ON S-3 LIKE FOUR-DIMENSIONAL FINSLER SPACES

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Abstract. In 1977, M. Matsumoto and R. Miron [9] constructed an orthonormal frame for an \( n \)-dimensional Finsler space, called ‘Miron frame’. The present authors [1, 2, 3, 10, 11] discussed four-dimensional Finsler spaces equipped with such frame. M. Matsumoto [7, 8] proved that in a three-dimensional Berwald space, all the main scalars are \( h \)-covariant constants and the \( h \)-connection vector vanishes. He also proved that in a three-dimensional Finsler space satisfying T-condition, all the main scalars are functions of position only and the \( v \)-connection vector vanishes [6, 7]. The purpose of the present paper is to generalize these results for an S-3 like four-dimensional Finsler space.

1. Preliminaries

Let \( M^4 \) be a four-dimensional smooth manifold and \( F^4 = (M^4, L) \) be a four-dimensional Finsler space equipped with a metric function \( L(x, y) \) on \( M^4 \). The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by
\[
l_i = \dot{\partial}_i L, \ g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \ h_{ij} = L \ddot{\partial}_i \dot{\partial}_j L \text{ and } C_{ijk} = \frac{1}{2} \ddot{\partial}_k g_{ij},
\]
respectively. The torsion vector \( C^i \) is defined by \( C^i = C^i_{jk} g^{jk} \). Throughout this paper, we use the symbols \( \dot{\partial}_i \) and \( \partial_i \) for \( \partial / \partial y^i \) and \( \partial / \partial x^i \) respectively. The Cartan connection in the Finsler space is given as \( \Gamma = (F^i_{jk}, G^i_j, C^i_{jk}) \). The \( h \)- and \( v \)-covariant derivatives of a covariant vector \( X_i(x, y) \) with respect to the Cartan connection are given by
\[
X_{ij} = \partial_j X_i - (\dot{\partial}_h X_i) G^h_j - F^r_{ij} X_r,
\]
and
\[
X_{i|j} = \dot{\partial}_j X_i - C^r_{ij} X_r.
\]
The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors \((e_1^i, e_2^i, e_3^i, e_4^i)\). The first vector \(e_1^i\) is the normalized supporting element \(l^i\) and the second \(e_2^i\) is the normalized torsion vector \(m^i = C^i / \tilde{c}\), where \(\tilde{c}\) is the length of the torsion vector \(C^i\). The third \(e_3^i = n^i\) and the fourth \(e_4^i = p^i\) are constructed by the method of Matsumoto and Miron [9]. With respect to this frame, the scalar components of an arbitrary tensor \(T_j^i\) are defined by

\[
T_{\alpha\beta} = T_j^i e_\alpha^j e_\beta^i.
\]

From this, we get

\[
T_j^i = T_{\alpha\beta} e_\alpha^i e_\beta^j,
\]

where summation convention is also applied to Greek indices. The scalar components of the metric tensor \(g_{ij}\) are \(\delta_{\alpha\beta}\). Therefore we get

\[
g_{ij} = l_i l_j + m_i n_j + n_i m_j + p_i p_j.
\]

Let \(H_{\alpha\beta\gamma}\) and \(V_{\alpha\beta\gamma} / L\) be scalar components of the \(h\)- and \(v\)-covariant derivatives \(e_\alpha^i|_j\) and \(e_\alpha^i|_j\) respectively of the vectors \(e_\alpha^i\), then

\[
e_\alpha^i|_j = H_{\alpha\beta\gamma} e_\beta^i e_\gamma^j,
\]

and

\[
Le_\alpha^i|_j = V_{\alpha\beta\gamma} e_\beta^i e_\gamma^j.
\]

\(H_{\alpha\beta\gamma}\) and \(V_{\alpha\beta\gamma}\) are called \(h\)- and \(v\)-connection scalars respectively and are positively homogeneous of degree 0 in \(y\).

Orthogonality of the Miron frame yields

\[
H_{\alpha\beta\gamma} = -H_{\beta\alpha\gamma} \quad \text{and} \quad V_{\alpha\beta\gamma} = -V_{\beta\alpha\gamma}.
\]

Also we have \(H_{1\beta\gamma} = 0\) and \(V_{1\beta\gamma} = \delta_{\beta\gamma} - \delta_1 \delta_{\beta\gamma} [7]\).

Now we define Finsler vector fields:

\[
h_i = H_{2\beta\gamma} e_{\gamma^i}, \quad j_i = H_{4\beta\gamma} e_{\gamma^i}, \quad k_i = H_{3\beta\gamma} e_{\gamma^i},
\]

and

\[
u_i = V_{2\beta\gamma} e_{\gamma^i}, \quad v_i = V_{4\beta\gamma} e_{\gamma^i}, \quad w_i = V_{3\beta\gamma} e_{\gamma^i}.
\]

The vector fields \(h_i, j_i, k_i\) are called \(h\)-connection vectors and the vector fields \(u_i, v_i, w_i\) are called \(v\)-connection vectors. The scalars \(H_{2\beta\gamma}, H_{4\beta\gamma}, H_{3\beta\gamma}\) and \(V_{2\beta\gamma}, V_{4\beta\gamma}, V_{3\beta\gamma}\) are considered as the scalar components \(h_\gamma, j_\gamma, k_\gamma\) and \(u_\gamma, v_\gamma, w_\gamma\) of the \(h\)- and \(v\)-connection vectors respectively with respect to the orthonormal frame.

Let \(C_{\alpha\beta\gamma}\) are the scalar components of \(LC_{ijk}\) then

\[
LC_{ijk} = C_{\alpha\beta\gamma} e_\alpha^i e_\beta^j e_\gamma^k.
\]
The main scalars of a four-dimensional Finsler space are given by \[1, 3, 11\]
\[
C_{222} = A, \quad C_{233} = B, \quad C_{244} = C, \quad C_{322} = D, \quad C_{333} = E, \quad C_{422} = F, \quad C_{433} = G, \quad C_{234} = H.
\]
We also have \(C_{344} = -(D + E), \quad C_{444} = -(F + G)\) and
\[
A + B + C = \tilde{L}c.
\]
\(\tilde{L}c\) is called the unified main scalar.

Taking \(h\)-covariant differentiation of (1.4), we get
\[
T_{\alpha\beta\gamma} = (\delta_k T_{\alpha\beta})^i e^i_{\alpha\beta\gamma} + T_{\mu\alpha\beta} H_{\mu\alpha\gamma} + T_{\alpha\mu} H_{\mu\beta\gamma}.
\]
then we obtain
\[
T_{\alpha\beta\gamma} = (\delta_k T_{\alpha\beta})^i e^i_{\alpha\beta\gamma} + \delta_{1\alpha} T_{\beta\gamma} e^i_{\alpha\beta\gamma}.
\]
Similarly, if \(T_{\alpha\beta\gamma}\) are scalar components of \(LT_{\alpha\beta\gamma}\), i.e.
\[
(1.12)
\]
\[
L^2 T_{\alpha\beta\gamma} = \delta_{1\alpha} T_{\beta\gamma} e^i_{\alpha\beta\gamma}.
\]
then we get
\[
(1.14)
\]
The scalar components \(T_{\alpha\beta\gamma}\) and \(T_{\alpha\beta\gamma}\) are respectively called \(h\)- and \(v\)-scalar derivatives of scalar components \(T_{\alpha\beta}\) of \(T\).

2. **T-condition**

The tensor \(T_{hijk}\) defined by
\[
(2.1)
\]
is called \(T\)-tensor in a Finsler space. It is completely symmetric in its indices. A Finsler space is said to satisfy \(T\)-condition if the \(T\)-tensor \(T_{hijk}\) vanishes identically.

We are concerned with the tensor \(C_{hij}\) of \(C\). From (1.8) and (1.13), it follows that
\[
L^2 C_{hij} = \delta_{1\alpha} T_{\alpha\beta\gamma} e^i_{\alpha\beta\gamma} + T_{\alpha\beta\gamma} e^i_{\alpha\beta\gamma} + T_{\alpha\beta\gamma} e^i_{\alpha\beta\gamma}.
\]
which implies
\[
(2.2)
\]
Therefore the scalar components \(T_{\alpha\beta\gamma}\) of \(LT_{hijk}\) are given by
\[
T_{\alpha\beta\gamma} = \delta_{1\alpha} C_{\alpha\beta\gamma} + \delta_{1\alpha} C_{\alpha\beta\gamma} + \delta_{1\alpha} C_{\alpha\beta\gamma} + \delta_{1\alpha} C_{\alpha\beta\gamma}.
\]
From \( T_{hijk}^{\ell k} = 0 \), we have \( T_{\alpha\beta\gamma\delta} = 0 \). Thus the surviving components \( T_{\alpha\beta\gamma\delta} \) are only

\[
T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta}; \quad \alpha, \beta, \gamma, \delta = 2, 3, 4. \tag{2.3}
\]

Using (1.14), the explicit forms of \( C_{\alpha\beta\gamma;\delta} \) are obtained as follows:

\[
\begin{align*}
\text{(a)} & \quad C_{222;\delta} = A_{\delta} - 3D u_{\delta} + 3F v_{\delta}, \\
\text{(b)} & \quad C_{233;\delta} = B_{\delta} + (2D - E) u_{\delta} + G v_{\delta} - 2H w_{\delta}, \\
\text{(c)} & \quad C_{244;\delta} = C_{\delta} + (D + E) u_{\delta} - (3F + G) v_{\delta} + 2H w_{\delta}, \\
\text{(d)} & \quad C_{322;\delta} = D_{\delta} + (A - 2B) u_{\delta} + 2H v_{\delta} - F w_{\delta}, \\
\text{(e)} & \quad C_{333;\delta} = E_{\delta} + 3Bu_{\delta} - 3Gw_{\delta}, \\
\text{(f)} & \quad C_{422;\delta} = F_{\delta} - 2H u_{\delta} - (A - 2C) v_{\delta} + Dw_{\delta}, \\
\text{(g)} & \quad C_{433;\delta} = G_{\delta} + 2H u_{\delta} - Bv_{\delta} + (2D + 3E) w_{\delta}, \\
\text{(h)} & \quad C_{444;\delta} = H_{\delta} + (F - G) u_{\delta} - (2D + 3E) v_{\delta} + (B - C) w_{\delta}, \\
\text{(i)} & \quad C_{344;\delta} = -D_{\delta} - E_{\delta} + Cu_{\delta} - 2H v_{\delta} + (F + 3G) w_{\delta}, \\
\text{(j)} & \quad C_{444;\delta} = -F_{\delta} - G_{\delta} - 3C v_{\delta} - (3D + 3E) w_{\delta}, \\
\text{(k)} & \quad C_{1\beta\gamma;\delta} = -C_{\beta\gamma;\delta},
\end{align*}
\]

where \( A_{\delta} = L(\partial_{k} A) e_{\delta}^{k} \). From (1.9) and (2.4), we get

\[
\begin{align*}
C_{222;\delta} + C_{233;\delta} + C_{244;\delta} &= A_{\delta} + B_{\delta} + C_{\delta} = (A + B + C)_{\delta} = (L\tilde{c})_{\delta}, \\
C_{322;\delta} + C_{333;\delta} + C_{344;\delta} &= L\tilde{c} u_{\delta}, \\
C_{422;\delta} + C_{433;\delta} + C_{444;\delta} &= -L\tilde{c} v_{\delta}.
\end{align*} \tag{2.5}
\]

Thus from (2.3), (2.4) and (2.5), we have

**Theorem 2.1.** In a four-dimensional Finsler space satisfying \( T \)-condition, the \( v \)-connection vectors \( u_{i} \) and \( v_{i} \) vanish identically. Also main scalar \( A \) and the unified main scalar \( L\tilde{c} \) are \( v \)-covariant constants (functions of position only). Furthermore, if \( v \)-connection vector \( w_{i} \) vanishes then all the main scalars are functions of position only.

3. **Berwald Space**

A Berwald space is characterized by \( C_{hijk}^{\ell k} = 0 \). From (1.8) and (1.11), it follows that

\[
LC_{hijk}^{\ell k} = C_{\alpha\beta\gamma;\delta} e_{\alpha}^{a} e_{\beta}^{b} e_{\gamma}^{c} e_{\delta}^{d} k,
\]

where \( C_{\alpha\beta\gamma;\delta} \) are given by

\[
C_{\alpha\beta\gamma;\delta} = (\delta_{\delta} C_{\alpha\beta\gamma} e_{\delta}^{k}) + C_{\mu\beta\gamma} H_{\mu}^{\alpha} a + C_{\alpha\mu\gamma} H_{\mu}^{\beta} \delta + C_{\alpha\beta\mu} H_{\mu}^{\gamma} \delta.
\]
The explicit forms of $C_{\alpha\beta\gamma,\delta}$ are obtained as follows:

\[
\begin{aligned}
  \text{(a)} \quad & C_{222,\delta} = A_{,\delta} - 3Dh_{\delta} + 3Fj_{\delta}, \\
  \text{(b)} \quad & C_{233,\delta} = B_{,\delta} + (2D - E)h_{\delta} + Gj_{\delta} - 2Hk_{\delta}, \\
  \text{(c)} \quad & C_{244,\delta} = C_{,\delta} + (D + E)h_{\delta} - (3F + G)j_{\delta} + 2Hk_{\delta}, \\
  \text{(d)} \quad & C_{322,\delta} = D_{,\delta} + (A - 2B)h_{\delta} + 2Hj_{\delta} - Fk_{\delta}, \\
  \text{(e)} \quad & C_{333,\delta} = E_{,\delta} + 3Bh_{\delta} - 3Gk_{\delta}, \\
  \text{(f)} \quad & C_{422,\delta} = F_{,\delta} - 2Hh_{\delta} - (A - 2C)j_{\delta} + Dk_{\delta}, \\
  \text{(g)} \quad & C_{433,\delta} = G_{,\delta} + 2Hh_{\delta} - B_{,\delta} + (2D + 3E)k_{\delta}, \\
  \text{(h)} \quad & C_{234,\delta} = H_{,\delta} + (F - G)h_{\delta} - (2D + 3E)j_{\delta} + (B - C)k_{\delta}, \\
  \text{(i)} \quad & C_{344,\delta} = -D_{,\delta} - E_{,\delta} + Ch_{\delta} - 2Hj_{\delta} + (F + 3G)k_{\delta}, \\
  \text{(j)} \quad & C_{444,\delta} = -F_{,\delta} - G_{,\delta} - 3Cj_{\delta} - (3D + 3E)k_{\delta}, \\
  \text{(k)} \quad & C_{1,\beta,\gamma,\delta} = 0.
\end{aligned}
\]

From (1.9) and (3.2), we get

\[
\begin{aligned}
  C_{322,\delta} + C_{333,\delta} + C_{344,\delta} &= (A + B + C)h_{\delta} = L\tilde{c}h_{\delta}, \\
  C_{422,\delta} + C_{433,\delta} + C_{444,\delta} &= -(A + B + C)j_{\delta} = -L\tilde{c}j_{\delta}, \\
  C_{222,\delta} + C_{233,\delta} + C_{244,\delta} &= (A_{,\delta} + B_{,\delta} + C_{,\delta}) = (A + B + C)_{,\delta}.
\end{aligned}
\]

Thus from (3.2) and (3.3), we have:

**Theorem 3.1** ([11]). In a four-dimensional Berwald space, the h-connection vectors $h_1$ and $j_1$ vanish identically. Also main scalar $A$ and the unified main scalar $L\tilde{c}$ are h-covariant constants. Furthermore, if h-connection vector $k_1$ vanishes then all the main scalars are h-covariant constants.

### 4. v-Curvature Tensor

The v-curvature tensor is defined by

\[
S_{hijk} = C_{hkr}^{\gamma}C_{ijr} - C_{hjr}^{\gamma}C_{ikr}.
\]

The scalar components $S_{\alpha\beta\gamma,\delta}$ of $L^2S_{hijk}$ are given by

\[
L^2S_{hijk} = S_{\alpha\beta\gamma,\delta}e_{\alpha}^{\gamma}e_{\beta}^{\gamma}e_{\gamma}^{\gamma}e_{\delta}^{\gamma}k.
\]
Theorem 4.1. In an S-3 like four-dimensional Finsler space satisfying T-condition, all the main scalars are functions of position only.
It is clear from (2.4) that if all the main scalars are functions of position only in a Finsler space satisfying $T$-condition, then the $v$-connection vectors $u_i$, $v_i$, and $w_i$ vanish. This leads to:

**Theorem 4.2.** In an $S$-3 like four-dimensional Finsler space satisfying $T$-condition, the $v$-connection vectors $u_i$, $v_i$, and $w_i$ vanish identically.

A Landsberg space is characterized by $C_{hijk} = C_{hiklj}$. H. Yasuda [12] proved that in an $S$-3 like Landsberg space, the $v$-curvature $S$ is constant. In view of this result, in an $S$-3 like four-dimensional Landsberg space, six independent functions


and

$$2FD + BH + CH - AH - DG + EF$$

are constants.

Since every Berwald space is a Landsberg space, these six functions are constant in an $S$-3 like Berwald space. From theorem 3.1 and equation (1.9), functions $A$ and $A + B + C$ are $h$-covariant constants in a four-dimensional Berwald space. Therefore in an $S$-3 like Berwald space, eight independent functions $A$, $A + B + C$,


and

$$2FD + BH + CH - AH - DG + EF$$

are $h$-covariant constants and therefore the main scalars $A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$ are $h$-covariant constants.

Thus, we have:

**Theorem 4.3.** In an $S$-3 like four-dimensional Berwald space, all the main scalars are $h$-covariant constants.

It is clear from (3.2) that if all the main scalars are $h$-covariant constants in a Berwald space, then the $h$-connection vectors $h_i$, $j_i$, and $k_i$ vanish.

This leads to:

**Theorem 4.4.** In an $S$-3 like four-dimensional Berwald space, the $h$-connection vectors $h_i$, $j_i$, and $k_i$ vanish identically.

In view of theorems 4.1, 4.2, 4.3 and 4.4, we can say

**Theorem 4.5.** In an $S$-3 like four-dimensional Berwald space satisfying $T$-condition, all the main scalars are constants and the $h$- and $v$-connection vectors vanish.

F. Ikeda [4] proved that a Landsberg space satisfying $T$-condition is a Berwald space. Thus, we may conclude:

**Theorem 4.6.** In an $S$-3 like four-dimensional Landsberg space satisfying $T$-condition, all the main scalars are constants and the $h$- and $v$-connection vectors vanish.
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