FIXED POINTS THEOREMS FOR \( n \)-VALUED MULTIFUNCTIONS

ABDELKADER STOUTI AND ABDELHAKIM MAADEN

Abstract. We first show that if \( Y \) is a nonempty AR space and \( F: Y \to Y \) is a compact \( n \)-valued multifunction, then \( F \) has at least \( n \) fixed point. We also prove that if \( C \) is a nonempty closed convex subset of a topological vector space \( E \) and \( F: C \to C \) is a continuous \( \Phi \)-condensing \( n \)-valued multifunction, then \( F \) has at least \( n \) fixed points.

1. Introduction and preliminaries

Let \( X \) and \( Y \) be two Hausdorff topological spaces.

A multifunction \( F: X \to Y \) is a map from \( X \) into the set \( 2^Y \) of nonempty subsets of \( Y \). The range of \( F \) is \( F(X) = \bigcup_{x \in X} F(x) \).

The multifunction \( F: X \to Y \) is said to be upper semi-continuous (usc) if for each open subset \( V \) of \( Y \) with \( F(x) \subset V \) there exists an open subset \( U \) of \( X \) with \( x \in U \) and \( F(U) \subset V \).

The multifunction \( F: X \to Y \) is called lower semi-continuous (lsc) if for every \( x \in X \) and open subset \( V \) of \( Y \) with \( F(x) \cap V \neq \emptyset \) there exists an open subset \( U \) of \( X \) with \( x \in U \) and \( F(x') \cap V \neq \emptyset \) for all \( x' \in U \).

A multifunction \( F: X \to Y \) is continuous if it is both upper semi-continuous and lower semi-continuous.

A multifunction \( F: X \to Y \) is compact if it is continuous and the closure of its range \( \overline{F(X)} \) is a compact subset of \( Y \).

A point \( x \) of \( X \) is said to be a fixed point of a multifunction \( F: X \to X \) if \( x \in F(x) \). We denote by \( \text{Fix}(F) \) the set of all fixed points of \( F \).

A multifunction \( F: X \to Y \) is said to be \( n \)-valued if for all \( x \in X \), the subset \( F(x) \) of \( Y \) consists of \( n \) points.

2000 Mathematics Subject Classification. 46A55, 52A07, 54H25.

Key words and phrases. AR spaces, \( n \)-valued multifunction, convex set, fixed point, \( \Phi \)-condensing multifunction.
A multifunction $F: X \to X$ is said to be an $n$-function if there exist $n$ continuous maps $f_i: X \to X$, where $i = 1, \ldots, n$, such that $F(x) = \{f_1(x), \ldots, f_n(x)\}$ and $f_i(x) \neq f_j(x)$ for all $x \in X$ and $i, j = 1, \ldots, n$ with $i \neq j$.

In this work, we shall use the following result due to H. Schirmer [11].

**Lemma 1.1.** [11]. Let $X$ and $Y$ be two compact Hausdorff topological spaces. If $X$ is path and simply connected and $F: X \to Y$ is a continuous $n$-valued multifunction, then $F$ is an $n$-function.

In [1], Borsuk first introduced the notion of AR spaces (for the general theory see [1, 2]).

**Definition 1.2.** [1, 2]. A space $Y$ is called an absolute retract space whenever

(i) $Y$ is metrizable and 
(ii) for any metrizable space $X$ and closed subset $A$ of $X$ each continuous map $f: A \to Y$ is extendable over $X$. The class of absolute retracts is denoted by AR.

By Dugundji’s extension Theorem [4], we know that every nonempty convex subset of a Banach space is an AR space. In [1], it is shown that every union of two AR spaces, which their intersection is an AR space is also an AR space. Recently, in [9], Park established the following result.

**Theorem 1.3.** [9]. Every nonempty compact convex subset of a metrizable topological vector space is an AR space.

In infinite dimension topology the Hilbert cube $I^\infty$ is an important tool. It is defined by

$$I^\infty = \left\{ (x_1, x_2, x_3, \ldots) : x_i \in \mathbb{R} \text{ and for all } i \in \mathbb{N}^*, |x_i| \leq \frac{1}{i} \right\}.$$

In [1], Borsuk proved the following result.

**Theorem 1.4.** [1]. Let $K$ be a nonempty compact metric space. Then, there is a closed subset $K_1$ of the Hilbert cube $I^\infty$ and a homeomorphic map $h: K \to K_1$.

In [11], Schirmer studied the fix-finite approximation property for $n$-valued multifunction defined on finite polyhedron. Later on, in [12, 13], the first author established some results concerning the fix-finite approximation property for $n$-valued multifunction defined in normed spaces and metrizable locally convex spaces. In the present work we are interesting to study the existence of fixed point of continuous $n$-valued multifunctions.

In [5, Theorem 10.8, p.94], one can find the proof of the generalized Schauder fixed point theorem.

**Theorem 1.5.** [5]. Let $Y$ be a nonempty AR space. Then, every compact map $f: Y \to Y$ has a fixed point.
In this note, we first prove that if $Y$ is an absolute retract and $F: Y \to Y$ is a compact $n$-valued multifunction, then $F$ has at least $n$ fixed points (see Theorem 2.1). That is a generalization of the generalized Schauder fixed point theorem [5]. By using the properties of AR spaces [1, 2], we shall show that if $C_i$, for $i = 1, \ldots, m$, is a finite family of nonempty convex compact subsets of a metrizable topological vector space such that $\cap_{i=1}^{m} C_i \neq \emptyset$, then every continuous $n$-valued multifunction $F: \bigcup_{i=1}^{m} C_i \to \bigcup_{i=1}^{m} C_i$ has at least $n$ fixed points (see Theorem 2.2).

The notion of measure of noncompactness was first introduced by Kuratowski in [6]. In Banach spaces he defined the set-measure of noncompactness, $\alpha$, as follows:

$$\alpha(A) = +\infty, \text{ if } A \text{ is unbounded. and if } A \text{ is bounded, then}$$

$$\alpha(A) = \inf\{d > 0 : A \text{ can be covered with finite number of sets of diameter less than } d\}.$$

Analogously, Gokhberg, Goldenstein and Markus (see Lloyd [7], Ch. 6) introduced the ball measure of noncompactness $\beta$. The notion of measure of noncompactness in the following definition is a generalization of the measure of noncompactness $\alpha$ and $\beta$ defined in terms of a family of seminorms or a norm.

Definition 1.6. Let $E$ be a topological vector space and $L$ be a lattice with a least element, which is denoted by 0. A function $\Phi: E \to L$ is called a measure of noncompactness on $E$ provided that the following conditions hold for any $X, Y \in 2^E$:

1. $\Phi(X) = 0$ if and only if $\overline{X}$ is compact,
2. $\Phi(\overline{co}X) = \Phi(X)$, where $\overline{co}$ denotes the convex closure of $X$,
3. $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

Definition 1.7. For $X \subset E$, a multifunction $F: X \to E$ is said to be $\Phi$-condensing provided that if $A \subset X$ and $\Phi(A) \leq \Phi(F(A))$, then $A$ is relatively compact; that is, $\Phi(A) = 0$.

Note that every multifunction defined on a compact set is $\Phi$-condensing.

In 2001, Cauty [3] obtained the affirmative solution of the Schauder conjecture as follows:

Theorem 1.8. [3]. Let $E$ be a Hausdorff topological vector space, $C$ a nonempty convex subset of $E$, and $f$ a continuous map from $C$ into $C$. If $f(C)$ is contained in a compact subset of $C$, then $f$ has a fixed point.

By using the last result, we prove that if $C$ is a nonempty closed convex subset of a Hausdorff topological vector space $E$ and $F: C \to C$ is a continuous $\Phi$-condensing $n$-valued multifunction, then $F$ has at least $n$ fixed points (see Theorem 2.5).
2. The Results

In this section, we shall establish some fixed point results for \( n \)-valued multifunctions. First, we shall show the following.

**Theorem 2.1.** Let \( Y \) be a nonempty AR space. Then, every compact \( n \)-valued multifunction \( F : Y \to Y \) has at least \( n \) fixed points.

**Proof.** Let \( Y \) be a nonempty AR space and \( F : Y \to Y \) be a compact \( n \)-valued multifunction. Let \( K = F(Y) \). Since \( K \) is a compact metric space, then by Theorem 1.4, there exists a closed subset \( K_1 \) of \( I^\infty \) and a homeomorphism \( h : K \to K_1 \). Let \( i : K \to Y \) and \( j : K_1 \to I^\infty \) be the inclusion maps. Then, the map \( i \circ h^{-1} : K_1 \to Y \) is continuous. From this and as \( K_1 \) is a closed subset of \( I^\infty \) and \( Y \) is an AR space, then there exists a continuous map \( g : I^\infty \to Y \) which extends the map \( i \circ h^{-1} \). Now, set \( G = j \circ h \circ F : Y \to I^\infty \).

**Claim 1.** The multifunction \( G : Y \to I^\infty \) is a \( n \)-valued continuous multifunction. Indeed, if \( x \in Y \), then \( F(x) = \{y_1, \ldots, y_n\} \) and \( y_i \neq y_j \) for all \( i, j = 1, \ldots, n \) with \( i \neq j \). So, we have

\[
G(x) = j(h(\{y_1, \ldots, y_n\})) = j(\{h(y_1), \ldots, h(y_n)\}) = \{h(y_1), \ldots, h(y_n)\}.
\]

As \( h \) is a homeomorphism, hence for every \( x \in Y \) the set \( G(x) \) has exactly \( n \) elements. Thus, \( G \) is an \( n \)-valued continuous multifunction and our claim is proved.

**Claim 2.** We have: \( F = g \circ G \). Indeed, if \( x \in Y \), then \( F(x) = \{y_1, \ldots, y_n\} \) and \( y_i \neq y_j \) for all \( i, j = 1, \ldots, n \) with \( i \neq j \). Then, we obtain,

\[
g(G(x)) = g(\{h(y_1), \ldots, h(y_n)\}) = \{g(h(y_1)), \ldots, g(h(y_n))\}.
\]

On the other hand, we know that for every \( i \in \{1, \ldots, n\} \), we have \( h(y_i) \in K_1 \). From this and as \( g/K_1 = i \circ h^{-1} \), then for every \( i \in \{1, \ldots, n\} \), we get

\[
g(h(y_i)) = i \circ h^{-1}(h(y_i)) = y_i.
\]

Therefore, \( F = g \circ G \) and our claim is proved.

**Claim 3.** The multifunction \( H = G \circ g : I^\infty \to I^\infty \) has at least \( n \) fixed point. Indeed, since \( G \) is an \( n \)-valued multifunction, then \( H \) is an \( n \)-valued multifunction. On the other hand \( G \) and \( g \) are continuous, so \( H \) is continuous. Since \( I^\infty \) is compact convex set, then by Lemma 1.1 \( H \) is an \( n \)-function. Hence, there exist \( n \) continuous maps \( h_i : I^\infty \to I^\infty \), where \( i = 1, \ldots, n \), such that \( H(x) = \{h_1(x), \ldots, h_n(x)\} \) and \( h_i(x) \neq h_j(x) \) for all \( x \in I^\infty \) and \( i, j = 1, \ldots, n \) with \( i \neq j \). By using the Schauder fixed point theorem [5], we deduce that we have \( Fix(h_i) \neq \emptyset \), for every \( i \in \{1, \ldots, n\} \). From this and as \( Fix(h_i) \cap Fix(h_j) = \emptyset \) for \( i, j = 1, \ldots, n \) and \( i \neq j \), we deduce that \( Fix(H) = \cup_{i=1}^n Fix(h_i) \), then \( H \) has at least \( n \) fixed points.

**Claim 4.** The multifunction \( F \) has at least \( n \) fixed point. Indeed, if \( x \) is a fixed point of \( H \), then \( g(x) \in (g \circ G)(g(x)) \). On the other hand, by Claim 2, we know that we have \( F = g \circ G \). Then,

\[
x \in Fix(H) \Rightarrow x \in H(x) \Rightarrow g(x) \in F(g(x)) \Rightarrow g(x) \in Fix(F).
\]
Thus, we have

\[ g(Fix(H)) \subseteq Fix(F). \]

Now, let \( x_i, x_j \in Fix(H) \) with \( i, j = 1, \ldots, n, i \neq j \) and \( x_i \neq x_j \). Let \( F(g(x_i)) = \{ z_i^1, \ldots, z_i^n \} \) and \( F(g(x_j)) = \{ z_j^1, \ldots, z_j^n \} \). As \( H = G \circ g \) and \( G = j \circ h \circ F \), then we have

\[ H(x_i) = \{ h(z_i^1), \ldots, h(z_n^n) \} \text{ and } H(x_j) = \{ h(z_j^1), \ldots, h(z_j^n) \}. \]

Since, \( x_i, x_j \in Fix(H) \), so there is \( k, l \in \{1, \ldots, n\} \) such that

\[ x_i = h(z_k^1) \text{ and } x_j = h(z_l^1). \]

From this and as \( h(z_k^1), h(z_l^1) \in K_i \) and \( g/\kappa_i = i \circ h^{-1} \), then we get

\[ g(x_i) = g(h(z_k^1)) = z_k^i = h^{-1}(x_i) \text{ and } g(x_j) = g(h(z_l^1)) = z_l^j = h^{-1}(x_j). \]

As \( x_i \neq x_j \) and \( h \) is a homeomorphism, hence we get \( g(x_i) \neq g(x_j) \) for \( i, j = 1, \ldots, n \) and \( i \neq j \). By Claim 3, we know that the set \( Fix(H) \) has at least \( n \) elements, so \( g(Fix(H)) \) has also at least \( n \) elements. On the other hand, we know that \( g(Fix(H)) \subseteq Fix(F) \). Therefore, \( F \) has at least \( n \) fixed points. \( \square \)

For finite unions of closed convex subsets of a metrizable topological vector space, we obtain the following result.

**Theorem 2.2.** Let \( C_i \), for \( i = 1, \ldots, m \), be a finite family of nonempty compact convex subsets of a metrizable topological vector space such that \( \bigcap_{i=1}^{m} C_i \neq \emptyset \). Then, every continuous \( n \)-valued multifunction \( F: \bigcup_{i=1}^{m} C_i \to \bigcup_{i=1}^{m} C_i \) has at least \( n \) fixed points.

**Proof.** Let \( C = \bigcup_{i=1}^{m} C_i \) and let \( F: C \to C \) be a continuous \( n \)-valued multifunction. By Theorem 1.3, we know that every nonempty convex subset of a metrizable topological vector space is an AR space. In addition, it is shown in [2] that every union of two AR, which their intersection is an AR is also an AR. From this it follows that \( C \) is an AR space. By using Theorem 2.1, we deduce that \( F \) has at least \( n \) fixed points in \( C \). \( \square \)

**Remark 2.3.** In Theorem 2.2, the condition \( \bigcap_{i=1}^{m} C_i \neq \emptyset \) is essential. Because if it is not the case, then there exists at least a continuous \( n \)-valued multifunction \( F: \bigcup_{i=1}^{m} C_i \to \bigcup_{i=1}^{m} C_i \) which is fixed free. Indeed, let \( C_1 = \overline{B((0,1), \frac{1}{2})} \) and \( C_2 = \overline{B((0,-1), \frac{1}{2})} \) be two compact convex in the Banach space \( \mathbb{R}^2 \) and let \( f: C_1 \cup C_2 \to C_1 \cup C_2 \) the continuous map defined by \( f(x) = -x \). If \( f(x) = x \), then \( x = 0 \). That is not possible. Therefore the map \( f \) is fixed point free.

Next, we shall show the following result.

**Theorem 2.4.** Let \( C \) be a nonempty closed convex subset of a Hausdorff topological vector space and \( F: C \to C \) a continuous \( \Phi \)-condensing \( n \)-multifunction. Then, \( F \) has at least \( n \) fixed points.

To prove Theorem 2.4, we recall the following result.
Lemma 2.5. [8]. Let $C$ be a nonempty closed convex subset of a topological vector space $E$, and $F: C \to C$ be a $\Phi$-condensing multifunction. Then, there exists a nonempty compact convex subset $K$ of $C$ such that $F(K) \subset K$.

Combining Theorems 1.3 and 1.8 and Lemma 2.5, we obtain the proof of Theorem 2.4.

Proof of Theorem 2.4. Let $C$ be a nonempty closed convex subset of a Hausdorff topological vector space and $F: C \to C$ be a continuous $\Phi$-condensing $n$-multifunction. By Lemma 2.5, there exists a nonempty compact convex subset $K$ of $C$ such that $F(K) \subset K$. From this and by using Lemma 1.1 and Theorems 1.3 and 1.8, we conclude that $F$ has at least $n$ fixed points. \qed

As a consequence of Theorem 2.4, we obtain the following result.

Corollary 2.6. Let $C$ be a nonempty closed convex subset of a Hausdorff topological vector space and $F: C \to C$ be a compact $n$-valued multifunction. Then, $F$ has at least $n$ fixed points.

References


Received 27 November, 2005.
Abelkader Stouti, Laboratoire de Mathématiques et Applications, UFR : Méthodes Mathématiques et Applications, Faculty of Sciences and Techniques, University Sultan Moulay Soulayman, P.O. Box 523. Beni-Mellal 23000, Morocco

E-mail address: stouti@yahoo.com

Abdelhakim Maaden, Laboratoire de Mathématiques et Applications, UFR : Méthodes Mathématiques et Applications, Faculty of Sciences and Techniques, University Sultan Moulay Soulayman, P.O. Box 523. Beni-Mellal 23000, Morocco

E-mail address: a.maaden@fstbm.ac.ma