ON CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

defined on the open unit disk $U = \{z : |z| < 1\}$. In this paper we define a subclass of $\alpha$-uniform starlike and convex functions by using the generalized Ruscheweyh derivatives operator introduced by authors in [9]. Several properties to this class are obtained.

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined on the open unit disk $U = \{z : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $U$. Let $S^*(\beta)$ and $C(\beta)$ be the classes of functions respectively starlike of order $\beta$ and convex of order $\beta$, $(0 \leq \beta < 1)$. Finally, let $\mathcal{T}$ be the subclass of $\mathcal{S}$, consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k.$$  

A function $f \in \mathcal{T}$ is called a function with negative coefficients. In this present paper, we study the following class of function:

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Definition 1.1. For $0 \leq \beta < 1$, $\alpha \geq 0$, $n \in \mathbb{N}_0$ and $\lambda \geq 0$, we let $M_n^\alpha(\alpha, \beta)$ consist of functions $f \in T$ satisfying the condition
\begin{equation}
\Re \left\{ \frac{z(D_n^\alpha f(z))'}{D_n^\alpha f(z)} \right\} > \alpha \left| \frac{z(D_n^\alpha f(z))'}{D_n^\alpha f(z)} - 1 \right| + \beta
\end{equation}
where $D_n^\alpha$ denote the operator introduced by authors [9] and given by
\[ D_n^\alpha f(z) = \frac{z(z^n - 1)^{D_\lambda f(z)}}{n!}, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \]
Note that if $f$ is given by (1), then we see that
\[ D_n^\alpha f(z) = z - \sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n, k) |a_k| z^k, \]
where $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $C(n, k) = \binom{k+n-1}{n}$. The family $M_n^\alpha(\alpha, \beta)$ is of special interest for it contains many well known, as well as new, classes of analytic univalent functions. In particular $M_1^\alpha(\alpha, \beta) \equiv U(k, \lambda, \beta)$ is the class of $\alpha$-uniformly convex function introduced and studied by Shanmugam et al. [8]. The classes $M_1^0(\alpha, 0) \equiv \alpha-ST$ $M_1^1(\alpha, 0) \equiv \alpha-UCV$ is respectively, the classes of $\alpha$-uniformly starlike function and $\alpha$-uniformly convex function introduced and studied by Kanas and Wnioswka [5, 4]. The classes $M_0^0(0, \beta) \equiv T^*(\beta)$ and $M_1^0(0, \beta) \equiv TC(\beta)$ is respectively the classes of starlike functions of order $\beta$ and classes of convex functions of order $\beta$ studied by Silverman [10]. Also, we note that the class $M_1^0(1, 1) \equiv UCV$ was studied by Rønning [6]. Finally, we remark that Goodman introduced the concept of uniformly starlike function and of uniformly convex function in [3] and proved some properties for such functions in [3] and [2].

In this paper we provide necessary and sufficient conditions, coefficient bounds, extreme points, radius of close-to-convexity, starlikeness and convexity for functions in $M_n^\alpha(\alpha, \beta)$. Inclusion theorem involving Hadamard products, convolution and integral operator are also obtained.

2. Characterization

We employ the technique adopted by Aqlan et al. [1] to find the coefficient estimates for our class.

Theorem 2.1. let $f$ given by (1) then, $f \in M_n^\alpha(\alpha, \beta)$ if and only if
\begin{equation}
\sum_{k=2}^{\infty} \left[ k - \beta + \alpha(k-1) \right] [1 + \lambda(k-1)] C(n, k) |a_k| \leq (1 - \beta),
\end{equation}
where $\alpha, \lambda \geq 0$, $0 \leq \beta < 1$ and $n \in \mathbb{N}_0$. The result is sharp.
Proof. We have \( f \in M_n^\alpha(\alpha, \beta) \) if and only if the condition (2) is satisfied. Upon the fact that
\[
\Re(w) > \alpha|w - 1| + \beta \iff \Re\left\{ w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta} \right\} > \beta, \quad -\pi \leq \theta < \pi.
\]
Equation (2) may be written as
\[
(4) \quad \Re\left\{ \frac{z(D_n^\alpha f(z))'}{D_n^\alpha f(z)}(1 + \alpha e^{i\theta}) - \alpha e^{i\theta} \right\} > \beta.
\]
Now, we let
\[
A(z) = z(D_n^\alpha f(z))'(1 + \alpha e^{i\theta}) - \alpha e^{i\theta} D_n^\alpha f(z), \quad B(z) = D_n^\alpha f(z).
\]
Then (4) is equivalent to
\[
|A(z) + (1 - \beta)B(z)| > |A(z) - (1 + \beta)B(z)| \quad \text{for} \quad 0 \leq \beta < 1.
\]
For \( A(z) \) and \( B(z) \) as above, we have
\[
|A(z) + (1 - \beta)B(z)| \geq (2 - \beta)|z| - \sum_{k=2}^{\infty} [k + 1 - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k) |a_k| |z|^k,
\]
and similarly
\[
|A(z) - (1 + \beta)B(z)| \leq \beta|z| - \sum_{k=2}^{\infty} [k - 1 - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k) |a_k| |z|^k.
\]
Therefore,
\[
|A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)|
\geq 2(1 - \beta) - 2 \sum_{k=2}^{\infty} [k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k) |a_k|,
\]
or
\[
\sum_{k=2}^{\infty} [k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k) |a_k| \leq (1 - \beta), \quad \text{which yields (3)}.
\]
On the other hand, we must have
\[
\Re\left\{ \frac{z(D_n^\alpha f(z))'}{D_n^\alpha f(z)}(1 + \alpha e^{i\theta}) - \alpha e^{i\theta} \right\} > \beta.
\]
Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq |z| = r < 1 \), the above inequality reduces to

\[
\Re \left\{ \frac{(1 - \beta)r - \sum_{k=2}^{\infty} [k - \beta + \alpha e^{i\theta}(k - 1)] [1 + \lambda(k - 1)] C(n, k) a_k r^k}{z - \sum_{k=2}^{\infty} [1 + \lambda(k - 1)] C(n, k) a_k r^k} \right\} \geq 0.
\]

Since \( \Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1 \), the above inequality reduces to

\[
\Re \left\{ \frac{(1 - \beta)r - \sum_{k=2}^{\infty} [k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k) a_k r^k}{z - \sum_{k=2}^{\infty} [1 + \lambda(k - 1)] C(n, k) a_k r^k} \right\} \geq 0.
\]

Letting \( r \to 1^- \), we get the desired result. Finally the result is sharp with the extremal function \( f \) given by

\[
f(z) = z - \frac{1 - \beta}{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k)} z^n.
\]

\( \square \)

3. Growth and Distortion Theorems

**Theorem 3.1.** Let the function \( f \) defined by (1) be in the class \( M_\lambda^n(\alpha, \beta) \). Then for \( |z| = r \) we have

\[
r - \frac{1 - \beta}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)} r^2 \leq |f(z)| \leq r + \frac{1 - \beta}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)} r^2.
\]

Equality holds for the function

\[
f(z) = z - \frac{1 - \beta}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)} z^2.
\]

**Proof.** We only prove the right hand side inequality in (6), since the other inequality can be justified using similar arguments. In view of Theorem 2.1, we have

\[
\sum_{k=2}^{\infty} |a_k| \leq \frac{1 - \beta}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)}.
\]

Since, \( f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \)

\[
|f(z)| = |z| - \sum_{k=2}^{\infty} |a_k| |z|^k \leq r + \sum_{k=2}^{\infty} |a_k| r^k
\]
\[ \leq r + r^2 \sum_{k=2}^{\infty} |a_k| \leq r + \frac{1 - \beta}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)} r^2, \]

which yields the right hand side inequality of (6).

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

**Theorem 3.2.** Let the function \( f \) defined by (1) be in the class \( M^\lambda_n(\alpha, \beta) \). Then for \( |z| = r \) we have

\[ 1 - \frac{2(1 - \beta)}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \beta)}{(n + 1)(2 - \beta + \alpha)(1 + \lambda)} r. \]

Equality holds for the function given by (7).

**Theorem 3.3.** \( f \in M^\lambda_n(\alpha, \beta) \), then \( f \in T^*(\gamma) \), where

\[ \gamma = 1 - \frac{(k - 1)(1 - \beta)}{[k - \beta + \alpha(k - 1)][1 + \lambda(k - 1)]C(n, k) - (1 - \beta)}. \]

The result is sharp, with function given by (7).

**Proof.** It is sufficient to show that (3) implies \( \sum_{k=2}^{\infty} (k - \gamma)|a_k| \leq 1 - \gamma \), that is,

\[ \frac{k - \gamma}{1 - \gamma} \leq \frac{[k - \beta + \alpha(k - 1)][1 + \lambda(k - 1)]C(n, k)}{1 - \beta}, \]

then

\[ \gamma \leq 1 - \frac{(k - 1)(1 - \beta)}{[k - \beta + \alpha(k - 1)][1 + \lambda(k - 1)]C(n, k) - (1 - \beta)}. \]

The above inequality holds true for \( n \in \mathbb{N}_0, k \geq 2, \alpha, \lambda \geq 0 \) and \( 0 \leq \beta < 1 \). \( \square \)

4. **Extreme points**

**Theorem 4.1.** Let \( f_1(z) = z \) and

\[ f_k(z) = z - \frac{1 - \beta}{[k - \beta + \alpha(k - 1)][1 + \lambda(k - 1)]C(n, k)} z^k, \quad (k \geq 2). \]

Then \( f \in M^\lambda_n(\alpha, \beta) \), if and only if it can be represented in the form

\[ f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \quad (\mu_k \geq 0, \sum_{k=1}^{\infty} \mu_k = 1). \]

**Proof.** Suppose \( f(z) \) can be expressed as in (8). Then

\[ f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) \]
\[ f(z) = z + \sum_{k=2}^{\infty} \mu_k \left\{ 1 - \beta \left[ \frac{k - \beta + \alpha(k-1)}{1 + \lambda(k-1)C(n,k)} \right] \right\} z^k \]

Thus

\[ f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k \left\{ 1 - \beta \left[ \frac{k - \beta + \alpha(k-1)}{1 + \lambda(k-1)C(n,k)} \right] \right\} z^k \]

\[ f(z) = \mu_1 z + \sum_{k=2}^{\infty} \mu_k z - \sum_{k=2}^{\infty} \mu_k \left\{ 1 - \beta \left[ \frac{k - \beta + \alpha(k-1)}{1 + \lambda(k-1)C(n,k)} \right] \right\} z^k \]

\[ f(z) = z - \sum_{k=2}^{\infty} \mu_k \left\{ 1 - \beta \left[ \frac{k - \beta + \alpha(k-1)}{1 + \lambda(k-1)C(n,k)} \right] \right\} z^k \]

So by Theorem 2.1, \( f \in M^\lambda_n(\alpha, \beta) \).

Conversely, we suppose \( f \in M^\lambda_n(\alpha, \beta) \). Since

\[ |a_k| \leq \frac{1 - \beta}{[k - \beta + \alpha(k-1)][1 + \lambda(k-1)C(n,k)]} \quad k \geq 2. \]

We may set

\[ \mu_k = \frac{[k - \beta + \alpha(k-1)][1 + \lambda(k-1)C(n,k)]}{1 - \beta} |a_k| \quad k \geq 2. \]

and \( \mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k \). Then

\[ f(z) = z - \sum_{k=2}^{\infty} a_k z^k = z - \sum_{k=2}^{\infty} \mu_k \left[ \frac{k - \beta + \alpha(k-1)}{1 + \lambda(k-1)C(n,k)} \right] z^k \]

\[ f(z) = z - \sum_{k=2}^{\infty} \mu_k z - \sum_{k=2}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \mu_k z + \sum_{k=2}^{\infty} \mu_k f_k(z) \]

\[ f(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) = \sum_{k=1}^{\infty} \mu_k f_k(z). \]

\[ \square \]

**Corollary 4.2.** The extreme points of \( M^\lambda_n(\alpha, \beta) \) are the functions

\[ f_1(z) = z \quad \text{and} \quad f_k(z) = z - \frac{1 - \beta}{[k - \beta + \alpha(k-1)][1 + \lambda(k-1)C(n,k)]} z^k, \quad k \geq 2. \]
5. Radii of Close-to-convexity, Starlikeness and Convexity

A function \( f \in M^\alpha_\lambda(\alpha, \beta) \) is said to be close-to-convex of order \( \delta \) if it satisfies
\[
\Re\{f'(z)\} > \delta, \quad (0 \leq \delta < 1; z \in U).
\]
Also a function \( f \in M^\alpha_\lambda(\alpha, \beta) \) is said to be starlike of order \( \delta \) if it satisfies
\[
\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta, \quad (0 \leq \delta < 1; z \in U).
\]
Further a function \( f \in M^\alpha_\lambda(\alpha, \beta) \) is said to be convex of order \( \delta \) if and only if \( zf'(z) \) is starlike of order \( \delta \), that is if
\[
\Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \delta, \quad (0 \leq \delta < 1; z \in U).
\]

**Theorem 5.1.** Let \( f \in M^\alpha_\lambda(\alpha, \beta) \). Then \( f \) is close-to-convex of order \( \delta \) in \(|z| < R_1\), where
\[
R_1 = \inf_{k \geq 2} \left[ \frac{(1-\delta)[k-\beta + \alpha(k-1)][1 + \lambda(k-1)]C(n,k)}{k(1-\beta)} \right]^\frac{1}{k-1}.
\]
The result is sharp with the extremal function \( f \) given by (5).

**Proof.** It is sufficient to show that \(|f'(z) - 1| \leq 1 - \delta\) for \(|z| < R_1\). We have
\[
|f'(z) - 1| = \left| - \sum_{k=2}^{\infty} ka_k z^{k-1} \right| \leq \sum_{k=1}^{\infty} ka_k |z|^{k-1}.
\]
Thus \(|f'(z) - 1| \leq 1 - \delta\) if
\[
(9) \quad \sum_{k=2}^{\infty} \left(\frac{k}{1-\delta}\right) |a_k| |z|^{k-1} \leq 1.
\]
But Theorem 2.1 confirms that
\[
(10) \quad \sum_{k=2}^{\infty} \left[\frac{k-\beta + \alpha(k-1)}{1-\beta}\right][1 + \lambda(k-1)]C(n,k)|a_k| \leq 1.
\]
Hence (9) will be true if
\[
\frac{k|z|^{k-1}}{1-\delta} \leq \left[\frac{k-\beta + \alpha(k-1)}{1-\beta}\right][1 + \lambda(k-1)]C(n,k).
\]
We obtain
\[
|z| \leq \left\{\frac{(1-\delta)[k-\beta + \alpha(k-1)][1 + \lambda(k-1)]C(n,k)}{k(1-\beta)}\right\}^\frac{1}{k-1}, \quad (k \geq 2)
\]
as required. \( \square \)
Theorem 5.2. Let \( f \in M^\mathcal{A}_n(\alpha, \beta) \). Then \( f \) is starlike of order \( \delta \) in \( |z| < R_2 \), where
\[
R_2 = \inf_{k \geq 2} \left[ \sqrt{(1 - \delta)[k - \beta + \alpha(k-1)][1 + \lambda(k-1)]C(n,k)} \right]^{1/\delta}.
\]
The result is sharp with the extremal function \( f \) given by (5).

Proof. We must show that \( \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta \) for \( |z| < R_2 \). We have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| - \sum_{k=2}^{\infty} (k-1)a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} \frac{(k-1)|a_k||z|^{k-1}}{1 - \sum_{k=2}^{\infty} |a_k||z|^{k-1}} \leq 1 - \delta.
\]
Hence (11) holds true if
\[
\sum_{k=2}^{\infty} (k-1)|a_k||z|^{k-1} \leq (1 - \delta) \left\{ 1 - \sum_{k=2}^{\infty} |a_k||z|^{k-1} \right\} \quad \text{or, equivalently,}
\]
\[
\sum_{k=2}^{\infty} \frac{(k-\delta)}{(1-\delta)} |a_k||z|^{k-1} \leq 1.
\]
Hence, by using (10) and (12) will be true if
\[
\frac{(k-\delta)}{(1-\delta)} |z|^{k-1} \leq \frac{[k - \beta + \alpha(k-1)][1 + \lambda(k-1)]C(n,k)}{1 - \beta}
\]
or
\[
|z| \leq \left\{ \frac{(1 - \delta)[k - \beta + \alpha(k-1)][1 + \lambda(k-1)]C(n,k)}{(k - \delta)(1 - \beta)} \right\}^{\frac{1}{1-\delta}}, \quad (k \geq 2)
\]
which completes the proof. \( \square \)

Theorem 5.3. Let \( f \in M^\mathcal{A}_n(\alpha, \beta) \). Then \( f \) is convex of order \( \delta \) in \( |z| < R_3 \), where
\[
R_3 = \inf_{k \geq 2} \left[ \sqrt{(1 - \delta)[k - \beta + \alpha(k-1)][1 + \lambda(k-1)]C(n,k)} \right]^{1/\delta}.
\]
The result is sharp with the extremal function \( f \) given by (5).

Proof. By using the same technique in the proof of Theorem 5.2, we can show that \( \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta \) for \( |z| \leq R_3 \), with the aid of Theorem 2.1. Thus we have the assertion of Theorem 5.3. \( \square \)
6. Inclusion theorem involving modified Hadamard products

For functions

\[ f_j(z) = z - \sum_{k=2}^{\infty} |a_{k,j}| z^k \quad (j = 1, 2) \]

in the class \( \mathcal{A} \), we define the modified Hadamard product \( f_1 \ast f_2(z) \) of \( f_1(z) \) and \( f_2(z) \) given by \( f_1(z) \ast f_2(z) = z - \sum_{k=2}^{\infty} |a_{k,1}||a_{k,2}| z^k \). We can prove the following.

**Theorem 6.1.** Let the functions \( f_j(z) \) \((j = 1, 2)\) given by (13) be on the class \( M^n_\alpha(\alpha, \beta) \) respectively. Then \( (f_1 \ast f_2)(z) \in M^n_\lambda(\alpha, \xi) \), where

\[ \xi = 1 - \frac{(1 - \beta)^2}{(n + 1)(2 - \beta)(2 - \beta + \alpha)(1 + \lambda) - (1 - \beta)^2}. \]

**Proof.** Employing the technique used earlier by Schild and Silverman [7], we need to find the largest \( \xi \) such that

\[ \sum_{k=2}^{\infty} \frac{[k - \xi + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n,k)}{1 - \xi} |a_{k,1}||a_{k,2}| \leq 1. \]

Since \( f_j(z) \in M^n_\alpha(\alpha, \beta) \) \((j = 1, 2)\), then we have

\[ \sum_{k=2}^{\infty} \frac{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n,k)}{1 - \beta} |a_{k,1}| \leq 1, \]

and

\[ \sum_{k=2}^{\infty} \frac{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n,k)}{1 - \beta} |a_{k,2}| \leq 1, \]

by the Cauchy-Schwartz inequality, we have

\[ \sum_{k=2}^{\infty} \frac{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n,k)}{1 - \beta} \sqrt{|a_{k,1}||a_{k,2}|} \leq 1. \]

Thus it is sufficient to show that

\[ \frac{[k - \xi + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n,k)}{1 - \xi} |a_{k,1}||a_{k,2}| \leq \frac{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n,k)}{1 - \beta} \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \geq 2), \]

that is,

\[ \sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(1 - \xi)[k - \beta + \alpha(k - 1)]}{(1 - \beta)[k - \xi + \alpha(k - 1)]}. \]
Note that
\[
\sqrt{|a_{k,1}|a_{k,2}} \leq \frac{(1 - \beta)}{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k)}.
\]
Consequently, we need only to prove that
\[
\frac{(1 - \beta)}{[k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k)}\leq \frac{(1 - \xi)[k - \beta + \alpha(k - 1)]}{(1 - \beta)[k - \xi + \alpha(k - 1)]} \quad (k \geq 2),
\]
or, equivalently, that
\[
\xi \leq 1 - \frac{(k - 1)(1 + \alpha)(1 - \beta)^2}{[k - \beta + \alpha(k - 1)]^2 [1 + \lambda(k - 1)] C(n, k) - (1 - \beta)^2} \quad (k \geq 2).
\]
Since
\[
A(k) = 1 - \frac{(k - 1)(1 + \alpha)(1 - \beta)^2}{[k - \beta + \alpha(k - 1)]^2 [1 + \lambda(k - 1)] C(n, k) - (1 - \beta)^2} \quad (k \geq 2).
\]
is an increasing function of \(k (k \geq 2)\), letting \(k = 2\) in last equation, we obtain
\[
\xi \leq A(2) = 1 - \frac{(1 + \alpha)(1 - \beta)^2}{[2 - \beta + \alpha]^2 (1 + \lambda(n + 1)) - (1 - \beta)^2}.
\]
Finally, by taking the function given by (7), we can see that the result is sharp. \(\square\)

7. Convolution and Integral Operators

Let \(f(z)\) be defined by (1), and suppose that \(g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k\). Then, the Hadamard product (or convolution) of \(f(z)\) and \(g(z)\) defined here by
\[
f(z) \ast g(z) = (f \ast g)(z) = z - \sum_{k=2}^{\infty} |a_k||b_k| z^k.
\]

**Theorem 7.1.** Let \(f \in M_n^\alpha(\alpha, \beta)\), and \(g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k\) \((0 \leq |b_n| \leq 1)\). Then \(f \ast g \in M_n^\alpha(\alpha, \beta)\).

**Proof.** In view of Theorem 2.1, we have
\[
\sum_{k=2}^{\infty} [k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k)|a_k||b_k| \\ \leq \sum_{k=2}^{\infty} [k - \beta + \alpha(k - 1)] [1 + \lambda(k - 1)] C(n, k)|a_k| \leq (1 - \beta).
\]
\(\square\)
Theorem 7.2. Let $f \in M^n_{\lambda}(\alpha, \beta)$ and let $v$ be real number such that $v > -1$, then the function $F(z) = \frac{v+1}{z^v} \int_0^z t^{v-1} f(t) dt$ also belongs to the class $M^n_{\lambda}(\alpha, \beta)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k,$$

where $A_k = \left( \frac{v+1}{v+k} \right) |a_k|$. Since $v > -1$, than $0 \leq A_k \leq |a_k|$. Which in view of Theorem 2.1, $F \in M^n_{\lambda}(\alpha, \beta)$. □

References