ON A CLASS OF LIE $p$-ALGEBRAS

CAMELIA CIOBANU

Abstract. In this paper we study the finite dimensional Lie $p$-algebras, \( L \) splitting on its abelian $p$-socle, the sum of its minimal abelian $p$-ideals. In addition, some properties of the Frattini $p$-subalgebra of \( L \) are pointed out.

1. Introduction

In this section, we recall some notions and properties necessary in the paper.

Definition 1.1. A Lie $p$-algebra is a Lie algebra \( L \) with a $p$-map \( a \rightarrow a^p \), such that:

\[
\begin{align*}
(\alpha x)^p &= \alpha^p x^p, \text{ for all } \alpha \in \mathbb{K}, x \in L, \\
x(ady)^p &= x(ady)^p, \text{ for all, } x, y \in L, \\
(x + y)^p &= x^p + y^p + \sum_{i=1}^{p-1} s_i(x, y) \text{ for all } x, y \in L,
\end{align*}
\]

where \( s_i(x, y) \) is the coefficient of \( X^{i-1} \) in the expansion of \( x(ad(Xu + y))^{p-1} \).

A subalgebra (respectively, ideal) of \( L \) is $p$-subalgebra (respectively, $p$-ideal) if it is closed under the $p$-map.

The notions of maximal $p$-subalgebra respectively maximal $p$-ideal of \( L \) are defined as usual. The intersection of $p$-subalgebras (respectively $p$-ideals) is a $p$-subalgebra (respectively a $p$-ideal) of \( L \).

We denote by \( \Phi_p(L) \) the $p$-subalgebra of \( L \) obtained by intersecting all maximal $p$-subalgebras of \( L \) and we call it the Frattini $p$-subalgebra of \( L \).

The largest $p$-ideal of \( L \) included into \( \Phi_p(L) \) is called the Frattini $p$-ideal and is denoted by \( F_p(L) \).

2000 Mathematics Subject Classification. 17B60, 17B66, 17B20.

Key words and phrases. Finite dimensional Lie $p$-algebra, $p$-ideal, $p$-subalgebra.

This paper was supported by the GAR 12/2007 - Contract 103/2008.
These are the corresponding notions to the Frattini subalgebra $\Phi(\mathcal{L})$ and the Frattini ideal $\mathcal{F}(\mathcal{L})$ for a Lie algebra.

We shall use the following notations:

- $[x, y]$ is the product of $x, y$ in $\mathcal{L}$;
- $\mathcal{L}^{(1)}$ the derived algebra of $\mathcal{L}$;
- $\mathcal{L}^{(n)} = \left(\mathcal{L}^{(n-1)}\right)^{(1)}$, for all $n \geq 2$;
- $\mathcal{A}$ is the subalgebra generated by the subset $\mathcal{A}$ of $\mathcal{L}$;
- $\mathcal{A}_p = \{x^p \mid x \in \mathcal{A}, p \in \mathbb{N}\}$, where $x^p = x^{(p)}$;
- $\mathcal{A}_p^n = \left(\mathcal{A}_p^{n-1}\right)^p$;
- $\mathcal{L}_1 = \bigcap_{i=1}^{\infty} \mathcal{L}^p$;
- $\mathcal{L}_0 = \{x \in \mathcal{L} \mid x^p = 0 \text{ for some } n\}$;
- $Z(\mathcal{L})$ is the center of $\mathcal{L}$;
- $\mathcal{N}(\mathcal{L})$ is the nilradical of $\mathcal{L}$.

Note that, if $\mathcal{L}$ is a $p$-algebra (finite dimensional), then $Z(\mathcal{L})$ is closed as $p$-ideal of $\mathcal{L}$.

2. Lie $p$-algebras which are $\mathcal{F}_p$-free

In [8], Stitzinger has proved the following

**Proposition 2.1.** If $\mathcal{L}$ is a finite dimensional Lie algebra over a field $\mathbb{K}$, then

$$\mathcal{L}^{(1)} \cap Z(\mathcal{L}) \subseteq \mathcal{F}(\mathcal{L}).$$

We may prove an analogue of this proposition for a Lie $p$-algebra.

**Lemma 2.2.** If $\mathcal{L}$ is a finite dimensional Lie $p$-algebra over a field $\mathbb{K}$, then we have

$$\left(\mathcal{L}^{(1)}\right)_p \cap Z(\mathcal{L}) \subseteq \mathcal{F}_p(\mathcal{L}).$$

**Proof.** Let $\mathcal{M}$ be a maximal $p$-subalgebra of $\mathcal{L}$ and suppose that $Z(\mathcal{L}) \nsubseteq \mathcal{M}$. Then $\mathcal{L} = \mathcal{M} + Z(\mathcal{L})$, so $\mathcal{L}^{(1)} = \mathcal{M}^{(1)} \subseteq \mathcal{M}$ and hence

$$\left(\mathcal{L}^{(1)}\right)_p \subseteq \left(\mathcal{M}\right)_p \subseteq \mathcal{M}.$$

The *abelian socle* $\text{Sa}(\mathcal{L})$ is the sum of all minimal ideals of $\mathcal{L}$.

We may define the abelian $p$-socle of the finite dimensional Lie $p$-algebra $\mathcal{L}$ as being the sum of all minimal abelian $p$-ideals of $\mathcal{L}$ and we denote it by $\text{Sap}(\mathcal{L})$.

The abelian socle (respectively, the abelian $p$-socle) of a finite dimensional Lie $(p)$-algebra is an ideal (a $p$-ideal) of $\mathcal{L}$, as one can show easily.
Definition 2.3. Let $\mathcal{L}$ be a finite dimensional Lie $p$-algebra and $I$ be a $p$-ideal of $\mathcal{L}$. We say that $\mathcal{L}$ $p$-splits over $I$ if there exists a $p$-subalgebra $B$ of $\mathcal{L}$ such that $\mathcal{L} = I + B$.

$B$ is called a $p$-complement of the $p$-ideal $I$.

Theorem 2.4. Let $\mathcal{L}$ be a finite dimensional Lie $p$-algebra such that $\mathcal{L}^{(1)} \neq 0$ and $\mathcal{L}^{(1)}$ is nilpotent. Then the following statements are equivalent:

(i) $\mathcal{F}_p(\mathcal{L}) = 0$.
(ii) $\text{Sap}(\mathcal{L}) = \mathcal{N}(\mathcal{L})$, and $\mathcal{L}$ $p$-splits over $\mathcal{N}(\mathcal{L})$.
(iii) $\mathcal{L}^{(1)}$ is abelian, $(\mathcal{L}^{(1)})^p = 0$, $\mathcal{L}$ $p$-splits over $\mathcal{L}^{(1)} \oplus Z(\mathcal{L})$, and

$$\text{Sap}(\mathcal{L}) = \mathcal{L}^{(1)} \oplus Z(\mathcal{L}).$$

In the same hypotheses, the Cartan subalgebra of $\mathcal{L}$ are exactly those subalgebras which have $\mathcal{L}^{(1)}$ as a $p$-complement.

Proof. (i) $\Rightarrow$ (ii): These implications are immediate from Theorems 4.1, 4.2 of [5].

(iii) $\Rightarrow$ (i): This also follows from Theorem 4.1 of [5].

(i) $\Rightarrow$ (iii): Suppose that $\mathcal{F}_p(\mathcal{L}) = 0$. Then $\mathcal{F}(\mathcal{L}) = 0$, and $\mathcal{L}^{(1)}$ is abelian. Now $(\mathcal{L}^{(1)})^p \subseteq Z(\mathcal{L})$ by Lemma 2.1 [6], and so

$$(\mathcal{L}^{(1)})^p \subseteq (\mathcal{L}^{(1)})^p \cap Z(\mathcal{L}) \subseteq (\mathcal{L}^{(1)})^p \cap Z(\mathcal{L}) \subseteq \mathcal{F}_p(\mathcal{L}) = 0,$$

by Lemma 2.2. Clearly $\mathcal{L}^{(1)} \oplus Z(\mathcal{L}) \subseteq \mathcal{N}(\mathcal{L}) = \text{Sap}(\mathcal{L})$.

Now let $\overline{m}$ be a minimal (and hence abelian) $p$-ideal of $\mathcal{L}$. Then $[\mathcal{L}, \overline{m}] = m$ is an ideal of $\mathcal{L}$ and

$$[\mathcal{L}, \overline{m}]^p \subseteq (\mathcal{L}^{(1)})^p \cap m^p \subseteq (\mathcal{L}^{(1)})^p \cap Z(\mathcal{L}) = 0$$

by Lemma 2.1 of [6] and by Lemma 2.2. Hence $[\mathcal{L}, \overline{m}]$ is $p$-closed, therefore $[\mathcal{L}, \overline{m}] = m$ or $[\mathcal{L}, \overline{m}] = 0$.

The former implies that $\overline{m} \subseteq \mathcal{L}^{(1)}$, and the latter that $\overline{m} \subseteq Z(\mathcal{L})$ hence $\text{Sap}(\mathcal{L}) = \mathcal{L}^{(1)} \oplus Z(\mathcal{L})$ and (iii) follows.

The last part of the theorem precises that the Cartan subalgebras are exactly those subalgebras having $\mathcal{L}^{(1)}$ as a $p$-complement. This follows from Proposition 1 of [8], or from Theorem 4.4.1.1. of [10] and from the fact that Cartan subalgebras are $p$-closed. □

Corollary 2.5. If $\mathcal{L}$ is a finite dimensional Lie $p$-algebra over $\mathbb{K}$ with $\mathcal{L}^{(1)}$ nilpotent, and nonzero $\mathcal{F}_p(\mathcal{L}) = 0$ and $\mathbb{K}$ is perfect, then the maximal toral subalgebras are precisely those having as $p$-complement $\mathcal{L}^{(1)} \oplus Z(\mathcal{L})$.

Proof. Take $\mathcal{L} = (\mathcal{L}^{(1)} \oplus Z(\mathcal{L})) + B$, with $Bp$-closed and $B^{(1)} = 0$ and let $B = B_0 \oplus B_1$ be the Fitting decomposition of $B$ relatively to the $p$-map. Then $\mathcal{L}^{(1)} \oplus Z(\mathcal{L}) = \text{Sap}(\mathcal{L}) = \mathcal{N}(\mathcal{L})$ from Theorem 2.4, (ii), (iii). But $\mathcal{L}^{(1)} \oplus Z(\mathcal{L}) + B_0$
is a nilpotent ideal of \( \mathcal{L} \) and so \( B_0 \subseteq \mathcal{N}(\mathcal{L}) \cap B = 0 \). Hence \( B = B_1 \) is toral. It is clear that \( B_1 + Z(\mathcal{L})_1 \) is a maximal toral subalgebra of \( \mathcal{L} \).

Finally, let \( T \) be any maximal torus of \( \mathcal{L} \), and let \( \mathcal{C} = Z_{\mathcal{L}}(T) \). Then \( \mathcal{C} \) is a Cartan subalgebra of \( \mathcal{L} \), (by Theorem 4.5.17 of [10]) and \( \mathcal{L} = \mathcal{L}^{(1)} + \mathcal{C} \) as above. Clearly we can write \( \mathcal{C} = C_0 \oplus T \). But now \( \mathcal{L}^{(1)} + C_0 \) is a nilpotent ideal of \( \mathcal{L} \), and so \( C_0 \subseteq \mathcal{N}(\mathcal{L}) \cap \mathcal{C} = Z(\mathcal{L}) \), making \( T \) a \( p \)-complement of \( \mathcal{L}^{(1)} \oplus Z(\mathcal{L})_0 \). \(\square\)

The condition “\( \text{Sap}(\mathcal{L}) = \mathcal{L}^{(1)} \oplus Z(\mathcal{L}) \)” in (iii) Theorem 2.4. cannot be weakened to “\( Z(\mathcal{L}) \subseteq \text{Sap}(\mathcal{L}) \)”, as the following example proves.

**Example 1.** We know which are the Lie algebras of dimension 2 over \( \mathbb{K} \) and we take \( \mathcal{L} = I + V \), where

\[
I = \mathbb{K}a + \mathbb{K}b, \quad V = \mathbb{K}v_1 + \mathbb{K}v_2, \\
v_1^p = v_2^p = b^p = 0, \quad a^p = 0, \\
[V, V] = 0, \quad [a, b] = 0, \quad [a, v_1] = v_1, \quad [a, v_2] = v_2, \quad [b, v_1] = v_2, \quad [b, v_2] = 0.
\]

Then \( \mathcal{L}^{(1)} = V \) is abelian, \( (\mathcal{L}^{(1)})^p = 0, Z(\mathcal{L}) = 0 \). Now

\[
\mathcal{N}(\mathcal{L}) = \mathbb{K}b + \mathbb{K}v_1 + \mathbb{K}v_2.
\]

Also \( \mathbb{K}v_2 \) is a maximal \( p \)-ideal. Let \( J \) be a minimal \( p \)-ideal contained in \( \mathcal{N}(\mathcal{L}) \). Since \( [\mathcal{N}(\mathcal{L}), \mathcal{N}(\mathcal{L})] = \mathbb{K}v_2 \), either \( J = \mathbb{K}v_2 \) or \( [\mathcal{N}(\mathcal{L}), J] = 0 \). Suppose that \( J \neq \mathbb{K}v_2 \). Then \( [b, J] = 0 \) so \( J \subseteq \mathbb{K}b + \mathbb{K}v_2 \), and \( [v_1, J] = 0 \) so \( J \subseteq \mathbb{K}v_1 + \mathbb{K}v_2 \). Thus \( J \subseteq \mathbb{K}v_2 \), a contradiction. Hence \( \mathcal{N}(\mathcal{L}) \neq \text{Sap}(\mathcal{L}) \).

E. L. Stitzinger has shown that, for any Lie algebra \( \mathcal{L} \) over the arbitrary field \( \mathbb{K} \), such that \( \mathcal{L}^{(1)} \) is nilpotent, \( \mathcal{L} \) is \( \mathcal{F} \)-free (that is \( \mathcal{F}(\mathcal{L}) = 0 \)) if and only if each subalgebra of \( \mathcal{L} \) is \( \mathcal{F} \)-free.

The complete analogue of this result does not hold if \( \mathcal{F}(\mathcal{L}) \) is replaced by \( \mathcal{F}_p(\mathcal{L}) \), as the following example proves.

**Example 2.** Let \( \mathcal{L} = \mathbb{K}a + \mathbb{K}b + \mathbb{K}v_1 + \mathbb{K}v_2 \) with \( \mathbb{K} = Z_2 \),

\[
a^2 = a, \quad b^2 = a + b, \quad [a, v_1] = v_1, \quad [a, v_2] = v_2, \quad [b, v_1] = v_2, \quad [b, v_2] = v_1 + v_2,
\]

\[
[a, b] = [v_1, v_2] = 0, \quad v_1^2 = v_2^2 = 0,
\]

and \( I = \mathbb{K}a + \mathbb{K}b \). We get \( \mathcal{F}_p(\mathcal{L}) = 0 \) where as \( \mathcal{F}_p(I) = \mathbb{K}a \).

However some partial results can be obtained.

**Theorem 2.6.** Let \( \mathcal{L} \) be a finite-dimensional \( p \)-Lie algebra. Then the following statements are equivalent:

(i) \( \mathcal{L}^{(1)} \) is nilpotent and \( \mathcal{F}_p(\mathcal{L}) = 0 \).

(ii) \( \mathcal{L} = I + B \) where \( B \) is an abelian subalgebra, \( I \) is an abelian \( p \)-ideal, the (adjoint) action of \( B \) on \( I \) is faithful and completely reducible, \( Z(\mathcal{L}) \) is completely reducible under the \( p \)-map, and the \( p \)-map is trivial on \([B, I]\).
ON A CLASS OF LIE $p$-ALGEBRAS

283

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.4, $\mathcal{L} = I + B$, where

$$I = \text{Sap}(\mathcal{L}) = I_1 \oplus \cdots \oplus I_n,$$

with $I_i$ is a minimal $p$-ideal of $\mathcal{L}$, for $i = 1, 2, \ldots, n$, and $B$ is a $p$-subalgebra of $\mathcal{L}$. Now $Z_B(I) = \{x \in B | [x, B] = 0\}$ is an ideal in the solvable Lie algebra $\mathcal{L}$. If $Z_B(I) \neq 0$, it follows that

$$0 \neq Z_B(I) \cap \text{Sap}(\mathcal{L}) \subseteq B \cap I = 0,$$

which is a contradiction. Hence $Z_B(I) = 0$ and the action of $B$ is faithful.

Now suppose that $I_i \not\subset Z(\mathcal{L})$. Then $I_i \cap Z(\mathcal{L}) \subset I_i$ and so, as $I_i \cap Z(\mathcal{L})$ is a $p$-ideal, $I_i \cap Z(\mathcal{L}) = 0$. If $a \in I_i$ then $(ad)a^p = 0$, and so $a^p = 0$, hence $a \in Z(\mathcal{L})$. Thus, if $a \in I_i \cap Z(\mathcal{L}) = 0$, and the minimality of $I_i$ implies that $I_i$ is an irreducible $B$-module but, of course, if $I_i \subseteq Z(\mathcal{L})$ then $I_i$ is a completely reducible $B$-module, so $I = I_1 \oplus \cdots \oplus I_n$ is a completely reducible $B$-module.

Now $\mathcal{L}^{(1)}$ is nilpotent, therefore $ad x$ is nilpotent, for every $x \in B^{(1)}$. It follows from Engel’s Theorem that $[B^{(1)}, I_i] \subseteq I_i$ for every $i = 1, 2, \ldots, n$. If $I_i \not\subset Z(\mathcal{L})$, this implies that $[B^{(1)}, I_i] = 0$, since $I_i$ is an irreducible $B$-module.

If $I_i \subseteq Z(\mathcal{L})$ then, clearly, $[B^{(1)}, I_i] = 0$ also. Thus $[B^{(1)}, I_i] = 0$, and so $B^{(1)} = 0$, as $Z_B(I) = 0$. Moreover, $Z(\mathcal{L}) \subseteq I$, since $Z_B(I) = 0$. If $a \in Z(\mathcal{L})$ and $a = a_1 + \cdots + a_n$, with $a_i \in I_i$, then $[x, a_1] + \cdots + [x, a_n] = 0$, for all $x \in \mathcal{L}$, so each $a_i \in Z(\mathcal{L})$. Hence $Z(\mathcal{L}) = \Sigma I_i$, where the sum is over all $I_i$ contained in $Z(\mathcal{L})$. Since $\mathcal{L}$ is a minimal $p$-ideal, $Z(\mathcal{L})$ must be irreducible under the $p$-map.

(ii) $\Rightarrow$ (i). In view of Theorem 4.1. of [5], it suffices to show that $I = \text{Sap}(\mathcal{L})$. Now we have $I = [B, I] \oplus Z(\mathcal{L})$, $[B, I]$ is a direct sum of irreducible $B$-modules (each of which is a minimal $p$-ideal) and $Z(\mathcal{L})$ is a direct sum of irreducible subspaces for the $p$-map (each of which is a minimal $p$-ideal). Thus, $I \subseteq \text{Sap}(\mathcal{L})$. But, as $B$ acts faithfully on $\mathcal{L}$, $I$ is a maximal abelian ideal. Hence $I = \text{Sap}(\mathcal{L})$, as required.

Corollary 2.7. Let $\mathcal{L}$ be a finite dimensional Lie $p$-algebra with $\mathcal{L}^{(1)}$ nilpotent and $\mathcal{F}_p(\mathcal{L}) = 0$. Let $P$ be a $p$-subalgebra of $\mathcal{L}$ containing $\text{Sap}(\mathcal{L})$. Then $\mathcal{F}_p(P) = 0$.

Proof. Write $\mathcal{L} = I + B$ as in Theorem 2.4 (ii). Then $P = I + (B \cap P)$ since $I = \text{Sap}(\mathcal{L}) \subseteq P$. Now $B$ acts completely reducibly on $[B, I]$, and hence so does $B \cap P$. It follows that $B \cap P$ acts completely reducibly on $[B \cap P, I]$. Moreover, $Z(P) = Z(\mathcal{L}) \oplus Z_{[B, I]}(B \cap P)$ and the $p$-map is trivial on $[B, I]$, so that $Z(P)$ is completely reducible under the $p$-map. The result now follows from Theorem 2.4.

Corollary 2.8. Let $\mathcal{L}$ be a finite dimensional Lie $p$-algebra such that $\mathcal{L}^{(1)}$ is nilpotent and $\mathcal{F}_p(\mathcal{L}) = 0$. If $J$ is an ideal of $\mathcal{L}$, then $\text{Sap}(J) = 0$. 

\[\Box\]
Proof. It suffices to show this for maximal ideals. By Corollary 2.5, we may assume that $I_1 \not\subseteq J$, where $\text{Sap}(\mathcal{L}) = I_1 \oplus \cdots \oplus I_n$, with $I_1, \ldots, I_n$ minimal abelian $p$-ideals. Then $\mathcal{L} = J + I_1$, since $J$ is maximal, and $J \cap I_1 = 0$. Thus $\mathcal{L} = J \oplus I_1$, $J \cong \mathcal{L}/I_1 \cong B + (I_2 \oplus \cdots \oplus I_n)$, and $I_1 \subseteq Z(\mathcal{L})$. Hence $Z_B(I_2 \oplus \cdots \oplus I_n) = Z_B(I) = 0$, and it is clear that all of the conditions of Theorem 2.4 (ii) hold. □

Corollary 2.9. If $\mathcal{L}$ is an abelian finite dimensional Lie $p$-algebra, then $\mathcal{F}_p(\mathcal{L}) = 0$, if and only if $\mathcal{L}$ is completely reducible under the $p$-map.

Proof. This statement can be proved by using Theorem 2.4 and the fact $B = 0$ and $\mathcal{L} = Z(\mathcal{L})$. □

Corollary 2.10. Let $\mathcal{L}$ be a finite dimensional Lie $p$-algebra such that $\mathcal{L} = \text{Sap}(\mathcal{L}) + B$ and that the conditions of Theorem 2.4 (ii) are satisfied. Assume in addition that $B$ is completely reducible under the $p$-map; that is $\text{Sap}(B) = B$. Then if $P$ is any $p$-subalgebra of $\mathcal{L}$, $P = \text{Sap}(P) \oplus B'$, the conditions of Theorem 2.4. (ii) are satisfied and $B'$ is completely reducible under the $p$-map.

Proof. If $\text{Sap}(\mathcal{L}) \subseteq P$, then $\text{Sap}(P) = \text{Sap}(\mathcal{L})$, and taking $B' = B \cap P$, we get the result.

It suffices to prove the Corollary for maximal $p$-subalgebras. So assume that $P$ is maximal and that $I_1 \not\subseteq P$, where $\text{Sap}(\mathcal{L}) = I_1 \oplus \cdots \oplus I_n$, with $I_1, \ldots, I_n$ minimal abelian $p$-ideals. Then $\mathcal{L} = I_1 + P$, with $P \cap I_1 = 0$. Hence $P \cong B + (I_2 \oplus \cdots \oplus I_n)$. As $B$ is completely reducible under the $p$-map, we have

$$B = B' \oplus Z_B(I_2 \oplus \cdots \oplus I_n).$$

Then $\text{Sap}(P) = Z_B(I_2 \oplus \cdots \oplus I_n) \oplus I_2 \oplus \cdots \oplus I_n, P = \text{Sap}(P) \oplus B'$, the conditions of Theorem 2.4 (ii) are satisfied and $B'$ is completely reducible under the $p$-map. □

Definition 2.11. A finite dimensional Lie $p$-algebra $\mathcal{L}$ is called $p$-elementary, if $\mathcal{F}_p(P) = 0$ for every $p$-subalgebra $P$ of $\mathcal{L}$.

Corollary 2.12. Assume $\mathcal{L}^{(1)}$ is a finite dimensional Lie $p$-algebra with nilpotent $\mathcal{L}^{(1)}$ and $\mathcal{F}_p(\mathcal{L}) = 0$. Let $\mathcal{L} = \text{Sap}(\mathcal{L}) \dotplus B$ as in Theorem 2.4 (ii). Then $\mathcal{L}$ is $p$-elementary, if and only if $B = \text{Sap}(B)$.

Proof. As $\mathcal{F}_p(\mathcal{L}) = 0$ and $\mathcal{L} = \text{Sap}(\mathcal{L}) \dotplus B$ (Theorem 2.4. (ii)), then $B$ has a faithful completely reducible representation on $\text{Sap}(\mathcal{L})$. This is equivalent to the fact that $B$ has a non-zero nilideals as in [7]. Since $B$ is abelian, this is equivalent to the injectivity of the $p$-map. Since $\mathbb{K}$ is algebraically closed, this is equivalent to $\text{Sap}(B) = B$ as in [4]. It follows from Corollary 2.14 that $\mathcal{L}$ is $p$-elementary. The converse is immediate from the definition. □
The result above cannot be extended to the case when $K$ is a perfect field. Let us see the following example.

**Example 3.** Let $L$ be any abelian Lie $p$-algebra for which the $p$-map is injective but $L$ is not completely reducible under the $p$-map. Then $L$ has a faithful completely reducible module $B$. Make $B$ into an abelian Lie $p$-algebra with trivial $p$-map. Then $F_p(B + L) = 0$, but $F_p(L) \neq 0$.

Now, if $K$ is not perfect, let $\lambda \in K \setminus K^p$and take $L = Ka + Kb$, with $a^p = a, b^p = \lambda a$. If $\lambda \in K$ and $\mu^p - \mu + \lambda = 0$ has no solution in $K$, take $L = Ka + Kb$ with $a^p = a, b^p = b + \lambda a$. Here we may take $B$ to be $p$-dimensional with $a$ represented by the identity matrix and $b$ represented by the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
& & & & \\
0 & 0 & 0 & \ldots & 1 \\
-\lambda & 1 & 0 & \ldots & 0
\end{pmatrix}
\]

(the companion matrix of $\mu^p - \mu + \lambda$). If $K = Z_p$ we may take $\lambda = -1$.

Putting $p = 2$, we get the example 2.7.

**References**


"Mircea cel Batran" Naval Academy, 1, Fulgerului Street, 900218, Constantza, Romania

E-mail address: cciobanu@anmb.ro