CERTAIN CLASS OF HARMONIC STARLIKE FUNCTIONS
WITH SOME MISSING COEFFICIENTS

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Abstract. In this paper we have introduced a new class $J_H(\alpha, \beta, \gamma)$ of Harmonic Univalent functions in the unit disk $E = \{z; |z| < 1\}$ on the lines of [3] and [4], but with some missing coefficient. We have studied various properties such as coefficient estimates, extreme points, convolution and their related results.

1. Introduction

The class of functions of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic univalent and normalized in the unit disc $E$, is denoted by $S$. The class $K$ of convex functions and class $S^*$ of starlike functions are two widely investigated subclasses of $S$.

A continuous function $f = u + iv$ defined in a domain $D \subseteq \mathbb{C}$ is harmonic in $D$ if $u$ and $v$ are real Harmonic in $D$. In any simply connected sub domain of $D$ we can write,

$$f = h + \overline{g}$$

(1.1)

where $h$ and $g$ are analytic, $h$ is called the analytic and $g$ the coanalytic part of $f$. In this paper we have introduced a new class $J_H(\alpha, \beta, \gamma)$ of functions of the form (1.1) namely $f = h + \overline{g}$ that are Harmonic Univalent and sense preserving

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in the unit disk $E$ with $f(0) = f'(0) - 1 = 0$, where $h$ and $g$ are of the form

$$ h(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1} $$

$J_H$ is the subclass of $J$.

For $0 \leq \alpha < 1$, $J_H$ denotes the subclass of $J$ consisting of harmonic starlike functions of order $\alpha$ satisfying,

$$ \frac{\partial}{\partial \theta} \left( \arg f(re^{i\theta}) \right) \geq \alpha; \quad |z| = r < 1. $$

Clunie and Sheil-Small [3] and Jahangiri [4] studied Harmonic starlike functions of order $\alpha$ and Rosey et al. [6] considered the Goodman-Ronning-Type harmonic univalent functions which satisfies the condition

$$ \Re \left\{ (1 + e^{i\alpha}) \frac{zf'}{f} - e^{i\alpha} \right\} \geq 0. $$

Definition. A function $f \in J_H(\alpha, \beta, \gamma)$ if it satisfies the condition

$$ \Re \left\{ (1 + e^{i\alpha}) \frac{zf'}{f} - \gamma e^{i\alpha} \right\} \geq \beta $$

$0 \leq \alpha < 1, 0 \leq \beta < 1, \frac{1}{2} < \gamma \leq 1$ where

$$ z' = \frac{\partial}{\partial \theta} (z = re^{i\theta}); \quad f'(z) = \frac{\partial}{\partial \theta} f(re^{i\theta}) $$

$\alpha, \beta, \gamma$ and $\theta$ are real.

Let $\overline{J}_H$ denote a subclass of $J(\alpha, \beta, \gamma)$ consisting of functions $f = h + \overline{g}$ such that $h$ and $g$ are of the form

$$ h(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1}, \quad a_{n+1} \geq 0, b_{n+1} \geq 0 $$

2. Coefficient Estimates

**Theorem 1.** Let $f = h + \overline{g}$, where $h$ and $g$ are given by (1.4). Furthermore let

$$ \sum_{n=2}^{\infty} \left\{ \frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{n+1}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1}| \right\} \leq 2 $$

where $a_1 = 1$, $0 \leq \beta < 1$ and $\frac{1}{2} < \gamma \leq 1$. Then $f$ is harmonic univalent in unit disc $E$ and $f \in \overline{J}_H(\alpha, \beta, \gamma)$. 
Proof. We first observe that $f$ is locally univalent and orientation preserving in unit disc $E$. This is because

$$|h'(z)| \geq 1 - \sum_{n=2}^{\infty} (n+1)|a_{n+1}|r^n > 1 - \sum_{n=2}^{\infty} (n+1)|a_{n+1}|$$

$$\geq 1 - \sum_{n=2}^{\infty} \frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}| \geq \sum_{n=2}^{\infty} \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}|$$

$$\geq \sum_{n=1}^{\infty} (n+1)|b_{n+1}| \geq \sum_{n=1}^{\infty} (n+1)|b_{n+1}|r^n \geq g'(z).$$

In order to show that $f$ is univalent in $E$ we show that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Since $E$ is simply connected and convex we have $z(\lambda) = (1-\lambda)z_1 + \lambda z_2 \in E$ if $0 \leq \lambda \leq 1$ and if $z_1, z_2 \in E$ so that $z_1 \neq z_2$. Then we write,

$$f(z_2) - f(z_1) = \int_0^1 [(z_2 - z_1)h'(z(t)) + (z_2 - z_1)g'(z(t))]dt.$$

Dividing by $z_2 - z_1 \neq 0$ and taking the real part we have,

$$\Re \left\{ \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right\} = \int_0^1 \Re \left[ \frac{h'(z(t)) + \frac{(z_2 - z_1)}{(z_2 - z_1)}g'(z(t))}{z_2 - z_1} \right]dt$$

(2.2)

$$> \int_0^1 \Re [h'(z(t)) - |g'(z(t))|]dt$$

on the other hand,

$$\Re (h'(z) - |g'(z)|) \geq \Re h'(z) - \sum_{n=1}^{\infty} (n+1)|b_{n+1}|$$

$$\geq 1 - \sum_{n=2}^{\infty} (n+1)|a_{n+1}| - \sum_{n=1}^{\infty} (n+1)|b_{n+1}|$$

$$\geq 1 - \sum_{n=2}^{\infty} \frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)}|a_{n+1}|$$

$$- \sum_{n=1}^{\infty} \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)}|b_{n+1}|$$

$$\geq 0$$

using (2.1). This along with inequality (2.2) leads to the univalence of $f$. According to the condition (1.2), it suffices to show that (2.1) holds if

$$\Re \left\{ \frac{(1 + e^{i\alpha})(z h'(z) - zg'(z)) - \gamma e^{i\alpha}(h(z) + g(z))}{h(z) + g(z)} \right\} = \Re \frac{A(z)}{B(z)} \geq \beta$$

where $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $0 \leq r < 1$, $\frac{1}{2} < \gamma \leq 1$. 
Let \( A(z) = (1 + e^{i\alpha})(zh'(z) - zg'(z)) - \gamma e^{i\alpha}(h(z) + g(z)) \) and \( B(z) = h(z) + g(z) \). Since \( Re(w) \geq \beta \) if and only if \(|\gamma - \beta + w| \geq |\gamma + \beta - w|\). It is enough to show that

\[
(2.3) \quad |A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)| \geq 0.
\]

Substitute for \( A(z) \) and \( B(z) \) in (2.3) to yield

\[
\begin{align*}
&|(1 - \beta)h(z) + (1 + e^{i\alpha})zh'(z) - \gamma e^{i\alpha}h(z) \nonumber \\
&\quad + (1 - \beta)g(z) - (1 + e^{i\alpha})zg'(z) - \gamma e^{i\alpha}g(z)| \\
&\quad - |(1 + \beta)h(z) - (1 + e^{i\alpha})zh'(z) + \gamma e^{i\alpha}h(z) \\
&\quad + (1 + \beta)g(z) + (1 + e^{i\alpha})zg'(z) + \gamma e^{i\alpha}g(z)| \\
&\quad = |(2 - \beta)z + ze^{i\alpha}(1 - \gamma) - \sum_{n=2}^{\infty} [(2n + 2 - \beta + e^{i\alpha}(n + 1 - \gamma)]a_{n+1}z^{n+1} \\
&\quad \quad - \sum_{n=2}^{\infty} [(n + 2 - \beta + e^{i\alpha}(1 + n + \gamma)]b_{n+1}z^{n+1}| \\
&\quad \quad - \beta z + ze^{i\alpha}(1 - \gamma) + \sum_{n=2}^{\infty} [(n - \beta) + e^{i\alpha}(1 + n - \gamma)]a_{n+1}z^{n+1} \\
&\quad \quad + \sum_{n=1}^{\infty} [(2 + \beta + n) + e^{i\alpha}(1 + n + \gamma)]b_{n+1}z^{n+1}| \\
&\quad \geq (3 - \beta - \gamma)|z| - \sum_{n=2}^{\infty} (3 + 2n - \beta - \gamma)|a_{n+1}|z^{n+1} \\
&\quad \quad - \sum_{n=1}^{\infty} (2n + \beta + 2n + \gamma + 1)|b_{n+1}|z^{n+1} \\
&\quad \quad - (\beta + \gamma - 1)|z| - \sum_{n=2}^{\infty} (2n - \beta - \gamma + 1)|a_{n+1}|z^{n+1} \\
&\quad \quad - \sum_{n=1}^{\infty} (3 + 2n + 2n + \beta + \gamma)|b_{n+1}|z^{n+1} \\
&\quad \geq 2(2 - \beta - \gamma)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{(2n - \beta - 2n + \gamma + 2)}{(2 - \beta - \gamma)}|a_{n+1}|z^{n} \\
&\quad \quad - \sum_{n=1}^{\infty} \frac{(2n + \beta + 2n + \gamma + 2)}{(2 - \beta - \gamma)}|b_{n+1}|z^{n} \right\} \\
&\quad \geq 2(2 - \beta - \gamma)|z| \left\{ 1 \right. \\
&\quad \quad - \left[ \sum_{n=2}^{\infty} \frac{(2n - \beta - 2n + \gamma + 2)}{(2 - \beta - \gamma)}|a_{n+1}| \right. \\
&\quad \quad \left. \left. - \sum_{n=1}^{\infty} \frac{(2n + \beta + 2n + \gamma + 2)}{(2 - \beta - \gamma)}|b_{n+1}| \right. \right\}.
\end{align*}
\]
\[ + \sum_{n=1}^{\infty} \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1}| \} \geq 0. \]

By (2.1), the functions

\[ (2.4) \quad f(z) = z + \sum_{n=2}^{\infty} \frac{2 - \beta - \gamma}{2n - \beta - \gamma + 2} x_{n+1} z^{n+1} + \sum_{n=1}^{\infty} \frac{2 - \beta - \gamma}{2n + \beta + \gamma + 2} y_{n+1} z^{n+1} \]

where

\[ \sum_{n=2}^{\infty} |x_{n+1}| + \sum_{n=1}^{\infty} |y_{n+1}| = 1 \]

shows that the coefficient bound given by (2.1) is sharp. \[ \square \]

The function of the form (2.4) are in \( J_H(\alpha, \beta, \gamma) \) because

\[ \sum_{n=2}^{\infty} \left\{ \frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{n+1}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1}| \right\} \]

\[ = 1 + \sum_{n=2}^{\infty} |x_{n+1}| + \sum_{n=1}^{\infty} |y_{n+1}| = 2 \]

where \( a_1 = 1 \) and some coefficients are missing. The restriction placed in Theorem (1) on the module of the coefficients of \( f \), enables us to conclude for arbitrary rotation of the coefficients of \( f \) that the resulting function would still be harmonic and univalent in \( J_H(\alpha, \beta, \gamma) \). The following theorem establishes that such coefficient bounds cannot be improved.

**Theorem 2.** Let \( f = h + g \), be so that \( h \) and \( g \) are

\[ (2.5) \quad h(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1}; \quad g(z) = \sum_{n=1}^{\infty} b_{n+1} z^{n+1} \]

Then \( f(z) \in J_H(\alpha, \beta, \gamma) \) if and only if

\[ (2.6) \quad \sum_{n=2}^{\infty} \left\{ \frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{n+1}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{n+1}| \right\} \leq 2 \]

where \( a_1 = 1, 0 \leq \beta < 1, \frac{1}{2} < \gamma \leq 1 \) and some coefficients are missing.

**Proof.** The “if” part follows from theorem [1] upon noting that if the analytic part \( h \) and co-analytic part \( g \) of \( f \in J_H \) are of the form (2.5) then \( f \in J_H \).

For the “only if” part, we show that \( f(z) \notin J_H \) if the condition (2.6) does not hold. Note that a necessary and sufficient condition for \( f = h + g \) given by (2.5) to be in \( J_H \) is that

\[ \text{Re} \left\{ (1 + e^{i\alpha}) z f'(z) - \gamma e^{i\alpha} \right\} \geq \beta. \]
This is equivalent to
\[
\text{Re}\left\{ \frac{(1 + e^{i\alpha})(zh'(z) - zg'(z)) - \gamma e^{i\alpha}(h(z) - g(z)) - \beta}{h(z) + g(z)} \right\} = \text{Re}\left\{ \frac{(2 - \beta - \gamma)z - \sum_{n=2}^{\infty} (2n - \beta - \gamma + 2)|a_{n+1}|z^{n+1} - \sum_{n=1}^{\infty} (2n + \beta + \gamma + 2)|b_{n+1}|z^{n+1}}{z - \sum_{n=2}^{\infty} |a_{n+1}|z^{n+1} + \sum_{n=1}^{\infty} |b_{n+1}|z^{n+1}} \right\}.
\]

The above condition must hold for all values of \(z, |z| = r < 1 \geq 0\). Choosing the values of \(z\) along \(+ve\) real axis where \(0 \leq z = r < 1\), we must have

\[
(2 - \beta - \gamma) - \sum_{n=2}^{\infty} (2n - \beta - \gamma + 2)|a_{n+1}|r^{n} - \sum_{n=1}^{\infty} (2n + \beta + \gamma + 2)|b_{n+1}|r^{n}
\]

If the condition (2.6) does not hold then the numerator in (2.7) is negative for \(r\) sufficiently close to 1. Thus, there exists \(z_0 = r_0\) in \((0, 1)\) for which the quotient in (2.7) is negative. This contradicts the required condition for \(f \in J_{H}\) and hence the required result. 

\[\square\]

3. Extreme Points

We obtain the extreme points of the closed convex hulls of \(J_{H}\), denoted by \(CLCHJ_{H}\).

**Theorem 3.** \(f(z) \in CLCHJ_{H}\) if and only if,

\[
f(z) = \sum_{n=2}^{\infty} (x_{n+1}h_{n+1} + y_{n+1}g_{n+1})
\]

where \(h_{1}(z) = z\);

\[
h_{n+1}(z) = z - \frac{(2 - \beta - \gamma)}{(2n - \beta - \gamma + 2)}z^{n+1}; \quad n = 2, 3, 4, \ldots
\]

\[
g_{n+1}(z) = z + \frac{(2 - \beta - \gamma)}{(2n + \beta + \gamma + 2)}z^{n+1}; \quad n = 1, 2, 3, \ldots
\]

\[
\sum_{n=2}^{\infty} (x_{n+1} + y_{n+1}) = 1; \quad x_{n+1} \geq 0 \text{ and } y_{n+1} \geq 0.
\]
In particular, the extreme points of $J_H$, are $\{h_{n+1}\}$ and $\{g_{n+1}\}$.

Proof. For function $f$ of the form (3.1) we have,

$$f(z) = \sum_{n=2}^{\infty} (x_{n+1} h_{n+1} + y_{n+1} g_{n+1})$$

Then

$$f(z) = \sum_{n=2}^{\infty} (x_{n+1} + y_{n+1})z - \sum_{n=2}^{\infty} \frac{(2 - \beta - \gamma)}{(2n - \beta - \gamma + 2)} x_{n+1} z^{n+1} + \sum_{n=1}^{\infty} \frac{(2 - \beta - \gamma)}{(2n + \beta + \gamma + 2)} y_{n+1} z^{n+1}$$

and so $f(z) \in CLCH J_H$.

Conversely, suppose that $f(z) \in CLCH J_H$. Set

$$x_{n+1} = \frac{(2n - \gamma - \beta + 2)}{(2 - \beta - \gamma)} |a_{n+1}|; \quad n = 2, 3, 4, \ldots$$

and

$$y_{n+1} = \frac{(2n + \gamma + \beta + 2)}{(2 - \beta - \gamma)} |b_{n+1}|; \quad n = 1, 2, 3, 4, \ldots$$

Then note that by theorem (2), $0 \leq x_{n+1} \leq 1, n = 2, 3, 4, \ldots$ and $0 \leq y_{n+1} \leq 1, n = 1, 2, 3, \ldots$.

Consequently, we obtain

$$f(z) = \sum_{n=2}^{\infty} (x_{n+1} h_{n+1} + y_{n+1} g_{n+1}).$$

Using Theorem 2 it is easily seen that $J_H$ is convex and closed and so

$$CLCH J_H = J_H.$$
4. Covolution Result

For harmonic functions,

\[ f(z) = z - \sum_{n=2}^{\infty} a_{n+1} z^{n+1} + \sum_{n=1}^{\infty} b_{n+1} \bar{z}^{n+1} \]

\[ G(z) = z - \sum_{n=2}^{\infty} A_{n+1} z^{n+1} + \sum_{n=1}^{\infty} B_{n+1} \bar{z}^{n+1} \]

we define the convolution of \( f \) and \( G \) as,

\[ (f \ast G)(z) = f(z) \ast G(z) \]

\[ = z - \sum_{n=2}^{\infty} a_{n+1} A_{n+1} z^{n+1} + \sum_{n=1}^{\infty} b_{n+1} B_{n+1} \bar{z}^{n+1} \]

**Theorem 4.** For \( 0 \leq \beta < 1 \) let \( f(z) \in \overline{J}_H(\alpha, \beta, \gamma) \) and \( G(z) \in \overline{J}_H(\alpha, \beta, \gamma) \).

Then

\[ f(z) \ast G(z) \in \overline{J}_H(\alpha, \beta, \gamma). \]

**Proof.** Let

\[ f(z) = z - \sum_{n=2}^{\infty} |a_{n+1}| z^{n+1} + \sum_{n=1}^{\infty} |b_{n+1}| \bar{z}^{n+1} \]

be in \( \overline{J}_H(\alpha, \beta, \gamma) \)

and

\[ G(z) = z - \sum_{n=2}^{\infty} |A_{n+1}| z^{n+1} + \sum_{n=1}^{\infty} |B_{n+1}| \bar{z}^{n+1} \]

be in \( \overline{J}_H(\alpha, \beta, \gamma) \)

Obviously, the coefficients of \( f \) and \( G \) must satisfy condition similar to the inequality (2.6). So for the coefficients of \( f \ast G \) we can write

\[
\sum_{n=2}^{\infty} \left[ \frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)} |a_{n+1}A_{n+1}| + \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)} |b_{n+1}B_{n+1}| \right] \\
\leq \sum_{n=2}^{\infty} \left[ \frac{(2n-\beta-\gamma+2)}{(2-\beta-\gamma)} |a_{n+1}| + \frac{(2n+\beta+\gamma+2)}{(2-\beta-\gamma)} |b_{n+1}| \right]
\]

The right side of this inequality is bounded by 2 because \( f \in \overline{J}_H(\alpha, \beta, \gamma) \). By the same token, we then conclude that

\[ f(z) \ast G(z) \in \overline{J}_H(\alpha, \beta, \gamma). \]

Finally, we show that \( f \in \overline{J}_H(\alpha, \beta, \gamma) \), is closed under convex combination of its members.

**Theorem 5.** The family \( \overline{J}_H(\alpha, \beta, \gamma) \) is closed under convex combination.
Proof. For \( i = 1, 2, 3 \ldots \) let \( f_i \in \mathcal{J}_H(\alpha, \beta, \gamma) \) where \( f_i \) is given by,
\[
f_i(z) = z - \sum_{n=2}^{\infty} |a_{i(n+1)}| z^{n+1} + \sum_{n=1}^{\infty} |b_{i(n+1)}| z^{n+1}
\]

Then by (2.6),
\[
\sum_{n=2}^{\infty} \left[ \frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} |a_{i(n+1)}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} |b_{i(n+1)}| \right] \leq 2.
\]

For \( \sum_{i=1}^{\infty} t_i = 1; 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as,
\[
\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left[ \sum_{i=1}^{\infty} t_i |a_{i(n+1)}| \right] z^{n+1} + \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{\infty} t_i |b_{i(n+1)}| \right] z^{n+1}.
\]

Then by (4.2)
\[
\sum_{n=2}^{\infty} t_i \left[ \frac{(2n - \beta - \gamma + 2)}{(2 - \beta - \gamma)} \sum_{i=1}^{\infty} t_i |a_{i(n+1)}| + \frac{(2n + \beta + \gamma + 2)}{(2 - \beta - \gamma)} \sum_{i=1}^{\infty} t_i |b_{i(n+1)}| \right]
\]
\[
\leq 2 \sum_{i=1}^{\infty} t_i = 2.
\]

This is the condition required by (2.6) and so,
\[
\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{J}_H(\alpha, \beta, \gamma).
\]

\[\square\]

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