THE UNIT GROUP OF $FS_3$

R.K. SHARMA, J.B. SRIVASTAVA, AND MANJU KHAN

ABSTRACT. In this paper we give a complete characterization of the unit group $U(FS_3)$ of the group algebra $FS_3$ of the symmetric group $S_3$ of degree 3 over a finite field $F$. Moreover, over the prime field $\mathbb{Z}_2$ and $\mathbb{Z}_3$, presentation of the unit groups of group algebras $\mathbb{Z}_2S_3$ and $\mathbb{Z}_3S_3$ in terms of generators and relators have also been obtained.

1. Introduction

Let $FG$ denote the group algebra of a group $G$ over a field $F$. For a normal subgroup $H$ of $G$, the natural homomorphism $g \mapsto gH : G \rightarrow G/H$ can be extended to an $F$-algebra homomorphism from $FG$ onto $F[G/H]$ defined by

$$\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g gH.$$ 

Kernel of this homomorphism, denoted by $\omega(H)$, is an ideal of $FG$ generated by \{ $h - 1 \mid h \in H$ \}. Thus, $FG/\omega(H) \cong F[G/H]$. The augmentation ideal, $\omega(FG)$, of the group algebra $FG$ is defined by

$$\omega(FG) = \left\{ \sum_{g \in G} a_g g \mid a_g \in F, \sum_{g \in G} a_g = 0 \right\}.$$ 

Clearly, $\omega(G) = \omega(FG)$. In general, $\omega(H) = \omega(FH)FG = F\omega(FH)$. Also $FG/\omega(G) \cong F$ implies that the Jacobson radical $J(FG) \subseteq \omega(FG)$. It is known that, the natural homomorphism $x \mapsto x + J(FG) : FG \rightarrow FG/J(FG)$ induces an epimorphism: $\mathcal{U}(FG) \rightarrow \mathcal{U}(FG/J(FG))$ with kernel $1 + J(FG)$ so that $\mathcal{U}(FG)/(1 + J(FG)) \cong \mathcal{U}(FG/J(FG))$.

This is also known that for any prime $p$ and for any positive integer $n$, there is a monic irreducible polynomial of degree $n$ over $\mathbb{Z}_p$ [7].

Here we shall use the presentation of $S_3$ as

$$S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau\sigma = \sigma^2\tau \rangle.$$ 

Thus, the elements of $S_3$ are \{ $1, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau$ \}. The alternating group $A_3$ of degree 3 is given by $A_3 = \{ 1, \sigma, \sigma^2 \}$. The distinct conjugacy classes of $S_3$
are \( C_0 = \{1\}, C_1 = \{\sigma, \sigma^2\} \) and \( C_2 = \{\tau, \sigma \tau, \sigma^2 \tau\} \). Hence, \( \{\hat{C}_0, \hat{C}_1, \hat{C}_2\} \) form a basis of center \( \mathcal{Z}(FS_3) \) of \( FS_3 \) (cf. Lemma 4.1.1 of [5]), where \( \hat{C}_i \) denotes the class sum.

We shall use \( V_1 \) for the unit subgroup \( 1 + J(FS_3) \).

The unit group of integral group ring \( \mathbb{Z}S_3 \) has been studied by Hughes and Pearson [2] and by Allen and Hobby [1]. The unit group has been discussed in terms of the bicyclic units by Jespers and Parmenter [3]. Sharma et al. [6] studied chains of subgroups of the unit group \( \mathcal{U}(\mathbb{Z}S_3) \). However, so far it seems the structure of the unit group \( \mathcal{U}(FS_3) \), for \( \text{char } F = p > 0 \) is not known.

This paper gives a complete characterization of the unit group \( \mathcal{U}(FS_3) \) over a finite field \( F \). Also we give the presentation of the unit groups of group algebras \( \mathbb{Z}_2S_3 \) and \( \mathbb{Z}_3S_3 \) over the prime field \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) in terms of generators and relators.

2. THE UNIT GROUP OF \( FS_3 \)

In this Section, the following theorems gives a complete structure of the unit group \( \mathcal{U}(FS_3) \) over an arbitrary finite field \( F \).

Let \( \text{char } F = p \) and \( |F| = p^n \).

**Theorem 2.1.** If \( p = 2 \), then \( \mathcal{U}(FS_3)/V_1 \cong GL(2, F) \times F^* \) and \( V_1 \) is central elementary abelian 2-group of order \( 2^n \), where \( GL(2, F) \) denotes the general linear group of degree 2 over \( F \).

**Theorem 2.2.** If \( p = 3 \) and \( Z(V_1) \) is the center of \( V_1 \), then \( Z(V_1) \) and \( V_1/Z(V_1) \) both are elementary abelian 3-groups.

**Theorem 2.3.** If \( p > 3 \), then
\[
\mathcal{U}(FS_3) \cong GL(2, F) \times F^* \times F^*.
\]

**Proof of the Theorem 2.1.** We define a matrix representation of \( S_3 \),
\[
\rho : S_3 \to \mathbb{M}(2, F) \oplus F
\]
by the assignment
\[
\sigma \mapsto \left( \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 1 \right)
\]
and
\[
\tau \mapsto \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)
\]
Thus, \( \rho \) can be extended to an \( F \)-algebra homomorphism
\[
\rho^* : FS_3 \to \mathbb{M}(2, F) \oplus F.
\]
Let \( x = \alpha_0 + \alpha_1 \sigma + \alpha_2 \sigma^2 + \alpha_3 \tau + \alpha_4 \sigma \tau + \alpha_5 \sigma^2 \tau \in \text{Ker } \rho^* \), where \( \alpha_i \)'s \( \in F \).

Therefore, \( \rho^*(x) = 0 \) gives the following system of equations:

\[
\begin{align*}
\alpha_0 + \alpha_2 + \alpha_3 + \alpha_5 &= 0 \\
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\
\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 &= 0 \\
\alpha_0 + \alpha_1 + \alpha_3 + \alpha_5 &= 0 \\
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 &= 0
\end{align*}
\]

By solving this system of equations we get all \( \alpha_i \)'s are same. Thus,

\[\text{Ker } \rho^* = \{ \alpha(1 + \sigma + \sigma^2 + \tau + \sigma \tau + \sigma^2 \tau) \mid \alpha \in F \}.\]

If \( \hat{S}_3 \) is the sum of all elements in \( S_3 \), then \( \hat{S}_3^2 = 0 \), because \( F \) is a field of characteristic 2. It follows that \( \text{Ker } \rho^* \) is a nilpotent ideal of \( FS_3 \). Hence, \( \text{Ker } \rho^* \subseteq J(FS_3) \). Since, \( \rho^* \) is onto, we have \( \rho^*(J(FS_3)) \subseteq J(M(2, F) \oplus F) = 0 \) and hence \( J(FS_3) \subseteq \text{Ker } \rho^* \). Hence, \( J(FS_3) = \text{Ker } \rho^* = \hat{S}_3 \) and so \( FS_3/J(FS_3) \cong M(2, F) \oplus F \). It follows that \( \mathcal{U}(FS_3)/V_1 \cong \mathcal{U}(FS_3/J(FS_3)) \cong GL(2, F) \times F^* \).

Further, assume \( f(X) \) is a monic irreducible polynomial of degree \( n \) over the field \( \mathbb{Z}_2 \). Then \( \mathbb{Z}_2[X]/(f(X)) \cong F \). Assume \( \xi \) is the residue class of \( X \mod (f(X)) \). So the structure of \( V_1 \) is

\[V_1 = \prod_{i=0}^{n-1} \langle 1 + \xi^i x \mid x = \hat{S}_3 \rangle ,\]

a central subgroup of order \( 2^n \).

**Proof of the Theorem 2.2.** Since \( A_3 \) is a normal subgroup of \( S_3 \) and \( [S_3 : A_3] = 2 \), which is invertible in \( F \), we have \( J(FS_3) = J(FA_3)FS_3 \) (cf. Lemma 7.2.7 of [5]). Further, since char \( F = 3 \) and \( A_3 \) is a 3-group, we get \( J(FA_3) = \omega(FA_3) \) (cf. Lemma 8.1.17 of [5]). Consequently,

\[J(FS_3) = \omega(FA_3)FS_3 = \omega(A_3) .\]

Hence,

\[FS_3/J(FS_3) = FS_3/\omega(A_3) \cong F[S_3/A_3] \cong FC_2 \cong F \oplus F .\]

Thus,

\[\mathcal{U}(FS_3)/V_1 \cong \mathcal{U}(FS_3/J(FS_3)) \cong F^* \times F^* .\]

Now, \( V_1 = 1 + J(FS_3) = 1 + \omega(A_3) = 1 + \omega(FA_3)FS_3 \) and \( \omega(FA_3)^3 = 0 \), then \( \omega(A_3)^3 = 0 \). Thus, every non identity element of \( V_1 \) is of order 3. For \( \alpha \in F \) and \( x = 1 + \sigma + \sigma^2 \), let \( u_\alpha = 1 + \alpha x \) and \( v_\alpha = 1 + \alpha x \tau \). Both \( u_\alpha \) and \( v_\alpha \) are central elements of \( FS_3 \) as well as elements of \( V_1 \). Take \( U = \{ u_\alpha \mid \alpha \in F \} \) and \( V = \{ v_\alpha \mid \alpha \in F \} \). Since, \( u_\alpha u_\beta = u_{\alpha + \beta} \) and \( v_\alpha v_\beta = v_{\alpha + \beta} \), it follows that both \( U \) and \( V \) are central subgroups of \( V_1 \). Further, since all the elements in \( U \) and \( V \) are distinct we have \( |U| = |V| = 3^n \). If possible, let \( u \in U \cap V \), i.e. \( u = u_\alpha = v_\beta \) for some \( \alpha, \beta \in F \). Thus, we have \( \alpha(1 + \sigma + \sigma^2) = \beta(1 + \sigma + \sigma^2) \tau ,\)
which implies that $\alpha = \beta = 0$ and so $U \cap V = \{1\}$. Then $U \times V \subseteq Z(V_1)$, which gives us that $|Z(V_1)| \geq 3^{2^n}$.

Assume $w_\alpha = 1 + \alpha(\sigma - 1)$ and $t_\alpha = 1 + \alpha(\sigma - 1)\tau$ are two noncommuting elements in $V_1 \setminus Z(V_1)$, where

\[
\begin{align*}
w_\alpha^2 &= 1 + 2\alpha(\sigma - 1) + \alpha^2(1 + \sigma + \sigma^2) = w_\alpha u_\alpha^2, \\
t_\alpha^2 &= 1 + 2\alpha(\sigma - 1)\tau + 2\alpha^2(1 + \sigma + \sigma^2) = t_\alpha u_\alpha^2.
\end{align*}
\]

It can be verified that $w_\alpha Z(V_1)w_\beta Z(V_1) = w_{\alpha + \beta} Z(V_1)$. Therefore, we get that $\{w_\alpha Z(V_1) \mid \alpha \in F\}$ is a subgroup of $V_1/Z(V_1)$. If possible, let $w_\alpha Z(V_1) = w_\beta Z(V_1)$. Then $w_\alpha w_\beta^2 \in Z(V_1)$, i.e. $w_\alpha w_\beta^2 \in Z(V_1)$. But, $w_\alpha w_\beta^2 = w_{\alpha + 2\beta}$ (mod $Z(V_1)$). Hence, $w_\alpha w_\beta^2 \in Z(V_1)$ implies $\alpha = \beta$. This shows that all the elements in $\{w_\alpha \mid \alpha \in F\}$ (mod $Z(V_1)$) are distinct. Thus, the number of elements in $\{w_\alpha Z(V_1) \mid \alpha \in F\}$ are $3^n$. Also, since $t_\alpha t_\beta = t_{\alpha + \beta} u_\alpha u_\beta$, by using the similar argument we get $\{t_\alpha Z(V_1) \mid \alpha \in F\}$ is a subgroup of $V_1/Z(V_1)$ with order $3^n$. Note that $w_\alpha Z(V_1)$ and $t_\beta Z(V_1)$ commute with each other.

Since, $\omega(A_3)$ is $F$-linear combination of $(\sigma - 1)$ and $(\sigma^2 - 1)$, we have $\omega(A_3)$ is $F$-linear combination of $(\sigma - 1), (\sigma^2 - 1), (\sigma - 1)\tau$ and $(\sigma^2 - 1)\tau$ so that any element $1 + x$ in $V_1$, for $x \in \omega(A_3)$, can be written as

\[
1 + x = 1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau + \alpha_3(\sigma^2 - 1)\tau,
\]

where $\alpha_i$'s $\in F$. Now,

\[
\begin{align*}
1 + \alpha_1(\sigma^2 - 1) &= 1 + 2\alpha_1(\sigma - 1) + \alpha_1(1 + \sigma + \sigma^2) \\
&= (1 + 2\alpha_1(\sigma - 1))(1 + \alpha_1(1 + \sigma + \sigma^2)) \\
&= w_{2\alpha_1} u_{\alpha_1},
\end{align*}
\]

and so,

\[
(1 + \alpha_0(\sigma - 1))(1 + \alpha_1(\sigma^2 - 1)) = 1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + 2\alpha_0\alpha_1(1 + \sigma + \sigma^2) \\
= (1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1))u_{2\alpha_0\alpha_1}.
\]

Thus, $(1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1)) = w_{\alpha_0} w_{2\alpha_1} u_{\alpha_1} u_{\alpha_0 \alpha_1}$. Further,

\[
\begin{align*}
(1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1))(1 + \alpha_2(\sigma - 1)\tau) &= (1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau) \times \\
&\times (1 + \alpha_0\alpha_2(1 + \sigma + \sigma^2)\tau)(1 + 2\alpha_1\alpha_2(1 + \sigma + \sigma^2)\tau) \\
&= (1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau)v_{\alpha_0 \alpha_2} v_{2\alpha_1 \alpha_2}.
\end{align*}
\]

Thus, $(1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau) = w_{\alpha_0} w_{2\alpha_1} u_{\alpha_1} u_{\alpha_0 \alpha_1} t_{\alpha_2} v_{2\alpha_0 \alpha_2} v_{\alpha_1 \alpha_2}$.

In similar way one can show that any element of $V_1$ can be expressed as a linear combination of $w_\alpha$ (mod $Z(V_1)$), $t_\alpha$ (mod $Z(V_1)$), for $\alpha \in F$. 

If possible, let \( w_\alpha \mathcal{Z}(V_1) = t_\beta \mathcal{Z}(V_1) \) for some \( \alpha, \beta \in F \). Then \( w_\alpha t_\beta^2 \in \mathcal{Z}(V_1) \), i.e. \( w_\alpha t_\beta^2 \in \mathcal{Z}(V_1) \). But,

\[
\begin{align*}
w_\alpha t_\beta^2 &= (1 + \alpha(\sigma - 1))(1 + 2\beta(\sigma - 1)\tau) \\
&= (1 + \alpha(\sigma - 1) + 2\beta(\sigma - 1)\tau) \pmod{\mathcal{Z}(V_1)}
\end{align*}
\]

Then \( w_\alpha t_\beta^2 \in \mathcal{Z}(V_1) \) when \( \alpha = \beta = 0 \). Thus,

\[
\{ w_\alpha \mathcal{Z}(V_1) \mid \alpha \in F \} \cap \{ t_\alpha \mathcal{Z}(V_1) \mid \alpha \in F \} = \mathcal{Z}(V_1).
\]

Hence, the order of \( V_1/\mathcal{Z}(V_1) \) is \( 3^{2n} \), so that the order of \( \mathcal{Z}(V_1) \) is \( 3^{2n} \).

Let \( f(X) \) be a monic irreducible polynomial of degree \( n \) in \( \mathbb{Z}_3[X] \). Therefore, \( \mathbb{Z}_3[X]/\langle f(X) \rangle \cong F \). Further, since order of each \( u_\alpha, v_\alpha \) is 3, \( \mathcal{Z}(V_1) \) is an elementary abelian 3-group and the structure of \( \mathcal{Z}(V_1) \) is given as

\[
\mathcal{Z}(V_1) = \prod_{i=0}^{n-1} \langle 1 + \alpha^i x \rangle \times \prod_{j=0}^{n-1} \langle 1 + \alpha^j x^\tau \rangle,
\]

where \( \alpha \) is residue class of \( X \) modulo \( \langle f(X) \rangle \).

The presentation of \( V_1/\mathcal{Z}(V_1) \) is given by

\[
V_1/\mathcal{Z}(V_1) = \prod_{i=0}^{n-1} \langle (1 + \alpha^i(\sigma - 1))\mathcal{Z}(V_1) \rangle \times \prod_{j=0}^{n-1} \langle (1 + \alpha^j(\sigma - 1)\tau)\mathcal{Z}(V_1) \rangle.
\]

Proof of the Theorem 2.3. Since \( p \nmid |S_3| \), by Maschke’s theorem \( FS_3 \) is a semi-simple Artinian algebra over \( F \). Then by Wedderburn structure theorem we get

\[
FS_3 \cong \bigoplus_{i=1}^r M(n_i, D_i),
\]

where \( D_i \)'s are finite dimensional division algebras over \( F \). Since \( F \) is a finite field, \( D_i \)'s are finite division algebras, and hence they are fields. In this case denote \( D_i \) by \( F_i \). Thus,

\[
FS_3 \cong \bigoplus_{i=1}^r M(n_i, F_i),
\]

where \( F_i \)'s are finite field extension of \( F \).

Since, \( \dim_F(FS_3) = 3 \), \( FS_3 \) is noncommutative, and not simple, the possible structures of the group algebra \( FS_3 \) are given by

\[
FS_3 \cong M(2, F) \oplus F \oplus F \quad \text{or} \quad FS_3 \cong M(2, F) \oplus F_2,
\]

where \( F_2 \) is a quadratic extension of \( F \). No other case is possible. Since, if \( M(2, F_2) \) occurs in the right hand side in the place of \( M(2, F) \), but then \( \dim_F(M(2, F_2)) = 8 \), a contradiction. Therefore, only \( M(2, F) \) will occur in the right hand side. Since \( \dim_F(FS_3) = 6 \), we get \( M(2, F) \) to be a direct
summand of $FS_3$ of codimension 2. So only two cases as mentioned above may arise.

We will prove that second case is not possible. If possible, let second case holds. In this case $U(FS_3) \cong GL(2, F) \times F^*$. In $F^*$, there is an element of order $p^{2n} - 1$, i.e. there is an element in the center of $U(FS_3)$ of order $p^{2n} - 1$. Now, $Z(FS_3)$ is $F$-linear combination of $\hat{C}_0$, $\hat{C}_1$ and $\hat{C}_2$, so any element $x \in Z(FS_3)$ can be written as $x = \alpha_0 \hat{C}_0 + \alpha_1 \hat{C}_1 + \alpha_2 \hat{C}_2$, where $\alpha_i \in F$. Since, $p > 3$, we get either $3|(p^n - 1)$ or $3|(p^n + 1)$. In both the cases it can be verified that $(\hat{C}_1)^{p^n} = \hat{C}_1$ and $(\hat{C}_2)^{p^n} = \hat{C}_2$. This gives $x^{p^n} = (\alpha_0 + \alpha_1 \hat{C}_1 + \alpha_2 \hat{C}_2)^{p^n} = \alpha_0 + \alpha_1 \hat{C}_1 + \alpha_2 \hat{C}_2 = x$. Hence, $x^{p^n} = x$ for all $x \in Z(FS_3)$. But then $U(Z(FS_3))$ is a group of exponent $(p^n - 1)$, a contradiction. Hence, second case does not arise. Thus, $FS_3 \cong M(2, F) \oplus F \oplus F$.

Hence,

$U(FS_3) \cong GL(2, F) \times F^* \times F^*$.  

\[ \square \]

3. Unit Groups of $\mathbb{Z}_2S_3$ and $\mathbb{Z}_3S_3$

In this section we give presentation of the unit group $U(\mathbb{Z}_p S_3)$ for the prime field $\mathbb{Z}_p$, when $p = 2, 3$.

**Theorem 3.1.** The unit group $U(\mathbb{Z}_2 S_3)$ is isomorphic to $D_{12}$, the dihedral group of order 12. In particular, if $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau \sigma = \sigma^2 \tau \rangle$ then $U(\mathbb{Z}_2 S_3) = \langle \omega, \tau \mid \omega^6 = \tau^2 = 1, \tau \omega = \omega^{\tau} \rangle$, where $\omega = 1 + \sigma^2 + \tau + \sigma \tau + \sigma^2 \tau$.

**Proof.** Any element of even length in $\mathbb{Z}_2 S_3$ cannot be a unit, since any such element belongs to the augmentation ideal $\omega(\mathbb{Z}_2 S_3)$. Elements of length 1 are trivial units in $\mathbb{Z}_2 S_3$. Let $x = g_1 + g_2 + g_3 \in \mathbb{Z}_2 S_3$, be an element of length 3. Then $x = g_1(1 + g_1^{-1}g_2 + g_1^{-1}g_3)$ is a unit if and only if $1 + g_1^{-1}g_2 + g_1^{-1}g_3$ is a unit. Hence, we can assume that any element of length 3 is of the form $x = 1 + g_1 + g_2$ for some non-identity elements $g_1, g_2 \in S_3$. The following two cases arise:

**Case 1.** Elements $g_1$ and $g_2$ commute with each other. First, note that, $x^2 = (1 + g_1 + g_2)^2 = 1 + g_1 + g_2$. Since $\sigma$ and $\sigma^2$ are the only elements of $S_3$ which commute each other, we get $x = 1 + g_1 + g_2 = 1 + \sigma + \sigma^2$. Since, $x$ is an idempotent, it can not be a unit.

**Case 2.** If $g_1$ and $g_2$ do not commute with each other, then also $x$ can not be a unit in $\mathbb{Z}_2 S_3$. For that, take $g_1, g_2 \in \{\tau, \sigma \tau, \sigma^2 \tau\}$, then $x^2 = 1 + g_1 g_2 + g_2 g_1 = 1 + \sigma + \sigma^2$, an idempotent; hence $x^2$ and therefore $x$ cannot be a unit. Next, assume $g_1 \in \{\tau, \sigma \tau, \sigma^2 \tau\}$ and $g_2 \in \{\sigma, \sigma^2\}$, then $x^2 = g_2 + g_1 g_2 + g_2 g_1 = g_2^2(1 + g_1 g_2 + g_2 g_1)$. If $x$ is a unit then $y = 1 + g_2 g_1 g_2 + g_2^2 g_1$ is also a unit. But, this is not possible, because $g_2 g_1 g_2$ and $g_2^2 g_1 \in \{\tau, \sigma \tau, \sigma^2 \tau\}$. Hence, no element of length 3 is a unit.
Hence, the unit group \( \mathcal{U}(\mathbb{Z}_3 S_3) \) of \( \mathbb{Z}_3 S_3 \) is
\[
\mathcal{U}(\mathbb{Z}_3 S_3) = \{ u_1, u_2, u_3, v, w, w^{-1}, 1, \sigma, \sigma^2, \tau, \sigma \tau, \sigma^2 \tau \}.
\]
Further, \( w^2 = \sigma^2 \), \( w^3 = \sigma^2 w = v \), \( w^4 = \sigma^4 = \sigma \), \( w^5 = w \sigma = w^{-1} \), \( w^6 = 1 \) and \( w \tau = u_3, w^3 \tau = u_1 \) and \( w^5 \tau = u_2 \). We get
\[
\mathcal{U}(\mathbb{Z}_3 S_3) = \langle w, \tau \mid w^6 = \tau^2 = 1, w \tau = \tau w^5 \rangle,
\]
which is a dihedral group of order 12. This completes the proof of this theorem.

Next, we will discuss about the unit group \( \mathcal{U}(\mathbb{Z}_3 S_3) \) over the prime field \( \mathbb{Z}_3 \). For the field \( \mathbb{Z}_3 \), structure of the unit group \( \mathcal{U}(\mathbb{Z}_3 S_3) \) is given as follows:

**Theorem 3.2.** Let \( V_1 = 1 + J(\mathbb{Z}_3 S_3) \) and let \( \mathcal{Z}(V_1) \) denotes the center of \( V_1 \). Then

(i) both the groups \( \mathcal{Z}(V_1) \) and \( V_1/\mathcal{Z}(V_1) \) are isomorphic to \( C_3 \times C_3 \).

(ii) the unit group \( \mathcal{U}(\mathbb{Z}_3 S_3)/V_1 \) is isomorphic to \( C_2 \times C_2 \). In particular, order of \( \mathcal{U}(\mathbb{Z}_3 S_3) \) is 324.

The above theorem is direct consequence of the Theorem 2.2.

Now, we give more precise presentations of the unit group \( \mathcal{U}(\mathbb{Z}_3 S_3) \). In fact, we present all units in their canonical forms.

In Example 8, Kulshammer and Sharma [4] showed that
\[
\omega(A_3) = \mathbb{Z}_3 u + \mathbb{Z}_3 v + \mathbb{Z}_3 uv + \mathbb{Z}_3 vu
\]
for some \( u, v \in \mathbb{Z}_3 S_3 \). Let \( u = (\sigma - \sigma^2)(1 - \tau) \) and \( v = (\sigma - \sigma^2)(1 + \tau) \). Thus, \( uv = 2(1 + \sigma + \sigma^2) + 2(1 + \sigma + \sigma^2) \tau \) and \( vu = 2(1 + \sigma + \sigma^2) + (1 + \sigma + \sigma^2) \tau \) and so \( \mathbb{Z}_3 u + \mathbb{Z}_3 v + \mathbb{Z}_3 uv + \mathbb{Z}_3 vu \subseteq \omega(A_3) \).

Further, \( \{ (1 - \sigma), (1 - \sigma^2), (1 - \sigma) \tau, (1 - \sigma^2) \tau \} \) form a basis of \( \omega(A_3) \). One can see that
\[
1 - \sigma = uv + vu - u - v,
\]
\[
1 - \sigma^2 = uv + vu + u + v,
\]
\[
(1 - \sigma) \tau = uv - vu - v + u,
\]
\[
(1 - \sigma^2) \tau = uv - vu + v - u.
\]
Thus, any element of $\omega(A_3)$ can be expressed as $\mathbb{Z}_3$-linear combination of $u, v, uv$ and $vu$. Hence $\omega(A_3) = \mathbb{Z}_3u + \mathbb{Z}_3v + \mathbb{Z}_3uv + \mathbb{Z}_3vu$.

Since $J(\mathbb{Z}_3S_3) = \omega(A_3)$, we have

$$V_1 = 1 + J(\mathbb{Z}_3S_3) = \{1 + \alpha_1u + \alpha_2v + \alpha_3uv + \alpha_4vu \mid 0 \leq \alpha_i \leq 2\}$$

for $i = 1, 2, 3, 4$. Let

$$x = uv + vu, \ y = uv - vu, \ \omega_1 = 1 + v, \ \omega_2 = 1 + u.$$

Assume $H_1 = \langle 1 + x, 1 + y \rangle$. Now, $1 + x, 1 + y \in Z(\mathbb{Z}_3S_3)$ and $u^2 = 0, \ v^2 = 0$ and $uvu = 0$, implies $x^2 = y^2 = 0$. Thus,

$$H_1 = \langle 1 + x \mid (1 + x)^3 = 1 \rangle \times \langle 1 + y \mid (1 + y)^3 = 1 \rangle \subseteq Z(\mathbb{Z}_3S_3).$$

Hence, $H_1 \subseteq Z(V_1)$. For the converse, observe that $uv, vu \in Z(\mathbb{Z}_3S_3)$. Therefore, if $z = 1 + \alpha_1u + \alpha_2v + \alpha_3uv + \alpha_4vu \in Z(V_1)$, then $\alpha_1u + \alpha_2v$ commutes with every element of $V_1$. In particular, $\alpha_1u + \alpha_2v$ commutes with $1 + v$ but, then it commutes with $v$ also. This implies that $\alpha_1u$ commutes with $v$. This gives that $\alpha_1y = \alpha_1(1 + y) = \alpha_1(1 + vu) = \alpha_1(uv) = \alpha_1(1) = \alpha_1$. Since, $1 + y$ is a unit, we get $\alpha_1 = 0$. Similarly, we get $\alpha_2 = 0$. Hence, $z = 1 + \alpha_3uv + \alpha_4vu$, i.e. $Z(V_1) = 1 + \mathbb{Z}_3uv + \mathbb{Z}_3vu$.

Since, $H_1 \subseteq Z(V_1)$ and $|H_1| = |Z(V_1)| = 9$ we get

$$Z(V_1) = 1 + \mathbb{Z}_3uv + \mathbb{Z}_3vu$$

$$= \langle 1 + x \mid (1 + x)^3 = 1 \rangle \times \langle 1 + y \mid (1 + y)^3 = 1 \rangle$$

$$= \langle 2 + \sigma + \sigma^2 \mid (2 + \sigma + \sigma^2)^3 = 1 \rangle \times$$

$$\times \langle (1 + (1 + \sigma + \sigma^2)\tau) \mid (1 + (1 + \sigma + \sigma^2)\tau)^3 = 1 \rangle.$$

We have so far got that

$$H_1 = \langle 1 + x \mid (1 + x)^3 = 1 \rangle \times \langle 1 + y \mid (1 + y)^3 = 1 \rangle = Z(V_1).$$

Next, $\omega_1, \omega_2 \notin Z(V_1)$ as $\omega_1\omega_2 \neq \omega_2\omega_1$. Otherwise,

$$(1 + v)(1 + u) = (1 + u)(1 + v) \Rightarrow uv - vu = y = x\tau = 0.$$ But, then $x = 1 + \sigma + \sigma^2 = 0$, a contradiction. Further, since $v^2 = 0$, $\omega_1^3 = (1 + v)^3 = 1$. Similarly, we get $\omega_2^3 = 1$. Also,

$$\omega_1\omega_2 = \omega_1^{-1}\omega_2^{-1}\omega_1\omega_2 = \omega_1^2\omega_2^2\omega_1\omega_2.$$ Observe that $\omega_1^2 = (1 + v)^2 = 1 + 2v + v^2 = 1 + 2v = 1 - v$. Similarly, $\omega_2^2 = 1 - u$.

So $\omega_1^2\omega_2^2 = (1 - v)(1 - u) = 1 - u - v + vu$ and

$$\omega_1\omega_2 = (1 + v)(1 + u) = 1 + u + v + vu.$$
and therefore,
\[ \omega_1^2 \omega_2 \omega_1 \omega_2 = (1 - u - v + vu)(1 + u + v + vu) \]
\[ = (1 - u - v)(1 + u + v) + vu + vu \text{ since } vu \in \mathbb{Z}(\mathbb{Z}_3 S_3), u^2 = v^2 = 0 \]
\[ = 1 - (u + v)^2 + 2vu \]
\[ = 1 - (uv + vu) - vu \]
\[ = 1 - 2vu - vu \]
\[ = 1 + vu - uv \]
\[ = 1 - y = (1 + y)^2. \]

The equation \((\omega_1, \omega_2) = (1 + y)^2 \in \mathbb{Z}(V_1)\) implies that \(\omega_1 \mathbb{Z}(V_1)\) and \(\omega_2 \mathbb{Z}(V_1)\) commute with each other. Also \((\omega_1 \mathbb{Z}(V_1))^3 = (\omega_2 \mathbb{Z}(V_1))^3 = \mathbb{Z}(V_1)\) as
\[ \omega_1^3 = \omega_2^3 = 1. \]

Since, \(|V_1/\mathbb{Z}(V_1)| = 9\), we get \(V_1/\mathbb{Z}(V_1) = \langle \omega_1 \mathbb{Z}(V_1) \rangle \times \langle \omega_2 \mathbb{Z}(V_1) \rangle\). This discussion summarizes the following:

**Lemma 3.3.** Let \(V_1\) be \(1 + J(\mathbb{Z}_3 S_3)\) and \(\mathbb{Z}(V_1)\) be its center. Then

(i) \(\mathbb{Z}(V_1) = \langle 1 + x \rangle \times \langle 1 + y \rangle\), where \(x = 1 + \sigma + \sigma^2\) and \(y = (1 + \sigma + \sigma^2)\tau; (1 + x)^3 = (1 + y)^3 = 1\).

(ii) \(\mathbb{Z}(V_1) = \{1 + \alpha uv + \beta vu \mid \alpha, \beta \in \mathbb{Z}_3\}, \text{ where } u = (\sigma - \sigma^2)(1 - \tau), v = (\sigma - \sigma^2)(1 + \tau)\)

(iii) \(V_1/\mathbb{Z}(V_1) = \langle \omega_1 \mathbb{Z}(V_1) \rangle \times \langle \omega_2 \mathbb{Z}(V_1) \rangle\), where \(\omega_1 = 1 + v, \omega_2 = 1 + u\).

This gives

**Theorem 3.4.** If \(x = 1 + \sigma + \sigma^2, y = (1 + \sigma + \sigma^2)\tau, u = (\sigma - \sigma^2)(1 - \tau)\) and \(v = (\sigma - \sigma^2)(1 + \tau)\), then

(i) \(V_1 = \{1 + \alpha_1 u + \alpha_2 v + \alpha_3 uv + \alpha_4 vu \mid \alpha_i \in \mathbb{Z}_3 \text{ for } i = 1, 2, 3, 4\}\)

(ii) \(V_1 = \langle 1 + x, 1 + y, 1 + v, 1 + u \rangle \mid (1 + x)^3 = (1 + y)^3 = (1 + v)^3 = (1 + u)^3 = 1, (1 + u)(1 + v) = (1 + y)(1 + v)(1 + u)\)

and 1 + x, 1 + y commute with every generator \};

(iii) \(V_1 = \{(1 + x)^i(1 + y)^j(1 + v)^k(1 + u)^l \mid 0 \leq i, j, k, l \leq 2\};\)

(iv) \(V_1 = [H] K, \text{ the semidirect product of } H \text{ by } K, \text{ where } H = \langle 1 + x \rangle \times \langle 1 + y \rangle \times \langle 1 + v \rangle\)

and \(K = \langle 1 + u \rangle \text{ or } \langle \sigma \rangle;\)

(v) \(V_1 = W \times \langle 1 + x \rangle \text{ where } W = \langle 1 + u, 1 + v \rangle = \langle 1 + y \rangle \times \langle 1 + v \rangle(1 + u).\)
Proof. Proof of Part (i) directly follows from our earlier discussion. First, we prove part (iv). Observe that \( H = \langle 1 + x, 1 + y, 1 + v \rangle = \langle 1 + x \rangle \times \langle 1 + y \rangle \times \langle 1 + v \rangle \) is an abelian subgroup of the form \( C_3 \times C_3 \times C_3 \) of \( V_1 \), because \( \langle 1 + x \rangle \times \langle 1 + y \rangle = H_1 = Z(V_1) \). It is known that for a finite group \( G \) of order \(|G|\), if \( p \) is the smallest prime such that \( p \) divides \(|G|\), then a subgroup of index \( p \) is normal in \( G \). Hence, \( H \leq V_1 \). Already we have checked that \( (1 + v)(1 + u) \neq (1 + u)(1 + v) \). Hence \( (1 + u) \notin H \). Thus, \( V_1 = HK \) and \( H \cap K = \{1\} \), where \( K = \langle 1 + u \rangle \).

Therefore, \( V_1 = [H] \langle 1 + u \rangle \), the semi direct product of \( H \) and \( \langle 1 + u \rangle \). Further, observe that

\[
(1 + u)(1 + v)(1 + y) = (1 + u + v + uv)(1 + uv - vu)
= 1 + (u + v + uv) + (uv - vu), \text{ since } u^2 = v^2 = 0 \text{ and } uv \in Z(\mathbb{Z}_3S_3)
= 1 + (\sigma - \sigma^2)(1 - \tau) + (\sigma - \sigma^2)(1 + \tau) + 2(1 + \sigma + \sigma^2)
= 1 + 2(\sigma - \sigma^2) + 2(1 + \sigma + \sigma^2)
= 1 + 2(1 + 2\sigma)
= \sigma.
\]

The equation \( \sigma = (1 + u)(1 + v)(1 + y) \) gives that \( \sigma \in \langle (1 + y), (1 + v), (1 + u) \rangle \).

Also \( \sigma(1 + v) \neq (1 + v)\sigma \Rightarrow \sigma \notin H \). Hence, \( \sigma \in [H] \langle 1 + u \rangle \). This proves that \( [H] \langle \sigma \rangle \subseteq [H] \langle 1 + u \rangle \).

For the converse, observe that \( (1 + u)(1 + v) = (1 + y)(1 + v)(1 + u) \).

\[
(1 + y)(1 + v) = (1 + uv - vu)(1 + v)
= 1 + v + (uv - vu) + (uv - vu)v
= 1 + v + uv - vu.
\]

Hence,

\[
(1 + y)(1 + v)(1 + u) = (1 + v + uv - vu)(1 + u)
= 1 + v + uv - vu + u + vu + (uv - vu)u
= 1 + u + v + uv
= (1 + u)(1 + v).
\]

Thus,

\[
(1 + y)(1 + v)^2\sigma = (1 + y)(1 + v)^2(1 + u)(1 + v)(1 + y)
= (1 + y)^2(1 + v)^2\{(1 + u)(1 + v)\}
= (1 + y)^2(1 + v)^2\{(1 + y)(1 + v)(1 + u)\}
= (1 + y)^3(1 + v)^3(1 + u)
= (1 + u).
\]

The equation \( (1 + u) = (1 + y)(1 + v)^2\sigma \) gives that \( 1 + u \in [H] \langle \sigma \rangle \). But, then \( [H] \langle 1 + u \rangle \subseteq [H] \langle \sigma \rangle \). Hence, \( V_1 = [H] \langle 1 + u \rangle = [H] \langle \sigma \rangle \). This proves part (iv).
Now, for part (ii), observe that each of \((1+x), (1+y), (1+v), (1+u)\) is a unit of order 3. Also \((1+u)(1+v) = (1+y)(1+v)(1+u)\) and that \((1+x), (1+y)\) commute with each generator. This proves part (ii) as

\[ V_1 = [H] \langle 1+u \rangle = \langle 1+x, 1+y, 1+u, 1+v \rangle. \]

The canonical form of part (iii) now follows from part (ii). For the proof of the part (v), observe that \(W = \langle 1+u, 1+v \rangle\) is a nonabelian normal subgroup of \(V_1\) of order 27. The following relations can be verified:

\[(1+u)^3 = (1+v)^3 = 1 \text{ and } 1+y = ((1+u), (1+v)) \in \mathcal{Z}(V_1).\]

Hence,

\[ W = \langle 1+u, 1+v \rangle = \langle 1+u, 1+v, 1+y \rangle \]

satisfies the following relations:

\[(1+u)^3 = (1+v)^3 = (1+y)^3 = 1,\]

\[(1+u)(1+v) = (1+v)(1+u)(1+y),\]

\[(1+u)(1+y) = (1+y)(1+u),\]

\[(1+v)(1+y) = (1+y)(1+v).\]

It can be easily seen that \(W = [(1+v, 1+y)](1+u)\), the semidirect product of \((1+v, 1+y)\) by \(1+u\). Further, \(1+x \notin W\) otherwise \(1+x \in \mathcal{Z}(W) = \langle 1+y \rangle\), a contradiction. Hence, \(V_1 = W \times \langle 1+x \rangle\). The proof of the theorem is now complete. \(\square\)

Further, \(V_1\) is a 3-group, \(\tau\) and \(-1\) are units in \(\mathbb{Z}_3S_3\) of order 2, we get \(\tau, -1 \notin V_1\). Also, \(V_1\) is a normal subgroup of \(\mathcal{U}(\mathbb{Z}_3S_3)\) of index 4 with \(\mathcal{U}(\mathbb{Z}_3S_3)/V_1 \cong C_2 \times C_2\). Hence we can explicitly write all the units as follows:

**Theorem 3.5.** The unit group

\[ \mathcal{U}(\mathbb{Z}_3S_3) = [V_1] \langle (-1) \times \langle \tau \rangle \rangle = (\pm V_1) \cup (\pm V_1 \tau) \]

\[ = \{ \pm (1 + \alpha_1 u + \alpha_2 v + \alpha_3 u v + \alpha_4 v u), \]

\[ \pm (1 + \alpha'_1 u + \alpha'_2 v + \alpha'_3 u v + \alpha'_4 v u) \tau \mid \alpha_1, \alpha'_1 \in \mathbb{Z}_3 \}. \]

We can write a presentation of the unit group as follows:

**Theorem 3.6.**

\[ \mathcal{U}(\mathbb{Z}_3S_3) = \{(1+x)^i(1+y)^j\omega_1^k\omega_2^l(-1)^m \tau^n \mid 0 \leq i, j, k, l \leq 2; 0 \leq m, n \leq 1 \}. \]

The canonical form obtained here uses 6 generators. Let \(u_1 = 2+u+v+uv+vu, u_2 = 1+u+v+uv+vu, u_3 = \tau+u+v+uv+vu, \) and \(u_4 = 1+u\). They can be re-written as \(u_1 = -\sigma^2, u_2 = -(1+\sigma^2), u_3 = 1-\sigma^2+\tau, u_4 = 1+(\sigma-\sigma^2)(1-\tau)\). The following relations can be verified:

\[ \omega_1 = u_4^2u_2^2u_3^4u_4^2, \quad \omega_2 = u_4, \quad 1+x = u_2^2u_2, \]

\[ (1+y) = u_2^2u_2^2u_3^4, \quad -1 = u_1^3, \quad \tau = u_2^2u_3u_4. \]
For example
\[
\begin{align*}
u_1^4 &= (-\sigma^2)^4 = \sigma^8 = \sigma^2 \\
u_2^4 &= \{-(1 + \sigma^2)\}^2 = (1 + \sigma^2)^2 = (1 + \sigma + 2\sigma^2) = 1 + \sigma - \sigma^2 \\
u_3^2 &= (1 - \sigma^2 + \tau)^2 = (1 - \sigma^2)^2 + \tau^2 + (1 - \sigma^2)\tau + \tau(1 - \sigma^2) \\
&= (1 + \sigma^4 - 2\sigma^2) + \tau^2 + (1 - \sigma^2)\tau + (1 - \sigma)\tau \\
&= (1 + \sigma + \sigma^2) + 1 + (2 - \sigma - \sigma^2)\tau \\
&= (2 + \sigma + \sigma^2) - (1 + \sigma + \sigma^2)\tau \\
&= 1 + (1 + \sigma + \sigma^2) - (1 + \sigma + \sigma^2)\tau \\
&= 1 + x - y. \\
u_3^4 &= (1 + x - y)^2 = 1 + x^2 + y^2 + 2x - 2y - 2xy \\
&= 1 - x + y = 1 - x + x\tau \\
&\quad \text{since } x, y \in \mathbb{Z}(\mathbb{Z}_3 S_3), \ x^2 = 0, y^2 = 0, \ \text{and } y = x\tau \\
&= 1 - x(1 - \tau) = 1 - (1 + \sigma + \sigma^2)(1 - \tau),
\end{align*}
\]

Since, \((1 - \tau)(\sigma - \sigma^2) = (\sigma - \sigma^2) - (\sigma^2 - \sigma)\tau = (\sigma - \sigma^2)(1 + \tau)\), we get
\[
\begin{align*}
u_2^4 &= \{1 + (\sigma - \sigma^2)(1 - \tau)\}^2 \\
&= 1 + 2(\sigma - \sigma^2)(1 - \tau) + (\sigma - \sigma^2)(1 - \tau)(\sigma - \sigma^2)(1 - \tau) \\
&= 1 + 2(\sigma - \sigma^2)(1 - \tau) = 1 - (\sigma - \sigma^2)(1 - \tau)
\end{align*}
\]

Now,
\[
\begin{align*}
u_1^4\nu_2^2 &= \sigma^2(1 + \sigma - \sigma^2) = \sigma^2 + 1 - \sigma = 1 - \sigma + \sigma^2, \\
\nu_1^4\nu_2^2\nu_3^4 &= (1 - \sigma + \sigma^2)\{1 - (1 + \sigma + \sigma^2)(1 - \tau)\} \\
&= (1 - \sigma + \sigma^2) - (1 - \sigma + \sigma^2)(1 + \sigma + \sigma^2)(1 - \tau) \\
&= (1 - \sigma + \sigma^2) - (1 + \sigma + \sigma^2)(1 - \tau), \\
\frac{\nu_1^4\nu_2^2\nu_3^4}{\nu_1^4\nu_2^2\nu_3^4} &= \{(1 - \sigma + \sigma^2) - (1 + \sigma + \sigma^2)(1 - \tau)\}\{1 - (\sigma - \sigma^2)(1 - \tau)\} \\
&= (1 - \sigma + \sigma^2) - (1 - \sigma + \sigma^2)(\sigma - \sigma^2)(1 - \tau) - (1 + \sigma + \sigma^2)(1 - \tau) \\
&\quad + (1 + \sigma + \sigma^2)(1 - \tau)(\sigma - \sigma^2)(1 - \tau).
\end{align*}
\]

Since, \((1 - \tau)(\sigma - \sigma^2) = (\sigma - \sigma^2)(1 + \tau)\), we get \((1 + \sigma + \sigma^2)(1 - \tau)(\sigma - \sigma^2)(1 - \tau) = 0\). Further \((1 - \sigma + \sigma^2)(\sigma - \sigma^2) = -1 + \sigma^2\).

Combining, we get
\[
\begin{align*}
u_1^4\nu_2^2\nu_3^4\nu_4^2 &= (1 - \sigma + \sigma^2) - (-1 + \sigma^2)(1 - \tau) - (1 + \sigma + \sigma^2)(1 - \tau) \\
&= (1 - \sigma + \sigma^2) - (\sigma - \sigma^2)(1 - \tau) \\
&= 1 - 2(\sigma - \sigma^2) + (\sigma - \sigma^2)\tau \\
&= 1 + (\sigma - \sigma^2) + (\sigma - \sigma^2)\tau = 1 + (\sigma - \sigma^2)(1 + \tau) \\
&= \omega_1.
\end{align*}
\]
Hence, \( u_1^4 u_2^3 u_3^4 u_4^2 = \omega_1 \).

This proves the first relation, namely \( u_1^4 u_2^3 u_3^4 u_4^2 = \omega_1 \). Similarly, other relations can be proved. Hence, \( \mathcal{U}(Z_3 S_3) \subseteq \langle u_1, u_2, u_3, u_4 \rangle \).

Further the following relations can be shown to hold among \( u_i \)’s:

\[
\begin{align*}
& u_1^6 = u_2^6 = u_3^6 = 1, \\
& u_1 u_2 = u_2 u_1, \\
& u_3 u_2 = u_2 u_3, \\
& u_4 u_2 = u_2 u_4, \\
& \text{and that } u_3^1, u_3^2 \text{ commute with each } u_i.
\end{align*}
\]

The group \( \langle u_1, u_2, u_3, u_4 \rangle \) is obviously contained in \( \mathcal{U}(Z_3 S_3) \). We have obtained canonical form presentation of the unit group \( \mathcal{U}(Z_3 S_3) \) as follows:

**Theorem 3.7.** \( \mathcal{U}(Z_3 S_3) = \{ u_1^i u_2^j u_3^k u_4^l \mid 0 \leq i, k \leq 5, 0 \leq j, l \leq 2 \} \), where \( u_1 = -\sigma^2 \), \( u_2 = -(1 + \sigma^2) \), \( u_3 = 1 - \sigma^2 + \tau \), \( u_4 = 1 + (\sigma - \sigma^2)(1 - \tau) \) and they satisfy the following relations:

\[
\begin{align*}
& u_1^6 = u_2^6 = u_3^6 = u_4^6 = 1, \\
& u_1 u_2 = u_2 u_1, \\
& u_3 u_2 = u_2 u_3, \\
& u_4 u_2 = u_2 u_4, \\
& \text{and } u_3^1, u_3^2 \text{ commute with each } u_i.
\end{align*}
\]

We can also write a presentation of the unit group in terms of 3- generators as follows:

**Theorem 3.8.** The unit group

\[
\mathcal{U}(Z_3 S_3) = \langle v_1, v_2, v_3 \mid v_1^6 = v_2^6 = v_3^6 = 1, \ v_3 v_2 = v_1 v_2 v_1 v_3^2, \ v_3 v_1 = v_2 v_1 v_3, \ v_2 v_1 = v_1 v_2 v_1 v_2 v_1 v_2 v_1, \ v_1^3 \text{ and } v_2^2 \text{ commute with each } v_i \rangle.
\]

This can be done by taking \( v_1 = u_1, v_2 = u_3, v_3 = u_4 \) in the presentation given in the earlier theorem.

**References**


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