ON A MULTIPLIER OF THE PROGRESSIVE MEANS AND CONVEX MAPS OF THE UNIT DISC

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Abstract. In this paper we are concerned with a multiplier $\mathfrak{m}(n)$ of the Progressive means, and convex maps of the unit disc. With this concern we would have brought up in a rather unified approach the results of G. Pólya and I. J. Schoenberg in [7], T. Başgöze, J. L. Frank, and F. R. Keogh in [3], and Ziad S. Ali in [1]. More theorems on the properties of the multiplier $\mathfrak{m}(n)$ are given, and a key lemma showing combinatorial trigonometric identities whose offsprings are: Several combinatorial, and combinatorial trigonometric identities, and a new method for generating the Chebyshev’s polynomials. Finally we present a different form of $\mathfrak{m}(n)$ as well as relating $\mathfrak{m}(n)$ to the subordination principle.

1. Introduction

Let $\sum_{k=0}^{\infty} u_k$ be a given series, and let $\{S_n\}_{0}^{\infty}$ denote the sequence of its partial sums. Let $\{q_n\}_{0}^{\infty}$ be a sequence of real numbers with $q_0 > 0$, and $q_n \geq 0$ for all $n > 0$, and let $Q_n = \sum_{k=0}^{n} q_k$. By G. H. Hardy [6] the sequence-to-sequence transformation

$$T_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k$$

is called the Norlund means of $\{S_n\}_{0}^{\infty}$, and is denoted by $(N, q_n)$.

The $(N, q_n)$ is regular if and only if $q_n = o(Q_n)$ as $n \to \infty$; furthermore, the sequence-to-sequence transformation

$$\overline{T_n} = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k$$

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is called the *progressive means* of \( \{S_n\} \), and is denoted by \((N, q_n)\). The \((N, q_n)\) is regular if and only if \(Q_n \to \infty\) as \(n \to \infty\). By Peter L. Duren [4] a function \(f\) analytic in a domain \(D\) is said to be simple, schlicht, or univalent if \(f\) is one-to-one mapping of \(D\) onto another domain. A domain \(E\) of the complex plane is said to be convex if and only if the line segment joining any two points of \(E\) lies entirely in \(E\). A function \(f\) which is analytic, univalent in the unit disc \(D = \{z : |z| < 1\}\), and is normalized by \(f(0) = f'(0) - 1 = 0\) is said to belong to the class \(S\). Now \(f \in S\) is said to belong to the class \(K\) if and only if it is a conformal mapping of the unit disc \(D = \{z : |z| < 1\}\) onto a convex domain. An analytic function \(g\) is said to be subordinate to an analytic function \(f\) (written \(g \prec f\)) if

\[
g(z) = f(\omega(z)) \quad |z| < 1
\]

for some analytic function \(\omega\) with \(|\omega(z)| \leq |z|\). It is known by the Koebe-One-Quarter theorem that the range of every function of the class \(S\) contains the disc \(\{w : |w| < \frac{1}{4}\}\), i.e. \(\frac{1}{4}z < f\). The strengthened version of the Koebe-One-Quarter theorem says that the range of every convex function \(f \in K\) contains the disc \(|w| < \frac{1}{2}\), i.e. \(\frac{1}{2}z < f\). The Chebychev’s polynomials of the first kind \(T_n(x)\), and of the second kind \(U_n(x)\) are respectively defined by:

\[
T_n(x) = \cos n\theta, \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad x = \cos \theta.
\]

### 2. MEANS CONNECTED WITH POWER SERIES

Suppose that \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) is regular for \(|z| < 1\). Let

- \(S_n(z, f) = \sum_{k=0}^{n} a_k z^k\) be the sequence of partial sums of \(f\),
- \(\sigma_n(z, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(z, f)\) be the Cesaro means or \((C, 1)\) means of \(f\),
- \(T_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k(z, f)\) be the Norlund means of \(f\),
- \(\overline{T}_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k(z, f)\) be the Progressive means of \(f\),
- \(V_n(z, f) = \frac{1}{(2^n)} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k\) be the de la Vallee Poussin means of \(f\).

### 3. KNOWN RESULTS

In [7] G. Pólya and I. J. Schoenberg proved the following theorem, and corollary:

**Theorem 3.1.** For \(f(z) \in K\), it is necessary and sufficient that \(V_n(z, f) \in K\) for \(n = 1, 2, \ldots\).

**Corollary 3.2.** For \(f(z) \in K\), \(V_n(z, f) \prec f\) for \(n = 1, 2, \ldots\).
In [3] T. Başgözü, J. L. Frank, and F. R. Keogh proved the following theorem:

**Theorem 3.3.**

(i) Suppose that the values taken by \( f(z) \) for \( z \) in a convex domain \( D_w \). Then the values taken by \( \sigma_n(z, f) \) also lie in \( D_w \) for all \( n \), and all \( z \) in \( D \).

(ii) Conversely, suppose that the values taken by \( \sigma_n(z, f) \) lie in a convex domain \( D_w \); then the values taken by \( f(z) \) lie in \( D_w \) for all \( z \) in \( D \).

In [1] Ziad S. Ali proved the following theorems:

**Theorem 3.4.**

(i) Let \( (N, q_n) \) be a regular Norlund transformation such that \( \{q_n\}_0^\infty \) is a non-decreasing sequence of positive numbers. Suppose that the values taken by \( f(z) \), for \( z \) in \( D \), lie in a convex domain \( D_w \), then the values taken by \( T_n(z, f) \), also lie in \( D_w \) for all \( n \), and all \( z \) in \( D \).

(ii) Conversely, suppose that the values taken by \( T_n(z, f) \) lie in a convex domain \( D_w \); then the values taken by \( f(z) \) lie in \( D_w \) for all \( z \) in \( D \).

**Theorem 3.5.**

(i) Let \( (N, q_n) \) be a regular Progressive transformation such that \( \{q_n\}_0^\infty \) is a non-increasing sequence of positive numbers. Suppose that the values taken by \( f(z) \), for \( z \) in \( D \), lie in a convex domain \( D_w \), then the values taken by \( T_n(z, f) \), also lie in \( D_w \) for all \( n \), and all \( z \) in \( D \).

(ii) Conversely, suppose that the values taken by \( T_n(z, f) \) lie in a convex domain \( D_w \); then the values taken by \( f(z) \) lie in \( D_w \) for all \( z \) in \( D \).

In [2] Ziad S. Ali proved the following theorem:

**Theorem 3.6.**

(i) Let \( f(z) = \sum_{k=1}^\infty a_kz^k \), \( (c_1 = 1) \) be regular in the unit disc \( |z| < 1 \).

(ii) Let \( T_n \) be a transformation of the Norlund type. Let

\[
Q^n_k = \sum_{r=0}^k q^n_r = \sum_{r=0}^k \frac{(2n - 2r + 1)}{(2n - r + 1)} \binom{2n}{r} q_0,
\]

and

\[
\omega(n) = \frac{-2}{Q^n_n} \sum_{k=1}^n (-1)^k Q^n_{n-k}.
\]

then \( \frac{1}{\omega(n)} T_n(z, f) \in K \) if and only if \( f \in K \).

4. **The Main Theorems**

In this section we prove the following theorems:
Theorem 4.1. (i) Let \( f(z) = \sum_{k=1}^{\infty} a_k z^k \), \((a_1 = 1)\) be regular in the unit disc \(|z| < 1\), and let \( T_n(z, f) = \frac{1}{q_n} \sum_{k=0}^{n} q_k S_k(z, f) \) be a transformation of the progressive type.

(ii) Let
\[
Q_{n-k} = Q^n_{n-k} = \sum_{r=0}^{n-k} q_r^n, \quad \text{and} \quad Q_n = Q_n^n = \sum_{r=0}^{n} q_r^n,
\]
\[
q_r^n = \begin{cases} \frac{(2n-2r+1)}{(2n-2r+1)} q_0 & \text{if} \; r = 0, 1, \ldots, (n-k), \\ q_{n-r} & \text{if} \; r = (n-k) + 1, (n-k) + 2, \ldots, n-1, n. \end{cases}
\]

(iii) Let
\[
\varpi(n) = \frac{-2}{Q_n^n} \sum_{k=1}^{n} (-1)^k (Q_n^n - Q_{k-1}^n),
\]
then \( \frac{1}{\varpi(n)} T_n(z, f) \in K \) if and only if \( f \in K \).

Proof. We begin first by noting that:
\[
\frac{1}{\varpi(n)} T_n(z, f) = \frac{1}{Q_n^n} \sum_{k=1}^{n} q_k^n S_k(z, f)
\]
expanding \( \sum_{k=1}^{n} q_k^n S_k(z, f) \), we can easily see:
\[
\frac{1}{\varpi(n)} T_n(z, f) = \frac{1}{-2} \sum_{k=1}^{n} (-1)^k (Q_n^n - Q_{k-1}^n) \sum_{k=1}^{n} (Q_n^n - Q_{k-1}^n) a_k z^k.
\]
Since
\[
Q_n^n = Q_{(n-k)}^n + (q_{(n-k)}^n + q_{(n-k)+1}^n + \cdots + q_{n-1}^n + q_n^0), \quad \text{and} \quad Q_{k-1}^n = q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n.
\]
Hence
\[
\frac{1}{\varpi(n)} T_n(z, f) = \frac{\sum_{k=1}^{n} \left( (Q_{(n-k)}^n + (q_{(n-k)}^n + \cdots + q_{n-1}^n + q_n^0)) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n) \right) a_k z^k}{-2 \sum_{k=1}^{n} (-1)^k \left( (Q_{(n-k)}^n + (q_{(n-k)}^n + \cdots + q_{n-1}^n + q_n^0)) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n) \right)}.
\]
Equivalently we have:
\[
\frac{1}{\varpi(n)} T_n(z, f) =
\]
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\[ \sum_{k=1}^{n} \left( Q_{n-k}^n + (q_{n-k-1}^n + \cdots + q_{n-1}^n + q_n^n) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n) \right) a_k z^k \]

\[ -2 \sum_{k=1}^{n} (-1)^k \left( Q_{n-k}^n + (q_{n-k-1}^n + \cdots + q_{n-1}^n + q_n^n) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n) \right) \]

Since \( q_r^n = q_{n-r}^n \) for \( r = n - (k - 1), n - (k - 2), \ldots, (n - 1), n \), it follows easily that:

\[ (q_{n-(k-1)}^n + q_{n-(k-2)}^n + \cdots + q_{n-1}^n + q_n^n) - (q_{k-1}^n + q_{k-2}^n + \cdots + q_1^n + q_0^n) = 0. \]

Hence

\[ \frac{1}{\mathcal{V}(n)} T_n(z, f) = \frac{1}{-2 \sum_{k=1}^{n} (-1)^k Q_{n-k}^n} \sum_{k=1}^{n} Q_{n-k}^n a_k z^k. \]

Now we can easily show that:

\[ Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n = \sum_{r=0}^{n-k} \left( \frac{2n - 2r + 1}{2n - r + 1} \right) \binom{2n}{r} q_0 = \binom{2n}{n-k} q_0. \]

Hence

\[ \frac{1}{\mathcal{V}(n)} T_n(z, f) = \frac{1}{-2 \sum_{k=1}^{n} (-1)^k \binom{2n}{n-k}} \sum_{k=1}^{n} \binom{2n}{n-k} a_k z^k. \]

Finally we can show that for \( n \) odd we have:

\[ -2 \sum_{k=1}^{n} (-1)^k \binom{2n}{n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} + \binom{2n}{n}, \]

and for \( n \) even we have:

\[ -2 \sum_{k=1}^{n} (-1)^k \binom{2n}{n-k} = - \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} + \binom{2n}{n}. \]

Now since:

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = 0. \]

It follows that

\[ \frac{1}{\mathcal{V}(n)} T_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k = V_n(z, f), \]

which are the de la Vallee Poussin means of \( f \), and the theorem follows by G. Pólya and I. J. Schoenberg [7].

\[ \square \]

**Theorem 4.2.** (i) Suppose that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is regular for \( |z| < 1 \), and suppose that \( T_n \) are the Progressive means.
(ii) Let $Q^n_n = n + 1$, and let

$$
\omega(n) = \begin{cases} 
\frac{2}{Q^n_n} \sum_{k=1}^{n} (-1)^k (Q^n_n - Q^n_{k-1}) & n \text{ is odd} \\
\frac{2}{Q^n_n} \sum_{k=1}^{n} (-1)^k (Q^n_n - Q^n_{k-1}) + 1 & n \text{ is even}, 
\end{cases}
$$

then

$$
\frac{1}{\omega(n)} T_n(z, f) \in K \quad \text{if and only if } f \in K.
$$

Proof. Clearly $Q^n_n - Q^n_{k-1} = n - k + 1$. Considering two separate cases for $n$ even, and $n$ odd we can easily see that

$$
n + 1 = \begin{cases} 
-2 \sum_{k=1}^{n} (-1)^k (n - k + 1) & n \text{ is odd} \\
-2 \sum_{k=1}^{n} (-1)^k (n - k + 1) + 1 & n \text{ is even}, 
\end{cases}
$$

Accordingly for any $n$ we have:

$$
\frac{1}{\omega(n)} T_n(z, f) = \frac{1}{n + 1} \sum_{k=0}^{n} S_k(z, f) = \sigma_n(z, f),
$$

which are the Cesaro means of $f$, and the result follows by T. Başgöze, J. L. Frank, and F. R. Keogh [3].

**Theorem 4.3.**

(i) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be regular in the unit disc $D = \{z : |z| < 1\}$.

(ii) Let $T_n(z, f)$ be a regular Progressive type transformation defined by a non-increasing sequence $\{q^n_r\}_{r=1}^{\infty}$ of positive real numbers such that

$$
\sum_{i \in \text{odd}} q^i_t = \sum_{i \in \text{even}} q^i_t, \quad \text{where } i \text{ is a non-negative integer then:}
$$

$$
\frac{1}{\omega(n)} T_n(z, f) \in K \quad \text{if and only if } f \in K.
$$

Proof. For $n$ odd integer, say $n = 2s + 1$ we have:

$$
-2 \sum_{i=1}^{n} (-1)^i (Q^n_n - Q^n_{i-1}) = -2 \sum_{i=1}^{2s+1} (-1)^i (Q^{2s+1}_{2s+1} - Q^{2s+1}_{2s+2})
$$

$$
= -2 \left( -\sum_{t=0}^{s} q^{2s+1}_{2t+1} \right) = 2 \sum_{i \in \text{odd}} q_i^n, \quad i = 1, 3, 5, \ldots.
$$
Similarly for \( n = 2s \) we have:

\[
-2 \sum_{i=1}^{n} (-1)^i(Q_n^i - Q_{i-1}^n) = -2 \sum_{i=1}^{2s} (-1)^i(Q_{2s}^i - Q_{i-1}^{2s})
\]

\[
= -2 \left( - \sum_{t=0}^{s-1} q_{2t+1}^{2s} \right) = 2 \sum_{i \in \text{odd}} q_i^n, \quad i = 1, 3, 5 \ldots .
\]

Therefore,

\[
\omega(n) = -2 \frac{Q_n^n}{Q_n^n} \sum_{i=1}^{n} (-1)^i(Q_n^i - Q_{i-1}^n) = \frac{1}{Q_n^n} \left( \sum_{i \in \text{odd}} q_i^n + \sum_{i \in \text{even}} q_i^n \right) = 1.
\]

Accordingly the result follows by Ziad S. Ali [1]. \( \square \)

5. THEOREMS ON \( \omega(n) \)

In this section we see more of the properties of \( \omega(n) \) through the following theorems.

**Theorem 5.1.** Let \( \{q_r^n\}_{r=1}^n \) be a sequence of positive real numbers, then

\[
\omega(n) = 1 \quad \text{if and only if} \quad \sum_{r \in \text{odd}} q_r^n = \sum_{r \in \text{even}} q_r^n .
\]

**Proof.** Let \( \omega(n) = 1 \), then

\[
-2 \sum_{r=1}^{n} (-1)_r (Q_n^r - Q_r^n) = Q_n^n \quad 2 \sum_{r \in \text{odd}} q_r^n = \left( \sum_{r \in \text{odd}} q_r^n + \sum_{r \in \text{even}} q_r^n \right).
\]

Now assume \( \sum_{r \in \text{odd}} q_r^n = \sum_{r \in \text{even}} q_r^n \), then

\[
\omega(n) = \frac{1}{Q_n^n} \left( \sum_{r \in \text{even}} q_r^n + \sum_{r \in \text{odd}} q_r^n \right) = 1. \quad \square
\]

Theorem 5.1 can be used as a tool to generate or prove new Combinatorial identities as seen by the following theorems:

**Theorem 5.2.** Let \( q_r^n = \binom{n}{r} \), then

\[
\sum_{r \in \text{odd}} q_r^n = \sum_{r \in \text{even}} q_r^n, \quad \text{and} \quad - \frac{1}{2^n-1} \sum_{r=1}^{n} (-1)^r \left( 2^n - \sum_{j=0}^{r-1} \binom{n}{j} \right) = 1.
\]

**Proof.** With \( q_r^n = \binom{n}{r} \), we have:

\[
\omega(n) = -2 \frac{Q_n^n}{Q_n^n} \left( \sum_{r=1}^{n} (-1)^r(Q_n^r - Q_{r-1}^n) \right) = \frac{1}{Q_n^n} \left( \sum_{r \in \text{odd}} \binom{n}{r} + \sum_{r \in \text{even}} \binom{n}{r} \right) = 1.
\]
Accordingly by theorem 5.1 we have the following combinatorial identity:

\[-\frac{1}{2^{n-1}} \sum_{r=1}^{n} (-1)^r \left( 2^n - \sum_{j=0}^{r-1} \binom{n}{j} \right) = 1.\]

The above newly generated combinatorial identity is implicitly saying for example when \( n \) is even: The sum of all combinations of \( n \) elements taken \( r \) at a time with \( r = 1, 3, 5, \ldots \) is \( 2^{n-1} \).

\[\square\]

**Theorem 5.3.** Let \( q_r^n = \frac{2n-2r+1}{2n-2r+1} \binom{2n}{r} q_0 \), and \( Q_r^n = \sum_{r=0}^{n} q_r^n \). Then we have:

\[\sum_{r \in \text{odd}} q_r^n = \sum_{r \in \text{even}} q_r^n.\]

**Proof.** With \( q_r^n = \frac{2n-2r+1}{2n-2r+1} \binom{2n}{r} q_0 \), we can show:

\[\overline{\omega}(n) = -\frac{2}{Q_r^n} \left( \sum_{r=1}^{n} (-1)^r Q_r^n - Q_r^{n-1} \right) = \frac{2}{Q_r^n} \sum_{r \in \text{odd}} Q_r^n = \frac{2n-2r+1}{2n-2r+1} \binom{2n}{r} = \sum_{r \in \text{even}} \frac{2n-2r+1}{2n-2r+1} \binom{2n}{r} = 1.\]

Now we may apply theorem 5.1. Moreover for \( n \) even we have:

\[\sum_{r \in \text{even}} 2n - 2r + 1 \binom{2n}{r} = \sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} - \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = \frac{1}{2} \binom{2n}{n}.\]

\[\sum_{r \in \text{odd}} 2n - 2r + 1 \binom{2n}{r} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} - \sum_{r=0}^{\frac{n-3}{2}} \binom{2n}{2r} = \frac{1}{2} \binom{2n}{n}.\]

For \( n \) odd we have:

\[\sum_{r \in \text{even}} 2n - 2r + 1 \binom{2n}{r} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} - \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = \frac{1}{2} \binom{2n}{n}.\]

\[\sum_{r \in \text{odd}} 2n - 2r + 1 \binom{2n}{r} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} - \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} = \frac{1}{2} \binom{2n}{n}.\]

This completes the proof of theorem 5.3. \(\square\)

We note from theorem 5.3. above that for \( n \) even

\[\sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} + \binom{2n-1}{n}; \quad \sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = 2^{2n-2} + \binom{2n-1}{n}.\]
For $n$ odd we can show
\[ \sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = 2^{2n-2}. \]

Accordingly for any $n$ we have:
\[ \sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = 2^{2n-2} + \frac{1 + (-1)^n}{2} \binom{2n-1}{n}, \quad n \geq 1, \]
which is identity 1.92 of Henry W. Gould [5]. Similarly for $n$ even we have from theorem 5.3:
\[ \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = 2^{2n-2}. \]

Now for $n$ odd we have:
\[ \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} + \binom{2n}{2r+1} = 2^{2n-2} + \binom{2n-1}{n}. \]

Accordingly for any $n$ we have:
\[ \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = 2^{2n-2} + \frac{1 - (-1)^n}{2} \binom{2n-1}{n}, \]
which is identity 1.98 of Henry W. Gould [5].

**Theorem 5.4.** For $n > 1$ we have:
\[ -2 \left( \sum_{k=1}^{n} (-1)^k \cdot \sum_{r=k}^{n} r^2 \binom{2n}{n-r} \right) = \sum_{r=1}^{n} r^2 \binom{2n}{n-r}. \]

*Proof.* Follows by theorem 5.1 and noting that for $n > 1$ we have:
\[ \sum_{r \in \text{odd}}^{n} r^2 \binom{2n}{n-r} = \sum_{r \in \text{even}}^{n} r^2 \binom{2n}{n-r}. \]

\[ \square \]

6. A Key Lemma

In this section we have the following lemma:

**Lemma 6.1.** For $1 \leq r \leq n$, and $\theta$ real we have:

(i) \[ \sum_{r=1}^{n} \binom{2n}{n-r} (\cos r \theta + (-1)^{r+1}) = 2^{n-1} (1 + \cos \theta)^n. \]

(ii) \[ \sum_{r=1}^{n} (-1)^r \binom{2n}{n-r} \cos r \theta + \frac{1}{2} \binom{2n}{n} = 2^{n-1} (1 - \cos \theta)^n. \]
Proof. (i) Using induction on $n$, and by repeated application of the recurrence formula
\[
\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1},
\]
the above lemma follows. Note now that the proof of the lemma also follows by noting that:
\[
\cos^2\frac{n\theta}{2} = \frac{1}{2^n}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{r=0}^{n-1} \binom{2n}{r} \cos(n-r)\theta,
\]
and where
\[
\sum_{r=1}^{n} (-1)^{r+1}\binom{2n}{n-r} = \frac{1}{2} \binom{2n}{n}.
\]
Furthermore note that the above lemma also follows from the following:
\[
\text{Re.}\left(e^{i\theta} (1 + e^{-i\theta})^{2n}\right) = \sum_{r=0}^{2n} \binom{2n}{r} \cos(n-r)\theta = 2^{2n} \cos^2\frac{\theta}{2}.
\]
We can also see that
\[
\sum_{r=0}^{n} \binom{2n}{r} \cos(n-r)\theta = \frac{1}{2} \sum_{r=0}^{2n} \binom{2n}{r} \cos(n-r)\theta + \frac{1}{2} \binom{2n}{n}.
\]
Now with $k = n - r$ it follows that
\[
\sum_{k=1}^{n} \binom{2n}{n-k} \cos k\theta + \frac{1}{2} \binom{2n}{n} = 2^{2n-1} \cos^2\frac{\theta}{2}.
\]
Accordingly
\[
\sum_{r=1}^{n} \binom{2n}{n-r} \left(\cos r\theta + (-1)^{r+1}\right) = 2^{n-1}(\cos \theta + 1)^n,
\]
and the lemma follows again.
(ii) Follows since
\[
(e^{-i\theta}(1 - e^{i\theta})^{2n}) = 2^{2n} \sin\frac{2n\theta}{2} \cdot (-1)^n
\]
\[
= (-1)^n \left(\binom{2n}{n} + 2 \sum_{r=1}^{n} (-1)^r \binom{2n}{n-r} \cos r\theta\right),
\]
\[
= \sum_{r=0}^{2n} (-1)^r \binom{2n}{r} \cos(n-r)\theta.
\]
This completes the proof of lemma 6.1. \qed
Remark 1. From above we have for $m = 2n$

$$\sum_{r=0}^{m} \binom{m}{r} \cos\left(\frac{m}{2} - r\right)\theta = 2^m \cos^m \frac{\theta}{2} \cdot 1.$$  

Accordingly we have:

$$\sum_{r=0}^{m} \binom{m}{r} r \cos\left(\frac{m}{2} - r\right)\theta = 2^m \cos^m \frac{\theta}{2} \cdot 1.$$  

For any $m$ the above two combinatorial identities which are 1.26, and 1.27 in the list of identities of Henry W. Gould [5] follow by considering $(1 + e^{i\theta})^m$.

Remark 2. Similarly for $m = 2n$ we have:

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \cos\left(\frac{m}{2} - r\right)\theta = (-1)^n 2^m \sin^m \frac{\theta}{2} \cdot 1$$

then we have:

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r \cos\left(\frac{m}{2} - r\right)\theta = (-1)^n \frac{m}{2} 2^m \sin^m \frac{\theta}{2} \cos \frac{m\theta}{2}$$

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r \sin\left(\frac{m}{2} - r\right)\theta = (-1)^n \frac{m}{2} 2^m \sin^m \frac{\theta}{2} \sin \frac{m\theta}{2}.$$  

For any $m$ the above two combinatorial identities which are 1.28, and 1.29 of the identities of Henry W. Gould [5] follow by considering $(1 - e^{i\theta})^m$. Now for $\theta = 0$ in lemma 6.1(i) we can show the following:

$$\sum_{r=0}^{n} \binom{2n}{r} = 2^{2n-1} + \binom{2n - 1}{n}$$

$$\sum_{r=0}^{n} (-1)^r \binom{2n}{r} = (-1)^n \binom{2n - 1}{n}$$

$$\sum_{r=0}^{n} \binom{2n + 1}{r} = 4^n$$

which are 1.85, 1.86, and 1.83 of Henry W. Gould [5].
Corollary 6.2. For $\theta \in \text{real}$, and $r \leq n$ we have the following combinatorial trigonometric identities:

\[
\sum_{r \in \text{even}}^{n} \binom{2n}{n-r} \cos r\theta = 2^{2n-2} \left( \frac{\cos 2n\theta}{2} + \frac{\sin 2n\theta}{2} \right) - \frac{1}{2} \binom{2n}{n}
\]
\[
\sum_{r \in \text{odd}}^{n} \binom{2n}{n-r} \cos r\theta = 2^{2n-2} \left( \frac{\cos 2n\theta}{2} - \frac{\sin 2n\theta}{2} \right).
\]

Proof. Follows from lemma 6.1. \qed

Corollary 6.3. For $r \leq n$ we have:

\[
\sum_{\substack{r \in \text{odd} \\
 r \geq 1}}^{n} \binom{2n}{n-r} r \cdot \sin r\theta = n2^{n-1} \sin \theta \cdot \left( \sum_{\substack{r \in \text{even} \\
 r \geq 0}}^{n-1} \binom{n-1}{r} \cos^r \theta \right), \quad n \geq 1
\]
\[
\sum_{\substack{r \in \text{even} \\
 r \geq 1}}^{n} \binom{2n}{n-r} r \cdot \sin r\theta = n2^{n-1} \sin \theta \cdot \left( \sum_{\substack{r \in \text{odd} \\
 r \geq 1}}^{n-1} \binom{n-1}{r} \cos^r \theta \right), \quad n \geq 1.
\]

Proof. Follows from lemma 6.1. \qed

Corollary 6.4. For $n \geq 2$ we have:

\[
\sum_{\substack{r \in \text{odd} \\
 r \geq 1}}^{n} \binom{2n}{n-r} r^2 \cdot \cos r\theta
\]
\[
= n \cdot 2^{n-1} \left( 1 - n \sin^2 \theta \right) \cdot \sum_{\substack{r \in \text{odd} \\
 r \geq 1}}^{n-2} \binom{n-2}{r} \cdot \cos^r \theta + \sum_{\substack{r \in \text{even} \\
 r \geq 0}}^{n-2} \binom{n-2}{r} \cos^{r+1} \theta
\]
\[
\sum_{\substack{r \in \text{even} \\
 r \geq 2}}^{n} \binom{2n}{n-r} r^2 \cdot \cos r\theta
\]
\[
= n \cdot 2^{n-1} \left( 1 - n \sin^2 \theta \right) \cdot \sum_{\substack{r \in \text{even} \\
 r \geq 0}}^{n-2} \binom{n-2}{r} \cdot \cos^r \theta + \sum_{\substack{r \in \text{odd} \\
 r \geq 1}}^{n-2} \binom{n-2}{r} \cos^{r+1} \theta.
\]

Proof. Follows from lemma 6.1. \qed
Corollary 6.5. For $0 \leq r \leq n$ we have:

$$
\sum_{r=0}^{n} r \binom{2n}{r} = n \cdot 2^{2n-1}
$$

$$
\sum_{r=0}^{n} r^2 \binom{2n}{r} = n \cdot 2^{2n-2} + n^2 2^{2n-1} - n^2 \binom{2n-1}{n}.
$$

Proof. Since from lemma 6.1 we have:

$$
\sum_{r=0}^{n} r \binom{2n}{n-r} = n \cdot 2^{2n-2}.
$$

Furthermore since we can also show that

$$
\sum_{r=0}^{n} r \binom{2n}{n+r} = \frac{n}{2} \binom{2n}{n},
$$

the corollary follows.

7. Generating the Chebyshev’s polynomials

Using lemma 6.1(i), then by the definition of the Chebyshev’s polynomials of the first kind $T_n(x)$, we see that $T_n(x)$ satisfies the following formula:

$$
\sum_{r=1}^{n} \binom{2n}{n-r} (T_r(x) + (-1)^{r+1}) = 2^{n-1}(x + 1)^n, \quad x = \cos \theta.
$$

Now by letting $r = 1, r = 2, r = 3, \ldots$ etc. we can respectively obtain

$$
T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots
$$

hence generating the Chebyshev’s polynomials of the first kind of degrees 1, 2, 3, \ldots etc. We can similarly see that $U_n(x)$, the Chebyshev’s polynomials of the second kind satisfy:

$$
\sum_{r=1}^{n} \binom{2n}{n-r} \cdot r \cdot U_{r-1}(x) = n \cdot 2^{n-1}(x + 1)^{n-1}, \quad x = \cos \theta.
$$

Again now for $r = 1, r = 2, r = 3, \ldots$ etc. we can respectively obtain

$$
U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \ldots,
$$

and hence generating the Chebyshev’s polynomials of the second kind of degrees 0, 1, 2, \ldots etc.
8. AN APPLICATION ON PROBABILITIES

Using lemma 6.1(i), we can show that the probability of \( n \) successes in \( 2n \) trials of a symmetric binomial distribution is given by:

\[
\frac{\left(\begin{array}{c} 2n \\ n \end{array}\right)}{2^{2n}} = \frac{1}{2^{2n-1}} \sum_{r=0}^{n} \left(\begin{array}{c} 2n \\ r \end{array}\right) \cos \frac{(n-r)\pi}{2} - \frac{1}{2^n}
\]

(1)

\[
\frac{\left(\begin{array}{c} 2n \\ n \end{array}\right)}{2^{2n}} = \frac{2^n}{2^{2n}} \sum_{r=0}^{\binom{2n}{r}} - 1
\]

(2)

\[
\frac{\left(\begin{array}{c} 2n \\ n \end{array}\right)}{2^{2n}} = \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{2n} t \, dt
\]

(3)

\[
\frac{\left(\begin{array}{c} 2n \\ n \end{array}\right)}{2^{2n}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}.
\]

(4)

9. A DIFFERENT FORM OF \( \omega(n) \)

A different form of \( \omega(n) \) is presented in this section, and this is seen by the following:

**Theorem 9.1.**

(i) Let \( f(z) = \sum_{k=1}^{\infty} c_k z^k \) \( (c_1 = 1) \) be regular in the unit disc \( |z| < 1 \).

(ii) Let

\[
Q_{n-k} = Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n, \quad \text{and} \quad Q_n = Q_n^n = \sum_{r=0}^{n} q_r^n,
\]

\[
q_r^n = \begin{cases} \left(\frac{2n-2r+1}{2n-2r+1} - \frac{2n}{r}\right) q_0, & r = 0, 1, \ldots, (n-k), \\ q_{n-r}^n, & r = (n-k) + 1, (n-k) + 2, \ldots, n-1, n. \end{cases}
\]

(iii) Let \( T_n \) be the Progressive means. With \( z = \rho e^{i\theta} \), let

\[
\omega_m(n, \theta) = \frac{-2}{Q_m^n} \min_{|z| \leq 1} |z| \sum_{r=1}^{n} (Q_n^n - Q_{n-r}^n) \cdot z^r, \text{ then}
\]

\[
\frac{1}{\omega_m(n, \theta)} T_n(z, f) \in K \text{ if and only if } f \in K.
\]

**Proof.** \( u(\rho, \theta) = \sum_{k=1}^{n} \left(\begin{array}{c} 2n-k \\ n \end{array}\right) \rho^k \cos k\theta \) is harmonic in

\[
D = \{ z : |z| < 1 \} \quad \text{as} \quad \nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} = 0.
\]

Furthermore \( u \) is continuous on \( D \); \( \{ z : |z| \leq 1 \} \). Accordingly by the minimum principle for harmonic functions \( u \) attains its minimum on the boundary of \( D \).

Now the proof of theorem 9.1 follows from lemma 6.1(i), and theorem 4.1.
Note that from lemma 6.1(i), or \( - \sum_{k=1}^{n} \binom{2n}{n-k} k \sin k \theta \) guarantees a minimum at \( \theta = \pi \in [0, 2\pi] \). □

10. The subordination principle and \( \bar{\omega}_m(n, \theta) \)

In this section we relate \( \bar{\omega}(n) \) to the subordination principle by the following theorem.

**Theorem 10.1.**

(i) Let \( K \) denote the class of “Schlicht” power series which map \( |z| < 1 \) onto some convex domain, and let \( f \in K \).

(ii) Let

\[
Q_{n-k} = Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n, \quad \text{and} \quad Q_n = Q_n^n = \sum_{r=0}^{n} q_r^n,
\]

\[
q_r^n = \begin{cases} 
(2n-2r+1) \binom{2n}{r} q_0 & r = 0, 1, \ldots, (n - k), \\
q_{n-r} & r = (n - k) + 1, (n - k) + 2, \ldots, n - 1, n.
\end{cases}
\]

(iii) Let \( T_n \) be a transformation of the Progressive type. With \( z = \rho e^{i\theta} \), let

\[
\bar{\omega}_m(n, \theta) = \frac{2}{Q_n^n} \min_{|z| \leq 1} \Re \sum_{r=1}^{n} \left( Q_n^n - Q_{r-1}^n \right) \cdot z^r, \quad \text{then}
\]

\[
\frac{1}{\bar{\omega}_m(n, \theta)} T_n(z, f) \prec f.
\]

**Proof.** Follows from the proof of 9.1, and corollary 3.2 of G. Pólya and I. J. Schoenberg [7]. Note that

\[
\frac{1}{\bar{\omega}_m(1, \theta)} T_1(z, f) = \frac{1}{2} z \prec f,
\]

which is the strengthened version of the Koebe-One-Quarter theorem, and

\[
\frac{1}{\bar{\omega}_m(2, \theta)} T_2(z, f) = \frac{2}{3} z + \frac{a_2}{6} z^2 = V_2(z, f) \prec f.
\]

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**References**


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