SECOND ORDER PARALLEL TENSORS ON $\alpha$ – SASAKIAN MANIFOLD

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Abstract. Levy had proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [12] has proved that a second order parallel tensor in a Kaehler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kaehlarian metric and the fundamental 2 – form. In this paper we show that a second order symmetric parallel tensor on an $\alpha$–K contact ($\alpha \in \mathbb{R}_0$) manifold is a constant multiple of the associated metric tensor and we also prove that there is no nonzero skew symmetric second order parallel tensor on an $\alpha$ – Sasakian manifold.

1. Introduction

In 1923, Eisenhart [10] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of the metric tensor is reducible. In 1926, Levy [11] had obtained the necessary and sufficient conditions for the existence of such tensors, Recently Sharma [12] has generalized Levy’s result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an $n$ – dimensional ($n \geq 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [12] that on a Sasakian manifold there is no nonzero parallel 2 – form. In this paper we have considered an almost contact metric manifold and have proved the following two theorems.

Theorem 1.1. On an $\alpha – K$ contact ($\alpha \in R_\alpha$) manifold a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.

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Now the question arises whether there is a skew symmetric second order parallel tensor on a \(\alpha - k\) contact manifold. We do not have an answer to it. However we do have an answer if the manifold is \(\alpha -\) Sasakian where \(\alpha \in \mathbb{R}_0\).

**Theorem 1.2.** On an \(\alpha -\) Sasakian manifold there is no nonzero parallel 2 – forms.

### 2. Preliminaries

A \(C^\infty\) manifold \(M\) of dimension \(2n + 1\) is called a contact manifold if it carries a global 1 – form \(A\) such that \(A \wedge (dA)^n \neq 0\). On a contact manifold there exists a unique vector field \(T\) called the characteristic vector field such that

\[
A(T) = 1, \quad (dA)(T, X) = 0
\]

for any vector field \(X\) on \(M\). By polarization we obtain a Riemannian metric \(g\) called an associated metric and a \((1,1)\) tensor field \(\phi\) on \(M\) such that

\[
\phi^2 = -I + A \otimes T
\]

(2.2)

\[
(dA)(X, Y) = g(X, \phi Y) \\
A(X) = g(X, T)
\]

for the arbitrary vector fields \(X\) and \(Y\) on \(M\). If in addition to (2.1) and (2.2), \(M^n\) admits a positive definite Riemannian metric \(g\) such that

\[
g(\phi X, \phi Y) = g(X, Y) - A(X)A(Y)
\]

(2.3)

\[
\phi(T) = 0, \quad A(\phi(X)) = 0, \forall X, Y \in \mathcal{X}(M) \\
\text{and rank}(\phi) = 2n \text{ everywhere on } M.
\]

Such a manifold satisfying (2.1), (2.2), and (2.3) is called an almost contact metric manifold. The structure endowed in \(M\) is called \((\phi, A, T, g)\) – structure.

For a \((\phi, A, T, g)\) – structure, the skew symmetric bilinear form

\[
\Phi(X, Y) = g(X, \phi Y)
\]

(2.4)

is called the fundamental 2 – form of the almost contact metric structure.

### 3. Some Definitions and Theorems

**Definition 3.1.** An almost contact metric structure is said to be an \(\alpha -K\) contact structure if the vector field \(T\) is killing with respect to \(g\).

In proving Theorems 1.1 and 1.2, we need the following theorems.

**Theorem 3.1.** On an \(\alpha -K\) contact structure the following holds.

\[
\nabla_X T = -\alpha \phi X \text{ for all } X \in \mathcal{X}(M)
\]

where \(\nabla\) is the Riemannian connection of \(g\).
**Theorem 3.2.** An almost contact metric structure \((\phi, A, T, g)\) is \(\alpha\)–Sasakian iff

\[(\nabla_x \phi) Y = \alpha \{ g(X, Y) T - A(Y) X \} \]

where \(\nabla\) denotes the Riemannian connection of \(g\).

**Proof.** The proofs of the above theorems follows in a similar fashion as in the Theorem 6.3 by Blair [3]. \(\square\)

**Definition 3.2 ([2]).** An almost \(\alpha\)–Sasakian manifold \(M\) is an almost contact metric manifold such that \(\phi(X, Y) = \frac{1}{\alpha} d\eta(X, Y), \alpha \in R_0\) and \(M\) is a \(\alpha\)–Sasakian manifold if the structure is normal.

**Theorem 3.3.** An almost contact metric manifold \(M\) is \(\alpha\)–Sasakian manifold iff for all \(X, Y \in X(M)\)

\[(3.3) R(X, Y) T = \alpha \{ A(Y) X - A(X) Y \} \]

**Proof.** The proof of the above theorem follows in view of Lemma 6.1 of Blair [3].

The two conditions of being normal and contact metric may be written as the following:

\[(3.4) R(T, X) Y = \alpha \{ g(X, Y) T - A(Y) X \} \]

\(\square\)

**Theorem 3.4.** For an \(\alpha\)–K contact manifold we have

\[(3.5) R(T, X) T = \alpha \{ -X + A(X) T \} \]

**Proof.** In view of (3.4), the proof follows immediately. \(\square\)

For a detailed study on a contact manifold the reader is referred to [2].

4. **Proofs of Theorems 1.1 and 1.2**

**Proof of Theorem 1.1.** Let \(h\) denote a \(0, 2\)–tensor field on an \(\alpha\)–K contact manifold \(M\) such that \(\nabla h = 0\). Then it follows that

\[(4.1) \ h(R(W, X) Y, Z) + h(Y, R(W, X) Z) = 0 \]

for arbitrary vector fields \(X, Y, Z, W\) on \(M\).

We write (4.1) as follows

\[g(R(W, X) Y, Z) + g(Y, R(W, X), Z) = 0.\]

On substituting \(W = Y = Z = T\) in (4.1) we get:

\[(4.2) g(R(T, X) T, T) + g(T, R(T, X), T) = 0.\]

In view of Theorem (3.4), the above equation becomes:

\[(4.3) g(-\alpha X + \alpha A(X) T, T) + g(T, -\alpha X + \alpha A(X) T) = 0.\]
In this equation, using (2.2) we get
\[ 2\alpha g (X, T) \ h(T, T) - \alpha h(X, T) - \alpha h(T, X) = 0. \]  
Differentiating (4.4) covariantly with respect to \( Y \) and using Theorem (3.1) we get
\[ 2\alpha h(T, T) \ g(\nabla_Y X, T) - 2\alpha^2 h(T, T) g(X, \phi Y) - \alpha g(\nabla_Y X, T) + \alpha^2 g(X, \phi Y) + \alpha^2 g(\phi Y, X) - \alpha g(T, \nabla_Y X) = 0. \]
Replacing \( Y \) by \( \phi Y \) and using equations (2.2), (2.3) and (4.4) we obtain
\[ h(X, Y) + h(Y, X) = 2h(T, T) g(X, Y). \]
But \( h \) is symmetric, thus on simplifying the above equation we get
\[ 2h(T, T) g(X, Y) = 2h(X, Y). \]
In view of the fact that \( h(T, T) \) is constant by differentiating it along any vector on \( M^{2n+1} \) we get
\[ h(T, T) g(X, Y) = h(X, Y) \]
which completes the proof.

Proof of Theorem 1.2. Let us consider \( h \) to be a parallel 2–form on an \( \alpha \)–Sasakian manifold \( M^{2n+1} \) and let \( H \) be a (1,1) tensor field metrically equivalent to \( h \) since \( h(X, Y) = g(HX, Y) \). Now (4.1) can be written as
\[ g(R(W, X) Y, Z) + g(Y, R(W, X) Z) = 0. \]
Let us put \( X = Y = T \) in (4.7) and using the fact that \( h(X, Y) = g(HX, Y) \) we get
\[ g(HR(W, T) T, Z) + g(HT, R(WT) Z) = 0. \]
Applying the skew symmetric property of \( R(X, Y) \) and using (3.3) and (3.4) in (4.8) and after simplifying, we obtain
\[ \alpha g(\nabla_X T) T + \alpha g(\nabla_Z T) HT = \alpha HZ. \]
Differentiating (4.9) along \( \phi X \) we obtain
\[ 2\alpha A(X) A(HZ) T - \alpha g(HZ, X) T - \alpha g(HZ, T) X = \alpha g(Z, X) HT - 2\alpha A(X) A(Z) HT + \alpha A(Z) HX. \]
Let \( \{e_i\}, i = 1, 2, \ldots, 2n + 1 \) be an orthonormal basis of the tangent space. In the above equation (4.10), we substitute \( X = e_i \) and take the inner product with \( e_i \) and eventually summing over \( i \) gives us
\[ \alpha (2n - 1) g(HZ, T) = 0. \]
Since \( \alpha (2n - 1) \neq 0 \), we have \( g(HZ, T) = 0 \). But \( g(HZ, T) = -g(HT, Z) \). Thus, \( HT = 0 \) and hence (4.9) shows that \( H = 0 \), which completes the proof.
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