ON \( \pi \)-IMAGES OF METRIC SPACES

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Abstract. In this paper, we prove that sequence-covering, \( \pi \)-images of metric spaces and spaces with a \( \sigma \)-strong network consisting of \( fcs \)-covers are equivalent. We also investigate \( \pi \)-images of separable metric spaces.

1. Introduction

A study of images of metric spaces is an important question in general topology ([2, 7, 9, 10, 16]). In recent years, \( \pi \)-images of metric spaces cause attention once again ([4, 13, 18, 19]). It is known that a space is a strong-sequence-covering (resp. sequentially-quotient), \( \pi \)-image of a metric space if and only if it has a \( \sigma \)-strong network consisting of \( cs \)-covers (resp. \( cs^* \)-covers) (see [13], for example). Note that strong-sequence-covering mapping \( \Rightarrow \) sequence-covering mapping \( \Rightarrow \) (if the domain is metric) sequentially-quotient mapping and that \( cs \)-cover \( \Rightarrow \) \( fcs \)-cover \( \Rightarrow \) \( cs^* \)-cover. It is natural to raise the following question.

Question 1.1. Can sequence-covering, \( \pi \)-images of metric spaces be characterized as spaces with a \( \sigma \)-strong network consisting of \( fcs \)-covers?

On the other hand, whether sequentially-quotient, \( \pi \)-images of metric spaces and sequence-covering, \( \pi \)-images of metric spaces are equivalent? This question is still open (see [13, Question 3.1.14] or [19, Question 4.4(2)], for example). This leads us to consider the following question.

Question 1.2. Are sequentially-quotient, \( \pi \)-images of separable metric spaces and sequence-covering, \( \pi \)-images of separable metric spaces equivalent?

In this paper, we give a positive answer for Question 1.1. We also investigate \( \pi \)-images of separable metric spaces, and answer Question 1.2 affirmatively.

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Throughout this paper, all spaces are assumed to be Hausdorff, and all mappings are continuous and onto. \( \mathbb{N} \) denotes the set of all natural numbers, \( \{x_n\} \) denotes a sequence, where the \( n \)-th term is \( x_n \). Let \( X \) be a space and let \( A \) be a subset of \( X \). We say that a sequence \( \{x_n\} \) converging to \( x \) in \( X \) is eventually in \( A \) if \( \{x_n : n > k\} \cup \{x\} \subset A \) for some \( k \in \mathbb{N} \). Let \( \mathcal{P} \) be a family of subsets of \( X \) and let \( x \in X \). \( \bigcup \mathcal{P} \), \( st(x, \mathcal{P}) \) and \( (\mathcal{P})_x \) denote the union \( \bigcup \{P : P \in \mathcal{P}\} \), the union \( \bigcup \{P \in \mathcal{P} : x \in P\} \) and the subfamily \( \{P \in \mathcal{P} : x \in P\} \) of \( \mathcal{P} \) respectively. For a sequence \( \{\mathcal{P}_n : n \in \mathbb{N}\} \) of covers of a space \( X \), we abbreviate \( \{\mathcal{P}_n : n \in \mathbb{N}\} \) to \( \mathcal{P}_n \). A point \( b = (\beta_n)_{n \in \mathbb{N}} \) of a Tychonoff-product space is abbreviated to \( (\beta_n) \), where \( \beta_n \) is the \( n \)-th coordinate of \( b \). If \( f : X \rightarrow Y \) is a mapping, then \( f(\mathcal{P}) \) denotes \( \{f(P) : P \in \mathcal{P}\} \).

2. \( \pi \)-Images of Metric Spaces

**Definition 2.1.** Let \( f : X \rightarrow Y \) be a mapping.

1. \( f \) is called a strong-sequence-covering mapping ([11]) if for every convergent sequence \( S \) in \( Y \), there exists a convergent sequence \( L \) in \( X \) such that \( f(L) = S \).

2. \( f \) is called a sequence-covering mapping ([6]) if for every sequence \( S \) converging to \( y \) in \( Y \), there exists a compact subset \( K \) of \( X \) such that \( f(K) = S \cup \{y\} \).

3. \( f \) is called a sequentially-quotient mapping ([1]) if for every convergent sequence \( S \) in \( Y \), there exists a convergent sequence \( L \) in \( X \) such that \( f(L) \) is a subsequence of \( S \).

4. \( f \) is called a compact-covering mapping([15]) if for every compact subset \( C \) of \( Y \), there exists a compact subset \( K \) of \( X \) such that \( f(K) = C \).

5. \( f \) is called a \( \pi \)-mapping ([16]), if for every \( y \in Y \) and for every neighborhood \( U \) of \( y \) in \( Y \), \( d(f^{-1}(y), X - f^{-1}(U)) > 0 \), where \( X \) is a metric space with a metric \( d \).

**Definition 2.2.** Let \( \mathcal{P} \) be a cover of a space \( X \).

1. \( \mathcal{P} \) is called an \( fcs \)-cover of \( X \) ([5]) if for every sequence \( S \) converging to \( x \) in \( X \), there exists a finite subfamily \( \mathcal{P}' \) of \( \mathcal{P} \) such that \( S \) is eventually in \( \bigcup \mathcal{P}' \).

2. \( \mathcal{P} \) is called a \( cs^* \)-cover ([13]) if for every convergent sequence \( S \) in \( X \), there exist \( P \in \mathcal{P} \) and a subsequence \( S' \) of \( S \) such that \( S' \) is eventually in \( P \).

**Definition 2.3.** (1) Let \( \mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\} \) be a cover of a space \( X \), where \( \mathcal{P}_x \subset (\mathcal{P})_x \). \( \mathcal{P} \) is called a network of \( X \) ([15]), if for every \( x \in U \) with \( U \) open in \( X \), there exists \( P \in \mathcal{P}_x \) such that \( x \in P \subset U \), where \( \mathcal{P}_x \) is called a network at \( x \) in \( X \).

2. Let \( \{\mathcal{P}_n\} \) be a sequence of covers of a space \( X \) and every \( \mathcal{P}_{n+1} \) is an refinement of \( \mathcal{P}_n \). \( \mathcal{P} = \bigcup \{\mathcal{P}_n : n \in \mathbb{N}\} \) is called a \( \sigma \)-strong network ([8]), if \( \{st(x, \mathcal{P}_n)\} \) is a network at \( x \) in \( X \) for every \( x \in X \).
(3) A $\sigma$-strong network $\mathcal{P} = \bigcup\{P_n : n \in \mathbb{N}\}$ is called a $\sigma$-strong network consisting of (countable) fcs-covers (resp. cs*-covers) if $\mathcal{P}_n$ is a (countable)
$fcs$-cover (resp. cs*-cover) for every $n \in \mathbb{N}$.

(4) A $\sigma$-strong network $\mathcal{P} = \bigcup\{P_n : n \in \mathbb{N}\}$ is called a $\sigma$-point-countable strong network if $\mathcal{P}_n$ is point-countable for every $n \in \mathbb{N}$.

**Theorem 2.4.** For a space $X$, the following are equivalent.

1. $X$ is a sequence-covering, $\pi$-image of a metric space.
2. $X$ has a $\sigma$-strong network consisting of fcs-covers.

**Proof.** (1)$\implies$(2): Let $M$ be a metric space with a metric $d$, and let $f : M \to X$ be a sequence-covering, $\pi$-mapping. We write $B(a, \varepsilon) = \{b \in M : d(a, b) < \varepsilon\}$ for every $a \in M$, where $\varepsilon > 0$. For every $n \in \mathbb{N}$, put $B_n = \{B(a, 1/n) : a \in M\}$, and put $P_n = f(B_n)$, then $P_n$ is a cover of $X$.

Claim 1. $\mathcal{P} = \bigcup\{P_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network of $X$.

It is clear that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for every $n \in \mathbb{N}$. We only need to prove that $\{st(x, P_n)\}$ is a network at $x$ in $X$ for every $x \in X$. Let $x \in U$ with $U$ open in $X$. Since $f$ is a $\pi$-mapping, there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) > 1/n$. Pick $m \in N$ such that $m > 2n$. It suffices to prove that $st(x, P_m) \subset U$. Let $a \in M$ and let $x \in f(B(a, 1/m)) \in P_m$.

We claim that $B(a, 1/m) \subset f^{-1}(U)$. If $B(a, 1/m) \not\subset f^{-1}(U)$, then $d(f^{-1}(x), M - f^{-1}(U)) = 0$, pick $c \in f^{-1}(x) \cap B(a, 1/m) \neq \emptyset$, then $d(f^{-1}(x), M - f^{-1}(U)) = d(c, b) - d(c, a) + d(a, b) < 2/m < 1/n$. This is a contradiction. So $B(a, 1/m) \subset f^{-1}(U)$, thus $f(B(a, 1/m)) \subset f^{-1}(U) = U$. This proves that $st(x, P_m) \subset U$.

Claim 2. $\mathcal{P}_n$ is an fcs-cover of $X$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Suppose $S$ is a sequence converging to $x$ in $X$. Since $f$ is sequence-covering, there exists a compact subset $K$ in $M$ such that $f(K) = S \cup \{x\}$. Note that $f^{-1}(x) \cap K$ is compact in $M$. There exists a finite subset $M$ of $M$ such that $f^{-1}(x) \cap K \subset \bigcup_{a \in M} B(a, 1/n)$. We can assume that $f^{-1}(x) \cap B(a, 1/n) \neq \emptyset$ for every $a \in M$.

Put $B = \{B(a, 1/n) : a \in M\}$ and $B = \bigcup B$, then $K - B$ is compact in $M$. Put $\mathcal{P} = \{f(B(a, 1/n)) : a \in M\}$. Then $\mathcal{P}$ is a finite subfamily of $(\mathcal{P}_n)$x. We prove that $S$ is eventually in $\bigcup \mathcal{P}$ as follows. If not, there exists a subsequence $\{x_k\}$ of $S$ converging to $x$ such that $x_k \not\in \bigcup \mathcal{P}'$ for every $k \in \mathbb{N}$. Thus there exists $a_k \in K - B$ such that $f(a_k) = x_k$ for every $k \in \mathbb{N}$. Since $K - B$ is compact in $M$, there exists a subsequence $\{a_{k_n}\}$ of $\{a_k\}$ such that the sequence $\{a_{k_n}\}$ converges to a point $a \in K - B$. Thus $f(a) \neq x$. This contradicts the continuity of $f$. So $S$ is eventually in $\bigcup \mathcal{P}'$. This proves that $\mathcal{P}_n$ is an fcs-cover of $X$.

By the above, $X$ has a $\sigma$-strong network $\mathcal{P} = \bigcup\{P_n : n \in \mathbb{N}\}$ consisting of fcs-covers.

(2)$\implies$(1): Let $X$ have a $\sigma$-strong network $\mathcal{P} = \bigcup\{P_n : n \in \mathbb{N}\}$ consisting of fcs-covers. For every $n \in \mathbb{N}$, put $P_n = \{P_a : a \in \Lambda_n\}$, and $\Lambda_n$ is endowed
with discrete topology. Put

\[ M = \{ a = (\alpha_n) \in \Pi_{n \in \mathbb{N}} \Lambda_n : \{ P_{\alpha_n} \} \text{ is a network at some } x_a \text{ in } X \}. \]

Then \( M \), which is a subspace of the product space \( \Pi_{n \in \mathbb{N}} \Lambda_n \), is a metric space with metric \( d \) described as follows.

Let \( a = (\alpha_n), b = (\beta_n) \in M \). If \( a = b \), then \( d(a, b) = 0 \). If \( a \neq b \), then
\[ d(a, b) = 1/\min\{ n \in \mathbb{N} : \alpha_n \neq \beta_n \}. \]

Define \( f : M \rightarrow X \) by choosing \( f(a) = x_a \) for every \( a = (\alpha_n) \in M \), where \( \{ P_{\alpha_n} \} \) is a network at \( x_a \) in \( X \). It is not difficult to check that \( f \) is continuous and onto.

Claim 1. \( f \) is a \( \pi \)-mapping.

Let \( x \in U \) with \( U \) open in \( X \). Since \( \{ P_n \} \) is a \( \sigma \)-strong network of \( X \), there exists \( n \in \mathbb{N} \) such that \( st(x, P_n) \subset U \). Then \( d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0 \). In fact, if \( a = (\alpha_n) \in M \) such that \( d(f^{-1}(x), a) < 1/2n \), then there is \( b = (\beta_n) \in f^{-1}(x) \) such that \( d(a, b) < 1/n \), so \( \alpha_k = \beta_k \) if \( k \leq n \).

Notice that \( x \in P_{\beta_n} \in P_n, P_{\alpha_n} = P_{\beta_n} \), so \( f(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, P_n) \subset U \), hence \( a \in f^{-1}(U) \). Thus \( d(f^{-1}(x), a) \geq 1/2n \) if \( a \in M - f^{-1}(U) \), so \( d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0 \). This proves that \( f \) is a \( \pi \)-mapping.

Claim 2. \( f \) is a sequence-covering mapping.

Let \( S = \{ x_n \} \) be a sequence converging to \( x \) in \( X \). For every \( n \in \mathbb{N} \), since \( P_n \) is an \( fcs \)-cover, there exists a finite subfamily \( F_n \) of \( (P_n)_x \) such that \( S \) is eventually in \( \bigcup F_n \). Note that \( S \cap \bigcup F_n \) is finite. There exists a finite subfamily \( G_n \) of \( P_n \) such that \( S \cap \bigcup F_n \subset \bigcup G_n \). Put \( F_n \cup G_n = \{ P_{\alpha_n} : \alpha_n \in \Gamma_n \} \), where \( \Gamma_n \) is a finite subset of \( \Lambda_n \).

For every \( \alpha_n \in \Gamma_n \), if \( P_{\alpha_n} \in F_n \), then \( S_{\alpha_n} = (S \cup \{ x \}) \cap P_{\alpha_n} \), otherwise, put \( S_{\alpha_n} = (S - \bigcup F_n) \cap P_{\alpha_n} \). It is easy to see that \( S \cup \{ x \} = \bigcup_{\alpha_n \in \Gamma_n} S_{\alpha_n} \) and \( \{ S_{\alpha_n} : \alpha_n \in \Gamma_n \} \) is a family of compact subsets of \( X \). Put \( K = \{ (\alpha_n) \in \Pi_{n \in \mathbb{N}} \Gamma_n : \bigcap_{n \in \mathbb{N}} S_{\alpha_n} \neq \emptyset \} \). Then

(i) \( K \subset M \) and \( f(K) \subset S \cup \{ x \} \): Let \( a = (\alpha_n) \in K \), then \( \bigcap_{n \in \mathbb{N}} S_{\alpha_n} \neq \emptyset \). Pick \( y \in \bigcap_{n \in \mathbb{N}} S_{\alpha_n} \), then \( y \in \bigcap_{n \in \mathbb{N}} P_{\alpha_n} \). Note that \( \{ P_{\alpha_n} : n \in \mathbb{N} \} \) is a network at \( y \) in \( X \) if and only if \( y \in \bigcap_{n \in \mathbb{N}} P_{\alpha_n} \). So \( a \in M \) and \( f(a) = y \in S \cup \{ x \} \).

This proves that \( K \subset M \) and \( f(K) \subset S \cup \{ x \} \).

(ii) \( S \cup \{ x \} \subset f(K) \): Let \( y \in S \cup \{ x \} \). For every \( n \in \mathbb{N} \), pick \( \alpha_n \in \Gamma_n \) such that \( y \in S_{\alpha_n} \). Put \( a = (\alpha_n) \), then \( a \in K \) and \( f(a) = y \). This proves that \( S \cup \{ x \} \subset f(K) \).

(iii) \( K \) is a compact subset of \( M \): Since \( K \subset M \) and \( \Pi_{n \in \mathbb{N}} \Gamma_n \) is a compact subset of \( \Pi_{n \in \mathbb{N}} \Lambda_n \). We only need to prove that \( K \) is a closed subset of \( \Pi_{n \in \mathbb{N}} \Gamma_n \).

It is clear that \( K \subset \Pi_{n \in \mathbb{N}} \Gamma_n \). Let \( a = (\alpha_n) \in \Pi_{n \in \mathbb{N}} \Gamma_n - K \). Then \( \bigcap_{n \in \mathbb{N}} S_{\alpha_n} = \emptyset \). There exists \( n_0 \in \mathbb{N} \) such that \( \bigcap_{n \leq n_0} S_{\alpha_n} = \emptyset \). Put \( W = \{ (\beta_n) \in \Pi_{n \in \mathbb{N}} \Gamma_n : \beta_n = \alpha_n \text{ for } n \leq n_0 \} \). Then \( W \) is open in \( \Pi_{n \in \mathbb{N}} \Gamma_n \) and \( a \in W \). It is easy to see that \( W \cap K = \emptyset \). So \( K \) is a closed subset of \( \Pi_{n \in \mathbb{N}} \Gamma_n \).

By the above (i), (ii) and (iii), \( f \) is a sequence-covering mapping.

By the above, \( X \) is a sequence-covering, \( \pi \)-image of a metric space. \( \square \)
Lemma 2.5. Let $P$ be a point-countable cover of a space $X$. Then $P$ is an $fcs$-cover if and only if $P$ is a $cs^*$-cover.

Proof. Necessity holds by Definition 2.2. We only need to prove sufficiency.

Let $P$ be a point-countable $cs^*$-cover of $X$. Let $S = \{x_n\}$ be a sequence converging to $x$ in $X$. Since $P$ is point-countable, put $(P)_x = \{P_n : n \in \mathbb{N}\}$. Then $S$ is eventually in $\bigcup_{n \leq k} P_n$ for some $k \in \mathbb{N}$. If not, then for any $k \in \mathbb{N}$, $S$ is not eventually in $\bigcup_{n \leq k} P_n$. So, for every $k \in \mathbb{N}$, there exists $x_{n_k} \in S - \bigcup_{n \leq k} P_n$.

We may assume $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. Put $S' = \{x_{n_k}\}$,

then $S'$ is a sequence converging to $x$ in $X$. Since $P$ is a $cs^*$-cover, there exists $m \in \mathbb{N}$ and a subsequence $S''$ of $S'$ such that $S''$ is eventually in $P_m$. This contradicts the construction of $S'$.

Recall a mapping $f : X \to Y$ is an $s$-mapping, if $f^{-1}(y)$ is a separable subset of $X$ for every $y \in Y$. Combining [13, Theorem 3.3.12] and [19, Lemma 2.2(2)], we have the following corollary.

Corollary 2.6. Let $X$ be a space. Then the following are equivalent.

(1) $X$ is a sequence-covering, $s$ and $\pi$-image of a metric space.
(2) $X$ is a sequentially-quotient, $s$ and $\pi$-image of a metric space.
(3) $X$ has a $\sigma$-point-countable strong network consisting of $fcs$-covers.
(4) $X$ has a $\sigma$-point-countable strong network consisting of $cs^*$-covers.

Proof. (1) $\implies$ (2): it is clear.
(2) $\implies$ (4): It holds by [13, Theorem 3.3.12].
(4) $\implies$ (1): It holds by [19, Lemma 2.2(2)].
(3) $\iff$ (4): It holds by Lemma 2.5.

3. $\pi$-Images of Separable Metric Spaces

Now we discuss sequence-covering (resp. sequentially-quotient), $\pi$-images of separable metric spaces.

Definition 3.1. Let $X$ be a space, and let $x \in X$. A subset $P$ of $X$ is called a sequential neighborhood of $x$ ([3]) if every sequence $\{x_n\}$ converging to $x$ in $X$ is eventually in $P$.

Definition 3.2. Let $P = \cup \{P_x : x \in X\}$ be a cover of a space $X$. $P$ is called an $sn$-network of $X$ ([14]), if $P_x$ satisfies the following (a),(b) and (c) for every $x \in X$, where $P_x$ is called an $sn$-network at $x$ in $X$.

(a) $P_x$ is a network at $x$ in $X$;
(b) if $P_1, P_2 \in P_x$, then $P \subset P_1 \cap P_2$ for some $P \in P_x$;
(c) every element of $P_x$ is a sequential neighborhood of $x$.

Remark 3.3. In [12], a sequential neighborhood of $x$ and an $sn$-network is called a sequence barrier at $x$ and a universal $cs$-network respectively.
Theorem 3.4. For a space $X$, the following are equivalent.

1. $X$ is a sequence-covering, $\pi$-image of a separable metric space;
2. $X$ is a sequentially-quotient, $\pi$-image of a separable metric space;
3. $X$ has a $\sigma$-strong network consisting of countable fcs-covers;
4. $X$ has a $\sigma$-strong network consisting of countable cs$^*$-covers.

Proof. The proofs of $(1) \iff (3)$ and $(2) \iff (4)$ are as the proof of Theorem 2.4. $(3) \iff (4)$ from Lemma 2.5. □

Ge proved that for a regular space $X$, conditions in Theorem 3.4 are equivalent to that $X$ has a countable sn-network ([4]). The following example shows that "regular" can not be omitted here.

Example 3.5. A space with a countable sn-network is not a sequentially-quotient, $\pi$-image of a metric space.

Proof. Let $R$ be the set of all real numbers, and let $\tau$ be the Euclidean topology on $R$. Put $X = R$ with the topology $\tau^* = \{ \{x\} \cup (D \cap U) : x \in U \in \tau \}$, where $D$ is the set of all irrational numbers. That is, $X$ is the pointed irrational extension of $R$. Then $X$ is Hausdorff, non-regular, and has a countable base ([17, Example 69]), so $X$ has a countable sn-network. Lin showed that $X$ is not a symmetric space ([13, Example 3.13(5)]), so $X$ is not a quotient, $\pi$-image of a metric space ([18]). Note that every sequentially-quotient mapping onto a first countable space is quotient ([1]). Thus $X$ is not a sequentially-quotient, $\pi$-image of a metric space. □

However, by the proofs of [14, Theorem 4.6 (3) $\Rightarrow$ (2)] and [4, Theorem 2.7(3) $\Rightarrow$ (1)], we have the following results without requiring the regularity of the spaces involved.

Proposition 3.6. For a space $X$, the following are true.

1. If $X$ is a sequentially-quotient, $\pi$-image of a separable metric space, then $X$ has a countable sn-network.
2. If $X$ has a countable closed sn-network, then $X$ is a compact-covering, compact image of a separable metric space.

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