PROJECTIVE RANDERS CHANGES OF SPECIAL FINSLER SPACES

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Abstract. A change of Finsler metric $L(x, y) \rightarrow \bar{L}(x, y)$, is called a Randers change of $L$ if $\bar{L}(x, y) = L(x, y) + b_i(x) y^i$. The purpose of this paper is to study the conditions for a Finsler space of weakly Berwald/Landsberg type which could be transformed by a Randers change to a Finsler space of the same type.

1. Introduction

Randers’s well-known method for giving examples of Finsler spaces has the form

$$L(x, y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$$

where $a_{ij}$ is a Riemannian metric and $\beta(y^i) = b_iy^i$ is a one form with the condition $\|b\| = \sqrt{a^{ij}b_ib_j} < 1$ ($a^{ij}$ is the inverse of $a_{ij}$). If we change $\alpha(x, y) = \sqrt{a_{ij}(x)y^iy^j}$ to a given Finsler metric, this method may lead to another Finsler metric.

Definition ([5]). A change of Finsler metric $L(x, y) \rightarrow \bar{L}(x, y)$, is called a Randers change of $L$ if

$$\bar{L}(x, y) = L(x, y) + b_i(x)y^i$$

where $\beta(x, y) = b_i(x)y^i$ is a one form on a smooth manifold $M^n$.

Thorough this paper we always suppose the regularity, positive homogeneity and strong convexity for the Finsler structure ([3]), thus we assume a priori that $\bar{L}$ satisfies the ordinary conditions as fundamental function.

Another important change of Finsler metrics is the so called projective change. A change of Finsler metric $L(x, y) \rightarrow \bar{L}(x, y)$, is called a projective change of $L$

2000 Mathematics Subject Classification. 53B40.

Key words and phrases. Randers change, weakly-Berwald space, Landsberg space.
if geodesic curves are preserved. It is a well-known fact that $L(x, y) \rightarrow \bar{L}(x, y)$ is projective if and only if there exists a scalar field $p(x, y)$ which is positive homogeneous of order one, called the projective factor, satisfying $G^i = G^i + p(x, y)y^i$ where $G^i$ are the geodesic spray coefficients.

Projective Randers changes are characterised by the following theorem:

**Theorem** ([4]). A Randers change is projective if and only if $b$ is a gradient vector field.

Randers changes of special Finsler spaces were studied e.g. in the papers [1], [7]. In [7] Park and Lee gave conditions for Finsler spaces changed by a Randers change to be of Douglas type.

**Theorem** ([7]). Let $F^n(M^n, L) \rightarrow F^n(M^n, \bar{L})$ a projective Randers change. If $F^n$ is a Douglas space, then $\bar{F}^n$ is also a Douglas space, and vice versa.

The terminology and notations are referred basically to monograph [6]. Let $M^n$ be an $n$-dimensional $(n > 2)$ differentiable manifold and $F^n$ be a Finsler space equipped with a fundamental function $L(x, y)$ on $M^n$. A short review of the basic notations:

- the Finsler metric tensor: $g_{ij} = \dot{\partial}_i \dot{\partial}_j L^2/2$ where $\dot{\partial}_i$ refers to the partial derivation with respect to $y^i$. $g^{ij}$ is the inverse of $g_{ij}$
- the distinguished section: $\ell^i = y^i/L$, $\ell_i = y_i/L$
- the angular metric tensor: $h_{ij} = g_{ij} - \ell_i \ell_j$
- the geodesic spray coefficients and successive $y$-derivatives:
  \[ 4G^i_j = (\dot{\partial}_i \partial_j L^2)y^i - \partial_j L^2, \quad G^i = g^{\alpha}\omega^\alpha_{ij}, \]
  \[ G^i_j = \dot{\partial}_i G^j, \quad G^i_{jk} = \dot{\partial}_k G^i_{j}, \quad G^i_{jkl} = \dot{\partial}_l G^i_{jkl}, \]
  \[ g_{\alpha\ell} G^\alpha_{ijk} = G_{lij} \]
- the hv-torsion
  \[ -2P_{ijk} = y_\alpha G^\alpha_{ijk}. \]

Throughout the paper we shall use the notation $L_i = \dot{\partial}_i L$, $L_{ij} = \dot{\partial}_j \dot{\partial}_i L$ etc. We use the following properties of the angular metric tensor freely:

- $h_{ij} = LL_{ij}$
- $h_{ij} \ell^j = 0$
- $g^{ij} h_{ik} = \delta^i_k - \ell_i \ell_k$
- $g^{ij} h_{ij} = n - 1$.

In the projective geometry of Finsler manifolds, there is an important projective invariant quantity, the *Douglas* tensor defined by

\[ D^h_{ijk} = G^h_{ijk} - \frac{1}{n + 1} \left( G_{ijk} y^h + \delta^i_k G_{jik} + \delta^i_j G_{ik} + \delta^i_k G_{ij} \right). \]
2. Projective Randers changes

Lemma 1. For a Randers change we have $\frac{1}{L} \cdot h_{ij} = \frac{1}{\bar{L}} \cdot \bar{h}_{ij}$.

Proof. It follows from (1) that $\bar{L}_i = L_i + b_i$, $\bar{L}_{ij} = L_{ij}$. The angular metric tensor satisfies $h_{ij} = LL_{ij}$, thus $\frac{\bar{h}_{ij}}{\bar{L}} = \bar{L}_{ij} = L_{ij} = \frac{h_{ij}}{L}$. □

Lemma 2. If $\bar{L}(x, y) = L(x, y) + \beta(x, y)$ is a projective Randers change, then

$$
\frac{1}{L} G_{lijk} + \frac{2}{L^2} \ell_l P_{ijk} - \frac{1}{(n+1)L} (h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij}) = \frac{1}{\bar{L}} G_{lijk} + \frac{2}{\bar{L}^2} \bar{\ell}_l \bar{P}_{ijk} - \frac{1}{(n+1)\bar{L}} (\bar{h}_{il} \bar{G}_{jk} + \bar{h}_{jl} \bar{G}_{ik} + \bar{h}_{kl} \bar{G}_{ij}).
$$

Proof. From (3) one obtains

$$
\frac{1}{L} h_{\alpha l} D_{ijk} = \frac{1}{L} (g_{\alpha l} - \ell_\alpha \ell_l) \cdot G_{ijk}^\alpha - \frac{1}{(n+1)L} (G_{ijk} h_{\alpha l} y^\alpha + h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij}).
$$

From the property $h_{\alpha l} y^\alpha = 0$ it follows that

$$
\frac{1}{L} h_{\alpha l} D_{ijk} = \frac{1}{L} (g_{\alpha l} - \ell_\alpha \ell_l) \cdot G_{ijk}^\alpha - \frac{1}{(n+1)L} (h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij}).
$$

From the definition of the hv-torsion (see (2)) we conclude that

$$
\frac{1}{L} h_{\alpha l} D_{ijk} = \frac{1}{L} (g_{\alpha l} - \frac{y_\alpha}{L} \ell_l) \cdot G_{ijk}^\alpha - \frac{1}{(n+1)L} (h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij})
$$

$$
= \frac{1}{L} G_{lijk} + \frac{2}{L^2} \ell_l P_{ijk} - \frac{1}{(n+1)L} (h_{il} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij}).
$$

The Douglas tensor $D_{ijk}$ is projective invariant. Moreover, by Lemma 1 we have $\frac{1}{L} h_{\alpha l} D_{ijk}^\alpha = \frac{1}{\bar{L}} \bar{h}_{\alpha l} \bar{D}_{ijk}^\alpha$ and this fact completes the proof. □

In the next two sections we give two consequences of the relation (4).

3. Projective Randers change between Landsberg spaces

Definition. If a Finsler space satisfies the condition $P_{ijk} = 0$, we call it a Landsberg space.
Theorem 1. Let \( F_n \) and \( \bar{F}_n \) be Landsberg spaces and let \( \bar{L}(x, y) = L(x, y) + \beta(x, y) \) be a projective Randers change between them. Then

\[
G_{lk} - \bar{G}_{lk} = \frac{1}{n-1} h_{kl} \lambda(x, y).
\]

where \( \lambda(x, y) \) is a scalar field.

Proof. Let \( F_n \) and \( \bar{F}_n \) be Landsberg spaces, i.e. \( \bar{P}_{ijk} = \bar{P}_{ijk} = 0 \). Then (4) becomes

\[
\frac{1}{L} G_{lijk} = \frac{1}{n+1} L \left( h_{ij} G_{jk} + h_{jl} G_{ik} + h_{kl} G_{ij} \right) \bigg|_{\bar{G}}.
\]

Moreover, for Landsberg spaces we have \( G_{lijk} - \bar{G}_{lijk} = 0, \ G_{lijk} - \bar{G}_{lijk} = 0 \). These properties lead to

\[
\frac{1}{(n+1) L} (h_{ij} G_{lk} + h_{ik} G_{lj} - h_{jl} G_{ik} - h_{kl} G_{ij}) = 0.
\]

Hence we see that

\[
h_{ji} (G_{lk} - \bar{G}_{lk}) + h_{ki} (G_{ij} - \bar{G}_{ij}) - h_{jl} (G_{ik} - \bar{G}_{ik}) - h_{kl} (G_{ij} - \bar{G}_{ij}) = 0.
\]

Contraction with \( g^{ij} \) gives

\[
(n-1) (G_{lk} - \bar{G}_{lk}) + h_{k}^{a} (G_{la} - \bar{G}_{la}) - h_{i}^{a} (G_{ak} - \bar{G}_{ak}) - h_{kl} g^{ji} (G_{ij} - \bar{G}_{ij}) = 0.
\]

Denoting \( g^{ji} (G_{ij} - \bar{G}_{ij}) \) by \( \lambda(x, y) \) we find that

\[
(n-1) (G_{lk} - \bar{G}_{lk}) + (G_{lk} - \bar{G}_{lk}) - (G_{lk} - \bar{G}_{lk}) - h_{kl} \lambda(x, y) = 0.
\]

\( \square \)

4. Projective Randers change between weakly-Berwald spaces

Definition \([2]\). If a Finsler space satisfies the condition \( G_{ij} = 0 \), we call it a weakly-Berwald space.

Theorem 2. Let \( F_n \) and \( \bar{F}_n \) be two weakly Berwald Finsler spaces which are related by a projective Randers change \( L \to \bar{L} \). Let \( p(x, y) \) denote the projective factor of the change, that is \( G^i = G^i + p(x, y) y^i \). Then \( \partial_i p(x, y) \) does not depend on \( y \).
Proof. The equation (4) for a weakly-Berwald space becomes:

\[
\frac{1}{L} G_{iijk} + \frac{2}{L^2} \ell_l P_{ijk} = \frac{1}{L} \bar{G}_{iijk} + \frac{2}{L^2} \ell_l \bar{P}_{ijk}.
\]

Because of

\[
\frac{1}{L} G_{iijk} + \frac{2}{L^2} \ell_l P_{ijk} = \frac{1}{L} g_{\alpha l} G_{i\alpha jk} - \ell_l g_{\alpha l} G_{i\alpha jk} = \frac{1}{L} \left[ (g_{\alpha l} - \ell_l \ell_{\alpha}) G_{i\alpha jk} \right]
\]

we have

\[
\frac{1}{L} h_{\alpha l} G_{i\alpha jk} = \frac{1}{L} \bar{h}_{\alpha l} G_{i\alpha jk}.
\]

Then it follows from \( h_{\alpha l}/L = \bar{h}_{\alpha l}/\bar{L} \) that

\[
\frac{1}{L} h_{\alpha l} G_{i\alpha jk} = \frac{1}{L} h_{\alpha l} \bar{G}_{i\alpha jk},
\]

that is

\[
0 = h_{\alpha l} \left( \bar{G}_{i\alpha jk} - G_{i\alpha jk} \right).
\]

After successive derivations we have:

\[
\begin{align*}
G^i &= G^i + p(x, y) y^i \\
\bar{G}^i &= \bar{G}^i + p_j y^i + p\delta^i \\
\bar{G}^j_k &= G^j_k + p_j y^i + p_j \delta^i + p_k \delta^i_j \\
\bar{G}^\alpha_{ijk} &= G^\alpha_{ijk} + p_j y^\alpha + p_k \delta^\alpha_j + p_{ij} \delta^\alpha_k
\end{align*}
\]

Substituting the last formula into (5) we have

\[
0 = h_{\alpha l} \left( p_{ijk} y^\alpha + p_{jk} \delta^\alpha_i + p_{ik} \delta^\alpha_j + p_{ij} \delta^\alpha_k \right)
\]

\[
0 = h_{\alpha l} p_{jk} + h_{ij} p_{ik} + h_{ik} p_{ij}.
\]

By contracting with \( g^{li} \) we obtain \( 0 = (n-1)p_{jk} + p_{jk} + p_{jk} \). This shows that \( (n+1)p_{jk} = 0 \) therefore \( p_j(x, y) \) does not depend on \( y \). \( \square \)

References

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