ORE EXTENSIONS OVER NEAR PSEUDO-VALUATION RINGS

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Abstract. We recall that a ring $R$ is called near pseudo-valuation ring if every minimal prime ideal is a strongly prime ideal.

Let $R$ be a commutative ring, $\sigma$ an automorphism of $R$. Recall that a prime ideal $P$ of $R$ is $\sigma$-divided if it is comparable (under inclusion) to every $\sigma$-stable ideal $I$ of $R$. A ring $R$ is called a $\sigma$-divided ring if every prime ideal of $R$ is $\sigma$-divided. Also a ring $R$ is almost $\sigma$-divided ring if every minimal prime ideal of $R$ is $\sigma$-divided.

We also recall that a prime ideal $P$ of $R$ is $\delta$-divided if it is comparable (under inclusion) to every $\delta$-invariant ideal $I$ of $R$. A ring $R$ is called a $\delta$-divided ring if every prime ideal of $R$ is $\delta$-divided. A ring $R$ is said to be almost $\delta$-divided ring if every minimal prime ideal of $R$ is $\delta$-divided.

We define a Min.Spec-type endomorphism $\sigma$ of a ring $R$ ($\sigma(U) \subseteq U$ for all minimal prime ideals $U$ of $R$) and a Min.Spec-type ring (if there exists a Min.Spec-type endomorphism of $R$). With this we prove the following. Let $R$ be a commutative Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ is the field of rational numbers), $\delta$ a derivation of $R$. Then:

1. $R$ is a near pseudo valuation ring implies that $R[x; \delta]$ is a near pseudo valuation ring.
2. $R$ is an almost $\delta$-divided ring if and only if $R[x; \delta]$ is an almost $\delta$-divided ring.

We also prove a similar result for $R[x; \sigma]$, where $R$ is a commutative Noetherian ring and $\sigma$ a Min.Spec-type automorphism of $R$.

1. Introduction

We follow the notation as in Bhat [10], but to make the note self contained, we have the following. All rings are associative with identity. Throughout this paper $R$ denotes a commutative ring with identity $1 \neq 0$. The nil radical of $R$ and the prime radical of $R$ are denoted by $N(R)$ and $P(R)$ respectively. The set of prime ideals of $R$ is denoted by $\text{Spec}(R)$, the set of minimal prime
ideals of $R$ is denoted by $\text{Min}\text{-}\text{Spec}(R)$, and the set of strongly prime ideals is denoted by $S\text{-}\text{Spec}(R)$. The center of $R$ is denoted by $Z(R)$. The field of rational numbers and the ring of integers are denoted by $\mathbb{Q}$ and $\mathbb{Z}$ respectively unless otherwise stated.

We recall that as in Hedstrom and Houston [15], an integral domain $R$ with quotient field $F$, is called a pseudo-valuation domain (PVD) if each prime ideal $P$ of $R$ is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). For example let $F = \mathbb{Q}(\sqrt{2})$ and $V = F + xF[[x]] = F[[x]]$. Then $V$ is a pseudo-valuation domain. We also note that $S = \mathbb{Q} + \mathbb{Q}x + x^2V$ is not a pseudo-valuation domain (Badawi [6]). For more details on pseudo-valuation rings, the reader is referred to Badawi [6].

In Badawi, Anderson and Dobbs [7], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way. A prime ideal $P$ of $R$ is said to be strongly prime if $\sigma P$ is strongly prime for all $\sigma \in \text{End}(R)$ (by Proposition (3.1) of Anderson [1], Proposition (4.2) of Anderson [2] and Proposition (3) of Badawi [3].

In Badawi [5], another generalization of PVDs is given in the following way. Let $R$ be a ring with total quotient ring $Q$ such that $N(R)$ is a divided prime ideal of $R$, let $\phi : Q \to R_{N(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then $\phi$ is a ring homomorphism from $Q$ into $R_{N(R)}$, and $\phi$ restricted to $R$ is also a ring homomorphism from $R$ into $R_{N(R)}$ given by $\phi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by $T$. A prime ideal $P$ of $\phi(R)$ is called a $T$-strongly prime ideal if $\phi(R)$ is a $T$-pseudo-valuation ring ($T$-PVR) if each prime ideal of $\phi(R)$ is $T$-strongly prime. A prime ideal $S$ of $R$ is called $\phi$-strongly prime ideal if $\phi(S)$ is a $T$-strongly prime ideal of $\phi(R)$. If each prime ideal of $R$ is $\phi$-strongly prime, then $R$ is called a $\phi$-pseudo-valuation ring ($\phi - \text{PVR}$).

This article concerns the study of skew polynomial rings over PVDs. Let $R$ be a ring, $\sigma$ an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ ($\delta : R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$). In case $\sigma$ is identity, $\delta$ is just called a derivation. For example let $R = F[x]$, $F$ a
field. Then \( \sigma: R \to R \) defined by \( \sigma(f(x)) = f(0) \) is an endomorphism of \( R \).

Also let \( K = \mathbb{R} \times \mathbb{R} \). Then \( g: K \to K \) by \( g(a, b) = (b, a) \) is an automorphism of \( K \).

Let \( \sigma \) be an automorphism of a ring \( R \) and \( \delta: R \to R \) any map. Let \( \phi: R \to M_2(R) \) defined by

\[
\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix},
\]

for all \( r \in R \) be a homomorphism. Then \( \delta \) is a \( \sigma \)-derivation of \( R \). Also let \( R = F[x], F \) a field. Then the usual differential operator \( \frac{d}{dx} \) is a derivation of \( R \).

We denote the Ore extension \( R[x; \sigma, \delta] \) by \( O(R) \). If \( I \) is an ideal of \( R \) such that \( I \) is \( \sigma \)-stable; i.e. \( \sigma(I) = I \) and \( I \) is \( \delta \)-invariant; i.e. \( \delta(I) \subseteq I \), then we denote \( I[x; \sigma, \delta] \) by \( O(I) \). We would like to mention that \( R[x; \sigma, \delta] \) is the usual set of polynomials with coefficients in \( R \), i.e. \( \{\sum_{i=0}^{n} x^{i}a_{i}, a_{i} \in R\} \) in which multiplication is subject to the relation \( ax = x\sigma(a) + \delta(a) \) for all \( a \in R \).

In case \( \delta \) is the zero map, we denote the skew polynomial ring \( R[x; \sigma] \) by \( S(R) \) and for any ideal \( II \) of \( R \) with \( \sigma(I) = I \), we denote \( I[x; \sigma] \) by \( S(I) \). In case \( \sigma \) is the identity map, we denote the differential operator ring \( R[x; \delta] \) by \( D(R) \) and for any ideal \( J \) of \( R \) with \( \delta(J) \subseteq J \), we denote \( J[x; \delta] \) by \( D(J) \).

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [10, 11, 12, 14, 16].

Recall that a ring \( R \) is called a near pseudo-valuation ring (NPVR) if each minimal prime ideal \( P \) of \( R \) is strongly prime (Bhat [12]). For example a reduced ring is NPVR.

Here the term near may not be interpreted as near ring (Bell and Mason [8]). We note that a near pseudo-valuation ring (NPVR) is a pseudo-valuation ring (PVR), but the converse is not true. For example a reduced ring is a NPVR, but need not be a PVR.

We recall that a prime ideal \( P \) of \( R \) is said to be divided if it is comparable (under inclusion) to every ideal of \( R \). A ring \( R \) is called a divided ring if every prime ideal of \( R \) is divided (Badawi [4]). It is known (Lemma (1) of Badawi, Anderson and Dobbs [7]) that a pseudo-valuation ring is a divided ring. Recall that a ring \( R \) is called an almost divided ring if every minimal prime ideal of \( R \) is divided (Bhat [12]).

We also recall that a prime ideal \( P \) of \( R \) is \( \sigma \)-divided if it is comparable (under inclusion) to every \( \sigma \)-stable ideal \( I \) of \( R \). A ring \( R \) is called a \( \sigma \)-divided ring if every prime ideal of \( R \) is \( \sigma \)-divided (see Bhat [10]). A ring \( R \) is said to be almost \( \sigma \)-divided ring if every minimal prime ideal of \( R \) is \( \sigma \)-divided (Bhat [12]).

A prime ideal \( P \) of \( R \) is said to be \( \delta \)-divided if it is comparable (under inclusion) to every \( \sigma \)-stable and \( \delta \)-invariant ideal \( I \) of \( R \). A ring \( R \) is called a \( \delta \)-divided ring if every prime ideal of \( R \) is \( \delta \)-divided (Bhat [10]). A ring \( R \) is said to be almost \( \delta \)-divided ring if every minimal prime ideal of \( R \) is \( \delta \)-divided (Bhat [12]).
The author of this paper has proved in Theorems (2.6) and (2.8) of [10] the following. Let $R$ be a ring and $\sigma$ an automorphism of $R$. Then:

1. If $R$ is a commutative pseudo-valuation ring such that $x \notin P$ for any $P \in \text{Spec}(S(R))$, then $S(R)$ is also a pseudo-valuation ring.
2. If $R$ is a $\sigma$-divided ring such that $x \notin P$ for any $P \in \text{Spec}(S(R))$, then $S(R)$ is also a $\sigma$-divided ring.

In Theorems (2.10) and (2.11) of [10] the following results have been proved. Let $R$ be a commutative Noetherian $\mathbb{Q}$-algebra and $\delta$ a derivation of $R$. Then:

1. If $R$ is a pseudo-valuation ring, then $D(R)$ is also a pseudo-valuation ring.
2. If $R$ is a divided ring, then $D(R)$ is also a divided ring.

An analogue of the above results for near pseudo-valuation rings, almost divided rings and almost $\delta$-divided rings has been proved in (Bhat [12]), where $R$ is a $\sigma(\ast)$-ring. Recall that a ring $R$ is said to be a $\sigma(\ast)$-ring (\sigma an endomorphism of $R$) if $a \sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ (Kwak [16]).

**Theorem** ([12, 2.5]). Let $R$ be a commutative Noetherian near pseudo valuation ring which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring and $\delta$ a $\sigma$-derivation of $R$. Then $O(R)$ is a Noetherian near pseudo-valuation ring.

**Theorem** ([12, 2.7]). If $R$ is a commutative Noetherian almost $\delta$-divided $\sigma(\ast)$-ring which is also an algebra over $\mathbb{Q}$, then $O(R)$ is a Noetherian almost $\delta$-divided ring.

In this paper we give a necessary and sufficient condition for $D(R)$ over a Noetherian $\mathbb{Q}$-algebra $R$ to be a near pseudo valuation ring. We also give a necessary and sufficient condition for $D(R)$ over a Noetherian $\mathbb{Q}$-algebra $R$ to be an almost divided ring. We prove similar results for $S(R)$ over a Noetherian ring $R$. These results have been proved in Theorems (2.5) and (2.7) respectively. But before that, we have the following definition:

**Definition 1.1.** Let $R$ be a ring. We say that an endomorphism $\sigma$ of $R$ is Min.Spec-type if $\sigma(U) \subseteq U$ for all minimal prime ideals $U$ of $R$. We say that a ring $R$ is Min.Spec-type ring if there exists a Min.Spec-type endomorphism of $R$.

**Example 1.2.** Let $R = \begin{pmatrix} F & F' \\ 0 & F \end{pmatrix}$, where $F$ is a field. Let $\sigma : R \to R$ be defined by $\sigma(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that $\sigma$ is a Min.Spec-type endomorphism of $R$, and therefore, $R$ is a Min.Spec-type ring.

**Proposition 1.3.** If $R$ is a Noetherian ring and $\sigma$ is an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring, then $\sigma$ is a Min.Spec-type automorphism of $R$; i.e. $R$ is a Min.Spec-type ring.
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Proof. Note that $\sigma$ is an automorphism, therefore, $\sigma(U) \subseteq U$ implies that $\sigma(U) = U$. Now let $R$ be a $\sigma$-ring. We will first show that $P(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in P(R)$. Then $\sigma(a)\sigma(a) = a\sigma(a)\sigma(a) = a\sigma(a) \sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. So $P(R)$ is completely semiprime. Now let $U = U_1$ be a minimal prime ideal of $R$. Let $U_2, U_3, \ldots, U_n$ be the other minimal primes of $R$. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of $R$. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n-1} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$. \qed

The converse of the above need not not be true. For example let $R = F[x]$, $F$ a field. Then $R$ is a commutative domain with $P(R) = 0$. Let $\sigma : R \rightarrow R$ be defined by $\sigma(f(x)) = f(0)$. Then $\sigma$ is a $\text{Min} \text{Spec}$-type endomorphism of $R$. Now let $f(x) = xa$, $0 \neq a \in F$. Then $f(x)\sigma(f(x)) \in P(R)$, but $f(x) \notin P(R)$. Therefore $R$ is not a $\sigma$-ring.

2. Ore extensions

We recall that Gabriel proved in Lemma (3.4) of [13] that if $R$ is a Noetherian $\mathbb{Q}$-algebra and $\delta$ is a derivation of $R$, then $\delta(U) \subseteq U$, for all $U \in \text{Min} \text{ Spec}(R)$. This result has been generalized in Theorem (2.2) of Bhat [9] for a $\sigma$-derivation $\delta$ of $R$ and the following has been proved:

**Theorem 2.1.** Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Then $\delta(U) \subseteq U$ for all $U \in \text{Min} \text{Spec}(R)$.

**Proof.** See Theorem (2.2) of Bhat [9]. \qed

**Theorem 2.2** ([11, Theorem 3.7]). Let $R$ be a Noetherian $\mathbb{Q}$-algebra and $\delta$ be a derivation of $R$. Then $P \in \text{Min} \text{Spec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Min} \text{Spec}(R)$.

Let $R$ be a Noetherian ring. Then since $\text{Min} \text{Spec}(R)$ is finite and for any automorphism $\sigma$ of $R$, $\sigma^j(U) \in \text{Min} \text{Spec}(R)$ for all $U \in \text{Min} \text{Spec}(R)$ and for all integers $j \geq 1$, it follows that there exists some positive integer $m$ such that $\sigma^m(U) = U$ for all $U \in \text{Min} \text{Spec}(R)$. We denote $\cap_{j=0}^{n-1} \sigma^j(U)$ by $U^0$. With this we have the following

**Theorem 2.3** ([11, Theorem 2.4]). Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$. Then $P \in \text{Min} \text{Spec}(S(R))$ if and only if there exists $U \in \text{Min} \text{Spec}(R)$ such that $S(P \cap R) = P$ and $P \cap R = U^0$.

**Theorem 2.4** (Hilbert Basis Theorem). Let $R$ be a right/left Noetherian ring. Let $\sigma$ and $\delta$ be as usual. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.

**Proof.** See Theorem (1.12) of Goodearl and Warfield [14]. \qed
Remark 1. We note if $R$ is a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $\sigma$ can be extended to an automorphism of $O(R)$ by $\sigma(x) = x$; i.e. $\sigma(xa) = x\sigma(a)$ for $a \in R$. Also $\delta$ can be extended to a $\sigma$-derivation of $O(R)$ by $\delta(x) = 0$; i.e. $\delta(xa) = x\delta(a)$ for $a \in R$.

It is known (Theorem (2.10) of Bhat [10]) that if $R$ is a commutative Noetherian $\mathbb{Q}$-algebra which is also a PVR. Then $D(R)$ is also a PVR. We generalize this result for NPVR and prove its converse also.

It is also known (Theorem (2.11) of Bhat [10]) that if $R$ is a commutative Noetherian $\mathbb{Q}$-algebra, and is also divided, then $D(R)$ is also divided. We generalize this result for almost divided rings and prove its converse also. Towards this we prove the following:

**Theorem 2.5.** Let $R$ be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let $\delta$ be a derivation of $R$. Further let any $U \in S.\text{Spec}(R)$ with $\delta(U) \subseteq U$ implies that $O(U) \in S.\text{Spec}(O(R))$. Then

1. $R$ is a near pseudo-valuation ring implies that $D(R)$ is a near pseudo-valuation ring.
2. $R$ is an almost $\delta$-divided ring if and only if $D(R)$ is an almost $\delta$-divided ring.

**Proof.** (1) Let $R$ be a near pseudo-valuation ring which is also an algebra over $\mathbb{Q}$. Now $D(R)$ is Noetherian by Theorem (2.4). Let $J \in \text{Min}.\text{Spec}(D(R))$. Then by Theorem (2.2) $J \cap R \in \text{Min}.\text{Spec}(R)$. Now $R$ is a near pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \subseteq S.\text{Spec}(R)$. Also $\delta(J \cap R) \subseteq J \cap R$ by Theorem (2.1). Now Theorem (2.2) implies that $D(J \cap R) = J$, and by hypothesis $D(J \cap R) \in S.\text{Spec}(D(R))$. Therefore $J \in S.\text{Spec}(D(R))$. Hence $D(R)$ is a near pseudo-valuation ring.

(2) Let $R$ be an almost $\delta$-divided which is also an algebra over $\mathbb{Q}$. Now $D(R)$ is Noetherian by Theorem (2.4). Let $J \in \text{Min}.\text{Spec}(D(R))$ and $K$ be an ideal of $D(R)$. Now by Theorem (2.2) $J \cap R \in \text{Min}.\text{Spec}(R)$. Now $R$ is an almost $\delta$-divided commutative Noetherian $\mathbb{Q}$-algebra, therefore $J \cap R$ and $K \cap R$ are comparable (under inclusion), say $J \cap R \subseteq K \cap R$. Now $\delta(K \cap R) \subseteq K \cap R$ by Lemma (2.18) of Goodearl and Warfield [14]. Therefore, $D(K \cap R)$ is an ideal of $D(R)$ and so $D(J \cap R) \subseteq D(K \cap R)$. This implies that $J \subseteq K$. Hence $D(R)$ is an almost $\delta$-divided ring.

Conversely suppose that $D(R)$ is almost $\delta$-divided (note that $\delta$ can be extended to a derivation of $D(R)$ by Remark (1)). Let $U \in \text{Min}.\text{Spec}(R)$ and $V$ be a $\delta$-invariant ideal of $R$. Now by Theorem (2.1) $\delta(U) \subseteq U$, and Theorem (2.2) implies that $D(U) \in \text{Min}.\text{Spec}(D(R))$. Now $D(R)$ is an almost $\delta$-divided ring, therefore $D(U)$ and $D(V)$ are comparable (under inclusion), say $D(U) \subseteq D(V)$. Therefore, $D(U) \cap R \subseteq D(V) \cap R$; i.e. $U \subseteq V$. Hence $R$ is an almost $\delta$-divided ring. \qed
We note that in above Theorem the hypothesis that any \( U \in S.Spec(R) \) with 
\( \delta(U) \subseteq U \) implies that \( O(U) \in S.Spec(O(R)) \) can not be deleted as extension of a strongly prime ideal of \( R \) need not be a strongly prime ideal of \( D(R) \).

**Example 2.6.** \( R = \mathbb{Z}_{(p)} \). This is in fact a discrete valuation domain, and therefore, its maximal ideal \( P = pR \) is strongly prime. But \( pR[x] \) is not strongly prime in \( R[x] \) because it is not comparable with \( xR[x] \) (so the condition of being strongly prime in \( R[x] \) fails for \( a = 1 \) and \( b = x \)).

It is known (Theorem (2.6) of Bhat [10]) that if \( R \) is a commutative PVR such that \( x \notin P \) for any \( P \in Spec(S(R)) \). Then \( S(R) \) is also a PVR. We generalize this result for NPVR and prove its converse also.

It is known (Theorem (2.8) of Bhat [10]) that if \( R \) is a \( \sigma \)-divided Noetherian ring such that \( x \notin P \) for any \( P \in Spec(S(R)) \). Then \( S(R) \) is also a \( \sigma \)-divided ring. We generalize this result for NPVR and prove its converse also. Towards this we have the following:

**Theorem 2.7.** Let \( R \) be a Noetherian ring. Let \( \sigma \) be a Min.Spec-type automorphism of \( R \). Further let any \( U \in S.Spec(R) \) with \( \sigma(U) = U \) implies that \( O(U) \in S.Spec(O(R)) \). Then

1. \( R \) is a near pseudo-valuation ring implies that \( S(R) \) is a near pseudo-valuation ring.
2. \( R \) is an almost \( \sigma \)-divided ring if and only if \( S(R) \) is an almost \( \sigma \)-divided ring.

**Proof.** (1) Let \( R \) be a near pseudo-valuation ring. Now \( S(R) \) is Noetherian by Theorem (2.4). Let \( J \in Min.Spec(S(R)) \). Then by Theorem (2.3) there exists \( U \in Min.Spec(R) \) such that \( S(P \cap R) = P \) and \( P \cap R = U^0 \). But \( \sigma \) being Min.Spec-type implies that \( \sigma(U) = U \), and so \( U^0 = U \). Now \( R \) is a near pseudo-valuation ring implies that \( U \in S.Spec(R) \). Now by hypothesis \( S(U) \in S.Spec(S(R)) \). But \( S(U) = P \). Therefore \( P \in S.Spec(S(R)) \). Hence \( S(R) \) is a near pseudo-valuation ring.

(2) Let \( R \) be a ring which is also almost \( \sigma \)-divided. Now \( S(R) \) is Noetherian by Theorem (2.4). Let \( J \in Min.Spec(S(R)) \) and \( K \) be an ideal of \( S(R) \) such that \( \sigma(K) = K \) (note that \( \sigma \) can be extended to an automorphism of \( S(R) \) by Remark (1)). Now by Theorem (2.3) there exists \( U \in Min.Spec(R) \) such that \( S(J \cap R) = J \) and \( J \cap R = U^0 \). But \( \sigma \) being Min.Spec-type implies that \( \sigma(U) = U \), and so \( U^0 = U \). Now \( R \) is an almost \( \sigma \)-divided, therefore \( U \) and \( K \cap R \) are comparable (under inclusion), say \( U \subseteq K \cap R \). Therefore, \( S(U) \subseteq S(K \cap R) \). This implies that \( J \subseteq K \). Hence \( S(R) \) is an almost \( \sigma \)-divided ring.

Conversely let \( R \) be a ring such that \( S(R) \) is almost \( \sigma \)-divided. Let \( U \in Min.Spec(R) \) and \( V \) be a \( \sigma \)-stable ideal of \( R \). Now \( \sigma \) being Min.Spec-type implies that \( \sigma(U) = U \) and Theorem (2.3) implies that \( S(U) \in Min.Spec(S(R)) \).
Now $S(R)$ is an almost $\sigma$-divided ring, therefore $S(U)$ and $S(V)$ are comparable (under inclusion), say $S(U) \subseteq S(V)$. Therefore, $S(U) \cap R \subseteq S(V) \cap R$; i.e. $U \subseteq V$. Hence $R$ is an almost $\sigma$-divided ring.

□

Problem. Let $R$ be a NPVR. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Is $O(R) = R[x; \sigma, \delta]$ a NPVR?

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References


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