvee_{1 \leq i \leq n-1} \lambda_{0,i}\), then the sequence \((x_n)_{n \geq 2}\) lies in \(B(0, \lambda, 1)\) and \(E_{\lambda, \mathcal{P}}(x_m - x_n) \geq 1\) for all \(\lambda_0 < \lambda \leq 1\). By Lemma 1.12, (ii), the sequence \((x_n)_{n \geq 2}\) has not any convergent subsequence in \(V\), which is a contradiction. This completes the proof.

**Theorem 2.12.** Let \((V, \mathcal{P})\) be a finite dimensional \(\text{CM}\mathcal{L}\)-fuzzy normed space and let \(A\) be a closed subspace of \(V\). Then, for all \(\lambda > \lambda_0\), there exists \(x_0 \in B(0, \lambda, 1)\) such that

\[
\inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y) = 1.
\]

**Proof.** By Theorem 2.9, for any \(y_n \in A\), there exist \(x_n \in V \setminus A\) and \(\lambda_0 \in L\) such that

\[
(2.3) \quad x_n \in B(0, \lambda, 1), \quad E_{\lambda, \mathcal{P}}(x_n - y_n) \geq 1
\]

for all \(\lambda > \lambda_0\). Since \(V\) is finite dimensional, by Theorem 2.11, \(B(0, \lambda, 1)\) is compact and so there exists \(x_0 \in B(0, \lambda, 1)\) such that

\[
\mathcal{P}(x_{nk} - x_0, t) \to 1
\]

for all \(t > 0\), where \((x_{nk})_{k \in \mathbb{N}}\) is a subsequence of \((x_n)_{n \in \mathbb{N}}\). Since \(x_0 \in B(0, \lambda, 1)\), \(E_{\lambda, \mathcal{P}}(x_0) \leq 1\). Since the null element \(0 \in A\), we have

\[
1 \geq E_{\lambda, \mathcal{P}}(x_0) = E_{\lambda, \mathcal{P}}(x_0 - 0) \geq \inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y).
\]

Next, we prove that \(\inf_{y \in A} E_{\lambda, \mathcal{P}}(x_0 - y) \geq 1\). By (2.1), \(\mathcal{P}(x_n - y_n, 1) \leq N(\lambda)\). Let \(\mathcal{P}(x_0 - y, 1) > L N(\lambda)\) for all \(y \in A\). Then, by continuity of \(\text{CM}\mathcal{L}\)-fuzzy norm \(\mathcal{P}\) and Remark 2.5, we can find \(\lambda_1 \in L\) such that, for \(\delta \in ]0, 1[,\)

\[
\mathcal{P}(x_0 - y, N_\delta(\delta)) > L N(\lambda_1),
\]

and

\[
N(\lambda_1) > L N(\lambda).
\]

Since \(x_{nk} \to x_0\), there exists \(k_0 \in \mathbb{N}\) such that, for every \(k \geq k_0,\)

\[
\mathcal{P}(x_{nk} - x_0, t) > L N(\lambda_1)
\]
for all \( t > 0 \). By triangle inequality 1.5, (d), we have

\[
\mathcal{N}(\lambda) \geq_L \mathcal{P}(x_{n_k} - y_{n_k}, t) \geq_L (\mathcal{P}(x_{n_k} - x_0, t/2), \mathcal{P}(x_0 - y_{n_k}, t/2)) \geq_L \mathcal{N}(\lambda_1), \mathcal{N}(\lambda_1)) >_L \mathcal{N}(\lambda),
\]

which is a contradiction. Then, for any \( y \in A \), we have \( \mathcal{P}(x_0 - y, 1) \leq_L \mathcal{N}(\lambda) \), which implies \( \inf_{y \in A} E_{\lambda,\mathcal{P}}(x_0 - y) \geq 1 \). This completes the proof. \( \square \)

**Definition 2.13.** A sequence \( (x_m)_{m \in \mathbb{N}} \) in a \( CML \)-fuzzy normed space \((V, \mathcal{P})\) is said to be *weakly convergent* if there exists \( x \in V \) such that, for all \( f \in \tilde{V} \) and \( t > 0 \),

\[
\mathcal{P}(f(x_m) - f(x), t) \to 1_L.
\]

This is written by

\[
x_m \overset{W}{\to} x.
\]

**Theorem 2.14.** Let \((V, \mathcal{P})\) be a \( CML \)-fuzzy normed space and let \((x_m)_{m \in \mathbb{N}}\) be a sequence in \( V \). Then we have the following:

(i) Convergence implies weak convergence with the same limit.

(ii) If \( \dim V < \infty \), then weak convergence implies convergence.

**Proof.** (i) Let \( x_m \to x \). Then, for all \( t > 0 \), we have

\[
\mathcal{P}(x_m - x, t) \to 1_L.
\]

By Definition 1.15, for every \( f \in \tilde{V} \),

\[
\mathcal{P}_0(f(x_m) - f(x), t) = \mathcal{P}_0(f(x_m - x), t) = \mathcal{P}(x_m - x, t/c)(c \neq 0).
\]

Then \( x_m \overset{W}{\to} x \).

(ii) Let \( x_m \overset{W}{\to} x \) and \( \dim V = n \). Let \( \{x_1, \ldots, x_n\} \) be a linearly independent set of \( V \). Then \( x_m = \alpha_1^{(m)} x_1 + \cdots + \alpha_n^{(m)} x_n \) and \( x = \alpha_1 x_1 + \cdots + \alpha_n x_n \). By assumption, for all \( f \in \tilde{V} \) and \( t > 0 \), we have

\[
\mathcal{P}_0(f(x_m) - f(x), t) \to 1_L.
\]

We take in particular \( f_1, \ldots, f_n \) defined by \( f_j x_j = 1 \) and \( f_j x_i = 0 \) \((i \neq j)\). Therefore, \( f_j(x_m) = \alpha_j^{(m)} \) and \( f_j(x) = \alpha_j \). Hence \( f_j(x_m) \to f_j(x) \) implies \( \alpha_j^{(m)} \to \alpha_j \). From this and Lemma 1.12 (ii), we obtain

\[
E_{\mu,\mathcal{P}}(x_m - x) = E_{\mu,\mathcal{P}}\left(\sum_{j=1}^n (\alpha_j^{(m)} - \alpha_j)x_j\right) \leq \sum_{j=1}^n |\alpha_j^{(m)} - \alpha_j|E_{\lambda,\mathcal{P}}(x_j) \to 0
\]

as \( m \to \infty \). This shows that \((x_m)_{m \in \mathbb{N}}\) convergence to \( x \). \( \square \)

**Theorem 2.15.** A \( CML \)-fuzzy normed space \((V, \mathcal{P})\) is locally convex.
Proof. It suffices to consider the family of neighborhoods of the origin, \( B(0, r, t) \), with \( t > 0 \) and \( r \in L \setminus \{0, 1\} \). Let \( t > 0, r \in L \setminus \{0, 1\}, x, y \in B(0, r, t) \) and \( \alpha \in [0, 1] \). Then we have
\[
P(\alpha x + (1 - \alpha)y, t) \geq L \wedge (P(\alpha x, \alpha t), P((1 - \alpha)y, (1 - \alpha)t))
\]
\[
= \wedge (P(x, t), P(y, t)) > L N(r).
\]
Thus, \( \alpha x + (1 - \alpha)y \) belongs to \( B(0, r, t) \) for all \( \alpha \in [0, 1] \). \( \square \)

3. Stability of Cubic Functional Equations in \( L \)-Fuzzy Normed Spaces

The study of stability problems for functional equations is related to a question of Ulam [41] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [22]. Subsequently, the result of Hyers was generalized by T. Aoki [3] for additive mappings and by Th.M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. The paper [34] of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers for more information on such problems to e.g. [5, 11, 23, 35, 36].

The functional equation
\[
3f(x + 3y) + f(3x - y) = 15f(x + y) + 15f(x - y) + 80f(y)
\]
is said to be the cubic functional equation since the function \( f(x) = cx^3 \) is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem for the cubic functional equation was proved by Jun and Kim [24] for mappings \( f : X \to Y \), where \( X \) is a real normed space and \( Y \) is a Banach space. Later a number of mathematicians worked on the stability of some types of the cubic equation [25, 34]. In addition, Mirmostafaei, Mirzavaziri and Moslehian [33, 32], Alsina [1], Mihe\c{t} and Radu [30], Mihe\c{t} et. al. [31] and Baktash et. al. [7] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

The aim of this note, is to provide a result on the stability of the cubic functional equation (3.1) in fuzzy normed spaces and give a better error estimation.

Now, we state our main result.

**Theorem 3.1.** Let \( X \) be a linear space, \( (Z, P') \) be a \( CML \)-fuzzy normed space, \( \varphi : X \times X \to Z \) be a function such that for some \( 0 < \alpha < 27 \),
\[
P'\varphi(3x, 0, t) \geq L \, P'(\alpha \varphi(x, 0, t)) \quad (x, y \in X, t > 0)
\]
and \( \lim_{n \to \infty} \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t) = 1 \) for all \( x, y \in X \) and \( t > 0 \). Let \((Y, \mathcal{P})\) be a complete fuzzy normed space. If \( f : X \to Y \) is a mapping such that

\[
(3.3) \quad \mathcal{P}(3f(x + 3y) + f(3x - y) - 15f(x + y) - 15f(x - y) - 80f(y), t) \\
\geq L \mathcal{P}'(\varphi(x, y), t)
\]

where \( x, y \in X, t > 0 \). Then there exists a unique cubic mapping \( C : X \to Y \) such that

\[
(3.4) \quad \mathcal{P}(f(x) - C(x), t) \geq L \mathcal{P}'(\varphi(x, 0), (27 - \alpha)t)).
\]

**Proof.** Putting \( y = 0 \) in (3.3) we get

\[
(3.5) \quad \mathcal{P}\left(\frac{f(3x)}{27} - f(x), t\right) \geq L \mathcal{P}(\varphi(x, 0), 27t) \quad (x \in X, t > 0).
\]

Replacing \( x \) by \( 3^n x \) in (3.5), and using (3.2) we obtain

\[
(3.6) \quad \mathcal{P}\left(\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^n x)}{27^n}, t\right) \geq L \mathcal{P}'(\varphi(3^n x, 0), 27 \times 27^n t) \\
\geq L \mathcal{P}'(\varphi(x, 0), \frac{27 \times 27^n}{\alpha^n} t).
\]

Since \( \frac{f(3^n) - f(x)}{27^n} = \sum_{k=0}^{n-1} \left( \frac{f(3^{k+1} x)}{27^{k+1}} - \frac{f(3^k x)}{27^k} \right) \), by (3.6) we have

\[
\mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t \sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}\right) \geq L \mathcal{P}'(\varphi(x, 0), t) = \mathcal{P}'(\varphi(x, 0), t),
\]

that is,

\[
(3.7) \quad \mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t\right) \geq L \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right).
\]

By replacing \( x \) with \( 3^m x \) in (3.7) we observe that:

\[
(3.8) \quad \mathcal{P}\left(\frac{f(3^{n+m} x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, t\right) \geq L \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{27 \times 27^k}}\right).
\]

Then \( \left\{ \frac{f(3^n x)}{27^n} \right\} \) is a Cauchy sequence in \((Y, \mathcal{P})\). Since \((Y, \mathcal{P})\) is a complete CM\(\text{L}\)-fuzzy normed space this sequence convergent to some point \( C(x) \in Y \). Fix \( x \in X \) and put \( m = 0 \) in (3.8) to obtain

\[
(3.9) \quad \mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t\right) \geq L \mathcal{P}'\left(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right),
\]
and so for every $\delta > 0$ we have

\begin{equation}
\mathcal{P}(C(x) - f(x), t + \delta) \geq_L \mathcal{P}(C(x) - f(3^n x), \delta), \mathcal{P}(\frac{f(3^n x) - f(x) - \frac{15}{8} f(3^n x)}{27^n}, t) \geq_L \mathcal{P}(C(x) - f(3^n x), \delta), \mathcal{P}'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27^n}}).
\end{equation}

Taking the limit as $n \to \infty$ and using (3.10) we get

\begin{equation}
\mathcal{P}(C(x) - f(x), t + \delta) \geq_L \mathcal{P}'(\varphi(x, 0), t(27 - \alpha)).
\end{equation}

Since $\delta$ was arbitrary, by taking $\delta \to 0$ in (3.11) we get

\[ \mathcal{P}(C(x) - f(x), t) \geq_L \mathcal{P}'(\varphi(x, 0), t(27 - \alpha)). \]

Replacing $x, y$ by $3^n x, 3^n y$ in (3.3) to get

\[ \mathcal{P}\left(\frac{f(3^n(x + y))}{27^n} + \frac{f(3^n(3x - y))}{27^n} - \frac{15 f(3^n(x + y))}{27^n} - \frac{15 f(3^n(x - y))}{27^n} - \frac{80 f(3^n(y))}{27^n}, t\right) \geq_L \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t), \]

for all $x, y \in X$ and for all $t > 0$. Since $\lim_{n \to \infty} \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t) = 1$ we conclude that $C$ fulfills (3.1). To prove the uniqueness of the cubic function $C$, assume that there exists a cubic function $D: X \to Y$ which satisfies (3.4). Fix $x \in X$. Clearly $C(3^n x) = 27^n C(x)$ and $D(3^n x) = 27^n D(x)$ for all $n \in \mathbb{N}$. It follows from (3.4) that

\[ \mathcal{P}(C(x) - D(x), t) = \mathcal{P}\left(\frac{C(3^n x) - D(3^n x)}{27^n}, t\right) \geq_L \mathcal{P}\left(\frac{C(3^n x) - f(3^n x)}{27^n}, \frac{t}{2}\right), \mathcal{P}\left(\frac{D(3^n x) - f(3^n x)}{27^n}, \frac{t}{2}\right) \geq_L \mathcal{P}'(\varphi(3^n x, 0), 27^n (27 - \alpha)\frac{t}{2}) \geq_L \mathcal{P}'\left(\varphi(x, 0), \frac{27^n (27 - \alpha)\frac{t}{2}}{\alpha^n}\right). \]

Since

\[ \lim_{n \to \infty} \frac{27^n (27 - \alpha)\frac{t}{2}}{\alpha^n} = \infty, \]

we get

\[ \lim_{n \to \infty} \mathcal{P}'(\varphi(x, 0), \frac{27^n (27 - \alpha)\frac{t}{2}}{\alpha^n}) = 1_L. \]

Therefore $\mathcal{P}(C(x) - D(x), t) = 1_L$ for all $t > 0$, whence $C(x) = D(x)$. \qed
Corollary 3.2. Let $X$ be a linear space, $L = [0, 1]$, $(Z, P')$ be a $\mathcal{CM}_L$-fuzzy normed space, $(Y, P)$ be a complete $\mathcal{CM}_L$-fuzzy normed space, $p, q$ be nonnegative real numbers and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that
\begin{equation}
P(3f(x + 3y) + f(3x - y) - 15f(x + y) - 15f(x - y) - 80f(y), t) \\
\geq P'(\|x\|^p + \|y\|^q z_0, t) \quad (x, y \in X, t > 0),
\end{equation}
f(0) = 0 and $p, q < 3$, then there exists a unique cubic mapping $C : X \to Y$ such that
\begin{equation}
P(f(x) - C(x), t) \geq P'(\|x\|^p z_0, (27 - 3^p)t)).
\end{equation}
for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \to Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 3^p$. □

Corollary 3.3. Let $X$ be a linear space, $L = [0, 1]$, $(Z, P')$ be a $\mathcal{CM}_L$-fuzzy normed space, $(Y, P)$ be a complete $\mathcal{CM}_L$-fuzzy normed space and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that
\begin{equation}
P(3f(x + 3y) + f(3x - y) - 15f(x + y) - 15f(x - y) - 80f(y), t) \\
\geq P'(\|x\|^p z_0, t)
\end{equation}
for $x, y \in X, t > 0$ and $f(0) = 0$, then there exists a unique cubic mapping $C : X \to Y$ such that
\begin{equation}
P(f(x) - C(x), t) \geq P'(\|x\|^p z_0, 26t)).
\end{equation}
for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \to Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 1$. □

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Received February 1, 2009.

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