SPECIAL REPRESENTATIONS OF SOME SIMPLE GROUPS WITH MINIMAL DEGREES

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Abstract. If $F$ is a subfield of $C$, then a square matrix over $F$ with non-negative integral trace is called a quasi-permutation matrix over $F$. For a finite group $G$, let $q(G)$ and $c(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational and the complex numbers, respectively. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of $G$. In this paper $q(G)$, $c(G)$ and $r(G)$ are calculated for Suzuki group and untwisted group of type $B_2$ with parameter $2^{2n+1}$.

1. Introduction

In [12] Wong defined a quasi-permutation group of degree $n$, to be a finite group $G$ of automorphisms of an $n$-dimensional complex vector space such that every element of $G$ has non-negative integral trace. The terminology drives from the fact that if $G$ is a finite group of permutations of a set $\Omega$ of size $n$, and we think of $G$ as acting on the complex vector space with basis $\Omega$, then the trace of an element $g \in G$ is equal to the number of points of $\Omega$ fixed by $g$. Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [2] Hartley with their colleague investigated further the analogy between permutation groups and quasi-permutation groups by studying the relation between the minimal degree of a faithful permutation representation of a given finite group $G$ and the minimal degree of a faithful quasi-permutation representation. They also worked over the rational field and found some interesting results. We shall often prefer to work over the rational field rather than the complex field.

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By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. Thus every permutation matrix over $C$ is a quasi-permutation matrix. For a given finite group $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $Q$, and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices.

By a rational valued character we mean a character $\chi$ corresponding to a complex representation of $G$ such that $\chi(g) \in Q$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of $G$ is then simply a complex representation of $G$ whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from $G$ to $GL(n, Q)$ a rational representation of $G$ and its corresponding character will be called a rational character of $G$. Let $r(G)$ denote the minimal degree of a faithful rational valued character of $G$. It is easy to see that for a finite group $G$ the following inequalities hold

$$r(G) < c(G) \leq q(G).$$

It is easy to see that if $G$ is a symmetric group of degree 6, then $r(G) = 5$ and $c(G) = q(G) = 6$. If $G$ is the quaternion group of order 8, then $r(G) = 2, c(G) = 4$ and $q(G) = 8$. Our principal aim in this paper is to investigate these quantities and inequalities further.

Finding the above quantities have been carried out in some papers, for example in [6, 5, 4] we found these for the groups $GL(2, q)$, $SU(3, q^2)$, $PSU(3, q^2)$, $SL(3, q)$ and $PSL(3, q)$.

In this paper we will apply the algorithms in [1] for the Suzuki group and untwisted group of type $B_2$ with parameter $2^{2n+1}$.

2. Background

Let $G$ be a finite group and $\chi$ be an irreducible complex character of $G$. Let $m_Q(\chi)$ denote the Schur index of $\chi$ over $Q$. Let $\Gamma(\chi)$ be the Galois group of $Q(\chi)$ over $Q$. It is known that

$$\sum_{\alpha \in \Gamma(\chi)} m_Q(\chi)\chi^\alpha$$

is a character of an irreducible $QG$-module ([9, Corollary 10.2 (b)]). So by knowing the character table of a group and the Schur indices of each of the irreducible characters of $G$, we can find the irreducible rational characters of $G$.

We can see all the following statements in [1].

**Definition 1.** Let $\chi$ be a character of $G$ such that, for all $g \in G$, $\chi(g) \in Q$ and $\chi(g) \geq 0$. Then we say that $\chi$ is a non-negative rational valued character.
**Definition 2.** Let $G$ be a finite group. Let $\chi$ be an irreducible complex character of $G$. Then we define

(1) $d(\chi) = |\Gamma(\chi)|\chi(1)$

(2) $m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ \min\{\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha(g) : g \in G\} & \text{otherwise} \end{cases}$

(3) $c(\chi) = \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha + m(\chi)1_G$.

**Lemma 1.** Let $\chi$ be a character of $G$. Then $\text{Ker } \chi = \text{Ker } \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$. Moreover $\chi$ is faithful if and only if $\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$ is faithful.

**Lemma 2.** Let $\chi \in \text{Irr}(G)$, then $\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$ is a rational valued character of $G$. Moreover $c(\chi)$ is a non-negative rational valued character of $G$ and $c(\chi)(1) = d(\chi) + m(\chi)$.

Now according to [1, Corollary 3.11] and above statements the following Corollary is useful for calculation of $r(G)$, $c(G)$ and $q(G)$.

**Corollary 1.** Let $G$ be a finite group with a unique minimal normal subgroup. Then

(1) $r(G) = \min\{d(\chi) : \chi$ is a faithful irreducible complex character of $G\}$

(2) $c(G) = \min\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$

(3) $q(G) = \min\{mQ(c(\chi)(1)) : \chi$ is a faithful irreducible complex character of $G\}$.

**Lemma 3.** Let $\chi \in \text{Irr}(G)$ $\chi \neq 1_G$. Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

**Lemma 4.** Let $\chi \in \text{Irr}(G)$. Then

(1) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

(2) $c(\chi)(1) \leq 2d(\chi)$. Equality occurs if and only if $Z(\chi)/\ker \chi$ is of even order.

**Lemma 5.** Let $G$ be a finite group. If the Schur index of each non-principal irreducible character is equal to $m$, then $q(G) = mc(G)$.

3. Calculation of $q(G), c(G)$ and $r(G)$ for the group $G = B_2(q)$

The group $G = B_2(q)$ is of order $\frac{q^4(q^4-1)(q^2-1)}{(2,q-1)}$ and if the characteristic of $K$ is two, the Lie algebras of type $B_n$ and of type $C_n$ are isomorphic. The complex character table of $B_2(q)$ is given in [7] as in Table 1.
Table 1. Character table of $B_2(q)$

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_{31}$</th>
<th>$A_{32}$</th>
<th>$A_{41}$</th>
<th>$A_{42}$</th>
<th>$B_1(i,j)$</th>
<th>$B_2(i)$</th>
<th>$B_3(i,j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>$q(q+1)^2/2$</td>
<td>$q(q+1)/2$</td>
<td>$q/2$</td>
<td>$q/2$</td>
<td>$-q/2$</td>
<td>$2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>$q^4$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>$q(q-1)^2/2$</td>
<td>$-q(q-1)/2$</td>
<td>$-q(q-1)/2$</td>
<td>$q/2$</td>
<td>$q/2$</td>
<td>$-q/2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_1(k,l)$</td>
<td>$(q+1)^2(q^2+1)$</td>
<td>$(q+1)^2$</td>
<td>$(q+1)^2$</td>
<td>$2q+1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\alpha_{ik}\alpha_{jl} + \alpha_{il}\alpha_{jk}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_4(k,l)$</td>
<td>$(q-1)^2(q^2+1)$</td>
<td>$(q-1)^2$</td>
<td>$(q-1)^2$</td>
<td>$-(2q-1)$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_k$</td>
<td>$(q^2-1)^2$</td>
<td>$-(q^2-1)^2$</td>
<td>$-(q^2-1)$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
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<td>$0$</td>
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</tbody>
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<table>
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<tr>
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<th>$C_2(i)$</th>
<th>$C_3(i)$</th>
<th>$C_4(i)$</th>
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</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
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<td>$q+1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\theta_4$</td>
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<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
</tr>
<tr>
<td>$\theta_5$</td>
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<td>$0$</td>
<td>$0$</td>
<td>$q-1$</td>
<td>$q-1$</td>
</tr>
<tr>
<td>$\chi_1(k,l)$</td>
<td>$0$</td>
<td>$(q+1)(\alpha_{ik} + \alpha_{il})$</td>
<td>$(q+1)\alpha_{ik}\alpha_{il}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_4(k,l)$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-(q-1)(\beta_{ik} + \beta_{il})$</td>
<td>$-(q-1)\beta_{ik}\beta_{il}$</td>
</tr>
<tr>
<td>$\chi_k$</td>
<td>$\tau^{ik} + \tau^{-ik} + \tau^{ikq} + \tau^{-ikq}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$B_4(i,j)$</th>
<th>$D_1(i)$</th>
<th>$D_2(i)$</th>
<th>$D_3(i)$</th>
<th>$D_4(i)$</th>
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</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_1(k,l)$</td>
<td>$0$</td>
<td>$\alpha_{ik} + \alpha_{il}$</td>
<td>$\alpha_{ik}\alpha_{il}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_4(k,l)$</td>
<td>$\beta_{ik}\beta_{jl} + \beta_{il}\beta_{jk}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\beta_{ik} + \beta_{il}$</td>
<td>$\beta_{ik}\beta_{il}$</td>
</tr>
<tr>
<td>$\chi_k$</td>
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<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
by Definition 2.2 and Lemma 2.4 we have

Theorem 2.

Let $\theta$ be a character and calculate $r(G)$, $c(G)$ for $G = B_2(2)$.

Proof. We know that $G = B_2(2)$ and by the Atlas of finite groups [6], it is easy to see that

$$r(B_2(2)) = 5, \quad c(B_2(2)) = 6.$$  

An overall picture is provided by the Table 2

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$d(\chi)$</th>
<th>$c(\chi)(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>$q(q+1)^2$</td>
<td>$\frac{q^2+2q+2}{2}$</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>$q^4$</td>
<td>$q(q^4+1)$</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>$q(q-1)^2$</td>
<td>$q^3(q-1)$</td>
</tr>
<tr>
<td>$\chi_{1}(k,l)$</td>
<td>$(q+1)^2(q^2+1)$</td>
<td>$(q+1)^2(q^2+1)+1$</td>
</tr>
<tr>
<td>$\chi_{4}(k,l)$</td>
<td>$(q-1)^2(q^2+1)$</td>
<td>$q^2(q^2-2q+2)$</td>
</tr>
<tr>
<td>$\chi_{5}(k)$</td>
<td>$(q^2-1)^2$</td>
<td>$q^2(q^2-1)$</td>
</tr>
</tbody>
</table>

Theorem 1. Let $G = B_2(2)$, then

$$r(B_2(2)) = 5, \quad c(B_2(2)) = 6.$$  

Proof. We know that $G = B_2(2)$ and by the Atlas of finite groups [6], it is easy to see that

$$r(B_2(2)) = 5, \quad c(B_2(2)) = 6.$$  

Theorem 2. Let $G = B_2(q)$, $q \neq 2$, then

1. $r(G) = \frac{q(q-1)^2}{2}$
2. $c(G) = \frac{q^2(q-1)^2}{2}$

Proof. The group $G = B_2(q)$, $q \neq 2$ is simple so their non-trivial irreducible characters are faithful and therefore we need to look at each faithful irreducible character $\chi$ and calculate $d(\chi), c(\chi)(1)$.

By the Table 1, we know that $\theta_1, \theta_4, \theta_5$ are rational valued characters, so by Definition 2.2 and Lemma 2.4 we have

$$d(\theta_1) = |\Gamma(\theta_1)|\theta_1(1) = \frac{q^2+2q+2}{2}$$
$$m(\theta_1) = -\frac{q}{2}$$
$$d(\theta_4) = |\Gamma(\theta_4)|\theta_4(1) = q^4$$
$$m(\theta_4) = -q$$
$$d(\theta_5) = |\Gamma(\theta_5)|\theta_5(1) = \frac{q^2(q-1)}{2}$$
$$m(\theta_5) = -\frac{q^2(q-1)}{2}$$

$$c(\theta_5)(1) = \frac{q^2(q-1)}{2}$$

For other characters by Lemmas 2.6, 2.7 we have

$$d(\chi_{1}(k,l)) = |\Gamma(\chi_{1}(k,l))|\chi_{1}(k,l)(1) \geq (q+1)^2(q^2+1)$$
and $m(\chi_{1}(k,l)) \geq 1$ and so $c(\chi_{1}(k,l))(1) \geq (q+1)^2(q^2+1)+1.$

$$d(\chi_{4}(k,l)) \geq (q-1)^2(q^2+1)$$
and $m(\chi_{4}(k,l)) \geq 2q-1$ and so

$$c(\chi_{4}(k,l))(1) \geq q^2(q^2-2q+2).$$

$$d(\chi_{5}(k)) \geq (q^2-1)^2$$
and $m(\chi_{5}(k)) \geq q^2-1$ and so $c(\chi_{5}(k))(1) \geq q^2(q^2-1).$

An overall picture is provided by the Table 2.
Now by Corollary 2.5 and above table we obtain

\[ \min \{ d(\chi) : \chi \text{ is a faithful irreducible complex character of } G \} = \frac{q(q-1)^2}{2} \]

and

\[ \min \{ c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G \} = \frac{q^2(q-1)}{2}. \]

\[ \square \]

4. Quasi-permutation representations of the group \( Sz(q) \)

A group \( G \) is called a \((ZT)\)-group if:

1. \( G \) is a doubly transitive group on \( 1 + N \) symbols,
2. the identity is the only element which leaves three distinct symbols invariant,
3. \( G \) contains no normal subgroup of order \( 1 + N \), and
4. \( N \) is even.

There is a unique \((ZT)\)-group of order \( q^2(q-1)(q^2+1) \) for any odd power \( q \) of \( 2 \) (see \([11, \text{Theorem 8}]\)). This group will be denoted here as \( Sz(q) \) and called a Suzuki group. The Suzuki groups are simple for all \( q > 2 \).

By \([10]\) the Suzuki group \( G(q) \) is isomorphic to a subgroup of \( SP_4(F_q) \) consisting of points left fixed by an involutive mapping of \( SP_4(F_q) \) onto itself.

Now we shall identify \( SP_4(K)^\sigma \) with the Suzuki group \( G(q) \), where \( SP_4(K)^\sigma \) is the set composed of all \( x \in SP_4(K) \) such that \( x^\sigma = x \).

Let \( K = F_q, q = 2^{2n+1} (n \geq 1) \) and let \( \theta \) be an automorphism of \( K \) defined by \( \alpha \to \alpha^{2^n}, \alpha \in K \). It is easy to see that \( \theta \) generates the Galois group of \( K \) over the prime field. Our purpose is to define an involutive mapping \( \sigma \) (which will not be an automorphism) of \( SP_4(K) \) onto itself by making use of \( \varphi \) and \( \theta \) so that the Suzuki group \( G(q) \) is isomorphic to the subgroup \( SP_4(K)^\sigma \) of \( SP_4(K) \) consisting of matrices left fixed by \( \sigma \).

Using Suzuki’s notation, \( G(q) \) is generated by \( S(\alpha, \beta), M(\xi) \) and \( T \):

\[ S(\alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha^\theta & 1 & 0 & 0 \\ \beta & \alpha & 1 & 0 \\ q(\alpha, \beta) p(\alpha, \beta) \alpha^\theta & 1 \end{pmatrix}, \]

\[ M(\xi) = \operatorname{diag}(\xi^\theta, \xi^{1-\theta}, \xi^{\theta-1}, \xi^{-\theta}), \]

\[ T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]
Define a matrix \( P \) by setting:
\[
P = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Then, one can easily verify that
\[
PS(\alpha, \beta)P^{-1} = R(\alpha, \beta)^{-1}, \quad PM(\xi)P^{-1} = h(\xi^\theta), \quad PTP^{-1} = J.
\]

Thus \( x \to PxP^{-1} \) gives an isomorphism \( G(q) \cong SP_4(K)^\sigma \). So Suzuki group is a simple group of order \( q^2(q - 1)(q^2 + 1) \).

Remark 1. The involution \( \sigma: SP_4(K) \to SP_4(K) \) can not be an automorphism. For, if \( \sigma \) is so, then \( \sigma \) can be expressed as
\[
x^\sigma = Ax^\omega A^{-1},
\]
with \( A \in GL_4(K) \) and an automorphism \( \omega \) of \( K \). Put \( x = x_a(t) = I + tX_a \).
Then \( x^\sigma = x_b(t^{2\theta}) = I + t^{2\theta}X_b = I + t^\omega AX_aA^{-1} \). If we take \( t = 1 \), then \( X_b = AX_aA^{-1} \). But this is absurd since \( X_a = E_{12} - E_{43} \) is of rank 2 and \( X_b = E_{24} \) is of rank 1.

The character table of \( Sz(q) \) is computed in [11], is as follows:
Table 3. Character table of \( Sz(q) \)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>( \sigma_0 )</th>
<th>( \rho_0 )</th>
<th>( \rho_0^{-1} )</th>
<th>( \pi_0^1 )</th>
<th>( \pi_1^1 )</th>
<th>( \pi_2^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi )</td>
<td>( \theta(q - 1) )</td>
<td>( \theta )</td>
<td>( \theta \sqrt{-1} )</td>
<td>( -\theta \sqrt{-1} )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>( \theta(q - 1) )</td>
<td>( -\theta )</td>
<td>( -\theta \sqrt{-1} )</td>
<td>( \theta \sqrt{-1} )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \psi_i )</td>
<td>( q^2 + 1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \varepsilon_j^i(\pi_0^1) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mu_j )</td>
<td>( (q - 2\theta + 1)(q - 1) )</td>
<td>( 2\theta - 1 )</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>( -\varepsilon_j^i(\pi_1^1) )</td>
<td>0</td>
</tr>
<tr>
<td>( \varphi_k )</td>
<td>( (q + 2\theta + 1)(q - 1) )</td>
<td>( -2\theta - 1 )</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>( -\varepsilon_j^k(\pi_2^1) )</td>
</tr>
</tbody>
</table>
Where $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$ are primitive $q - 1$, $q + 2\theta + 1$, $q - 2\theta + 1$-th root of $1$, respectively.

In this table $q = 2\theta^2$ and the $\varepsilon_i^j$ are defined as follows:

$$
\varepsilon_0^i(\xi_0^j) = \varepsilon_0^{ij} + \varepsilon_0^{-ij} \text{ for } i = 1, 2, \ldots, q/2 - 1,
$$

where $\xi_0$ is a generator of cyclic group of order $q - 1$.

$$
\varepsilon_1^i(\xi_1^j) = \varepsilon_1^{ik} + \varepsilon_1^{-ik} + \varepsilon_1^{-ikq} \text{ for } i = 1, 2, \ldots, q + 2\theta
$$

where $\xi_1$ is a generator of cyclic group of order $q + 2\theta + 1$.

$$
\varepsilon_2^i(\xi_2^j) = \varepsilon_2^{ik} + \varepsilon_2^{-ik} + \varepsilon_2^{-ikq} \text{ for } i = 1, 2, \ldots, q + 2\theta
$$

where $\xi_2$ is a generator of cyclic group of order $q - 2\theta + 1$.

**Lemma 6.** Let $G = Sz(q)$, $q = 2^{2n+1}$, then all characters of $G$ have Schur index 1.

**Proof.** See [8, Theorem 9].

**Theorem 3.** Let $G = Sz(q)$, $q = 2^{2n+1}$, then $r(G) = 2\theta(q - 1)$, $c(G) = q(G) = 2\theta q$, where $\theta = 2^n$ and $q = 2\theta^2$.

**Proof.** Let $G = Sz(q)$, $q = 2^{2n+1}$, by Lemma 4.1 the Schur index of every irreducible character is 1, therefore $c(G) = q(G)$. The groups $G = Sz(q)$ is simple, so their non-trivial irreducible characters are faithful and therefore we need to look at each faithful irreducible character $\vartheta$ say and calculate $d(\vartheta), c(\vartheta)(1)$.

By Table 3 we know $\chi$ is a rational valued character, so by Definition 2.2 and Lemma 2.4 we have:

$$
d(\chi) = |\Gamma(\chi)|\chi(1) = q^2,
$$

and $m(\chi) = 1$, and so $c(\chi)(1) = q^2 + 1$.

For the character $\zeta$ we have $|\Gamma(\zeta)| = 2$ and therefore:

$$
d(\zeta) = |\Gamma(\zeta)|\zeta(1) = 2\theta(q - 1),
$$

and $m(\zeta) = 2\theta$, and so $c(\zeta)(1) = 2\theta q$.

In this way, by Lemmas 2.6, 2.7 we have

$$
d(\psi_i) \geq q^2 + 1
$$

and $c(\psi_i) \geq q^2 + 2$,

$$
d(\mu_j) \geq (q - 2\theta + 1)(q - 1)
$$

and $c(\mu_j) \geq q^2 - 2\theta q + 2\theta$, $d(\varphi_k) \geq (q + 2\theta + 1)(q - 1)$ and $c(\varphi_k) \geq q(q + 2\theta)$.

The values are set out in the following table:

<table>
<thead>
<tr>
<th>Character</th>
<th>$d(\chi)$</th>
<th>$c(\chi)(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_i$</td>
<td>$q^2 + 1$</td>
<td>$q^2 + 2$</td>
</tr>
<tr>
<td>$\mu_j$</td>
<td>$q^2 - 2\theta q + 2\theta$</td>
<td>$q\theta$</td>
</tr>
<tr>
<td>$\varphi_k$</td>
<td>$q(q + 2\theta)$</td>
<td>$q\theta$</td>
</tr>
</tbody>
</table>

By observing the Corollary 2.5 and Table 4 we have:

$$
\min\{d(\chi) : \chi \text{ is a faithful irreducible complex character of } G\} = 2\theta(q - 1)
$$

and

$$
\min\{c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G\} = 2\theta q.
$$
Table 4

<table>
<thead>
<tr>
<th>d(ϑ)</th>
<th>c(ϑ)(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ϑd(ϑ)</td>
<td>ϑ(ϑ+1)</td>
</tr>
<tr>
<td>χ + q^2</td>
<td>q^2 + 1</td>
</tr>
<tr>
<td>ζ + 2θ(q-1)</td>
<td>2θq</td>
</tr>
<tr>
<td>ψi ≥ q^2 + 1</td>
<td>q^2 + 1</td>
</tr>
<tr>
<td>µj ≥ (q−2θ+1)(q−1)</td>
<td>≥ q^2−2θq+2θ</td>
</tr>
<tr>
<td>φk ≥ (q + 2θ)(q−1)</td>
<td>≥ q(q + 2θ)</td>
</tr>
</tbody>
</table>

Hence r(G) = 2θ(q−1), c(G) = q(G) = 2θq.

References


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