SYMMETRIC UNITS AND GROUP IDENTITIES IN GROUP ALGEBRAS. I

VICTOR BOVDI

Dedicated to Professor L.G. Kovács on his 70th birthday

Abstract. We describe those group algebras over fields of characteristic different from 2 whose units symmetric with respect to the classical involution, satisfy some group identity.

1. Introduction

Let $U(A)$ be the group of units of an algebra $A$ with involution $*$ over the field $F$ and let $S_*(A) = \{u \in U(A) \mid u = u^*\}$ be the set of symmetric units of $A$.

Algebras with involution have been actively investigated. In these algebras there are many symmetric elements, for example: $x + x^*$ and $xx^*$ for any $x \in A$. This raises natural questions about the properties of the symmetric elements and symmetric units. In [10] Ch. Lanski began to study the properties of the symmetric units in prime algebras with involution, in particular when the symmetric units commute. Using the results and methods of [4], in [5] we classified the cases when the symmetric units commute in modular group algebras of $p$-groups. The solution of this question for integral group rings and for some modular group rings of arbitrary groups was obtained in [6, 3].

Several results on the group of units $U(R)$ show that if $U(R)$ satisfies a certain group theoretical condition (for example, it is nilpotent or solvable), then $R$’s properties are restricted and a polynomial identity on $R$ holds. This suggests that there may be some general underlying relationship between group identities and polynomial identities. In this topic Brian Hartley made the following:

Conjecture 1. Let $FG$ be a group algebra of a torsion group $G$ over the field $F$. If $U(FG)$ satisfies a group identity, then $FG$ satisfies a polynomial identity.

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The theory of $PI$-algebras has been established for a long time. On the contrary, the study of algebras with units satisfying a group identity has emerged only recently [11, 12]. Our goal here is to show that with a few extra assumptions, these algebras are actually $PI$-algebras. In fact, these classes of algebras are quite special, because if the group of units is too small in an algebra, a group identity condition can not limit the structure of the whole algebra. In view of Hartley’s conjecture, as a natural generalization the works [5, 6, 3, 10] it is a natural question when does the symmetric units satisfy a group identity in group algebra. Note that the structure theorem of the algebras with involution whose symmetric elements satisfy a polynomial identity was obtained earlier by S.A. Amitsur in [1]. A. Giambruno, S.K. Sehgal and A. Valenti in [8] obtained the following result for group algebras of torsion groups:

**Theorem 1.** Let $FG$ be a group algebra of a torsion group $G$ over an infinite field $F$ of characteristic $p > 2$ and assume that the involution $*$ on $FG$ is canonical. The symmetric units $S_*(FG)$ satisfy a group identity if and only if $G$ has a normal subgroup $A$ of finite index, the commutator subgroup $A'$ is a finite $p$-group and one of the following conditions holds:

(i) $G$ has no quaternion subgroup of order 8 and $G'$ has of bounded exponent $p^k$ for some $k$.

(ii) $G$ has of bounded exponent $4p^s$ for some $s \geq 0$, the $p$-Sylow subgroup of $G$ is normal and $G/P$ is a Hamiltonian $2$-subgroup.

In the present paper we extend the result of A. Giambruno, S.K. Sehgal and A. Valenti. For non-torsion groups $G$ we describe the group algebras $FG$ over the field $F$ of characteristic different from 2 whose symmetric units $S_*(FG) = \{u = \sum_{g \in G} \alpha_g g \in U(FG) \mid u = u^* = \sum_{g \in G} \alpha_g g^{-1}\}$ satisfy a group identity. The present result was announced at the International Workshop Polynomial Identities in Algebras, 2002, Memorial University of Newfoundland.

2. Main results

In the sequel of this paper $d(\omega)$ denotes a positive integer, which depends on the group identity $\omega$ and it is defined in the next section. Our results are the following:

**Theorem 2.** Let $G$ be a non-torsion nonabelian group and $\text{char}(F) = p \neq 2$ and assume that the symmetric units of $FG$ satisfy some group identity $\omega = 1$. Assume that $|F| > d(\omega)$, where $d(\omega)$ is an integer which depends only on the word $\omega$. Let $P$ be a $p$-Sylow subgroup of $G$ and let $t(G)$ be the torsion part of $G$.

(1) If $p > 2$ then $P$ and $t(G)$ are normal subgroups of $G$ such that:
(a) $B = t(G)/P$ is an abelian $p'$-subgroup and its subgroups are normal in $G$;
(b) if $B$ is noncentral in $G/P$ then the algebraic closure $L$ of the prime subfield $F_p$ in $F$ is finite and for all $g \in G/P$ and for any $a \in B$ there exists an $r \in \mathbb{N}$ such that $a^g = a^p$ and $|L : F_p|$ is a divisor of $r$;
(c) the $p$-Sylow subgroup $P$ is a finite group;
(d) the $p$-Sylow subgroup $P$ is infinite and $G$ has a subgroup $A$ of finite index, such that $A'$ is a finite $p$-group and the commutator subgroup $H'$ of $H = AP$ is a bounded $p$-group. Moreover, if $P$ is unbounded, then $G'$ is a bounded $p$-group;

(II) If $\text{char}(F) = 0$ then $t(G)$ is a subgroup, every subgroup of $t(G)$ is normal in $G$ and one of the following conditions holds:
(a) $t(G)$ is abelian and each idempotent of $Ft(G)$ is central in $FG$;
(b) $t(G)$ is a Hamiltonian 2-group, and each symmetric idempotent of $Ft(G)$ is central in $FG$.

3. Notation, preliminary results and the proof

Let $FG$ be the group algebra of $G$ over $F$. We introduce the following notation:

- $(g, h) = g^{-1}h^{-1}gh$ for all $g, h \in G$;
- $|g|$ and $C_G(g)$ are the order and the centralizer of $g \in G$, respectively;
- $G', \text{Syl}_p(G)$ are the commutator subgroup and the Sylow $p$-subgroup of $G$;
- $t(G)$ is the set of elements of finite order in $G$;
- $\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$ is the FC-radical of $G$;
- $\Delta_p(G) = \{g \in \Delta(G) \mid g$ has order of a power of $p$\};
- $T(G/H)$ is a left transversal of the subgroup $H$ in $G$;
- $N(FG)$ is the sum of all nilpotent ideals of the group algebra $FG$;
- $A(FG)$ is the augmentation ideal of the group algebra $FG$.

Let $A$ be an algebra over a field $F$, let $F_0$ be the ring of integers of the field $F$, and suppose that $U(A)$ satisfies a group identity $\omega = 1$. Then, as it was proved in Lemma 3.1 of [11], there exists a polynomial $f(x)$ over $F_0$ of degree $\delta(\omega)$ which is determined by the word $\omega$. In several papers (see for example [8]) the authors assumed that the field $F$ is infinite so they could apply the “Vandermonde determinant argument”. We shall use some lemmas from [8], which are easy to prove using the method of the paper [11] even without the assumption that the field $F$ is infinite.

In our proof we will use the following facts:

**Lemma 1.** ([1]) Let $A$ be an algebra with involution over $F$ of $\text{char}(F) \neq 2$, such that the set of symmetric units of $A$ satisfy a group identity $\omega = 1$. If $I$ is a stable nil ideal of $A$ then the symmetric units of $A/I$ satisfy a group identity.
Lemma 2. (see [8]) Let $A$ be an algebra over the field $F$ of characteristic $p \neq 2$, such that the set of symmetric units of $A$ satisfy a group identity $\omega = 1$ and $|F| > \delta(\omega)$, where $\delta(\omega)$ is an integer which depends only on the word $\omega$. Then:

(i) if $A$ is semiprime, then $asa = 0$ for every nilpotent element $s \in S_*(A)$ and square-zero $a \in S_*(A)$;

(ii) if $a \in A$ is square-zero, then $(aa^*)^m = 0$, for some $m \in \mathbb{N}$;

(iii) if $A$ is semiprime and $u, v \in A$ such that $uv = 0$, then $usv = 0$ for any square-zero symmetric element $s$;

(iv) if the subring $L$ of $A$ is nil, then $L$ satisfy a polynomial identity;

(v) each symmetric idempotent is central;

(vi) if $A$ is artinian, then $A$ is isomorphic to a direct sum of division algebras and $2 \times 2$ matrices algebras over a field with symplectic involution. Each nilpotent element of $A$ has index at most 2;

(vii) if $A = FG$ is the group algebra of the group $G = Q_8 \langle \langle c \rangle \rangle$, where $Q_8$ is the quaternion group of order 8, then the order of the cyclic subgroup $\langle c \rangle$ is finite.

Lemma 3. (see [8]) Let $A$ be a normal abelian subgroup of $G$ of finite index such that $G = A \cdot H$, where $H$ is a finite group. Let $\text{char}(F) = p$ and assume that the set of symmetric units of $FG$ satisfy a group identity $\omega = 1$. If $|F| > \delta(\omega)$, where $\delta(\omega)$ is an integer which depends only on the word $\omega$, then $G'$ has bounded exponent $p^m$, where $m$ depends only on $\delta$.

Now we are ready to prove the following

Lemma 4. Let $\text{char}(F) = p > 2$ and let the set of symmetric units of $FG$ satisfy a group identity $\omega = 1$. Assume that $|F| > \delta(\omega)$, where $\delta(\omega)$ is an integer which depends only on the word $\omega$. Then the $p$-Sylow subgroup $P$ of $\Delta(G)$ is normal in $G$ and the set of symmetric units of $F[G/P]$ satisfy a group identity.

Proof. Let $H$ be a finite subgroup of $\Delta(G)$ and let $J = J(F_pH)$ be the radical of the finite group algebra $F_pH$ over the prime subfield $F_p$. According to Lemma 2(vi), the factor algebra $F_pH/J$ is isomorphic to a direct sum of fields and $2 \times 2$ matrices algebras over a finite field with symplectic involution and a nilpotent element $\bar{u} = u + J \in F_pH/J$ has index at most 2. Moreover, from this decomposition follows that $\bar{u}\bar{u}^*$ is central. By Lemma 2(ii) the element $\bar{u}\bar{u}^*$ is nilpotent and central in the semiprime algebra $F_pH/J$. Therefore $\bar{u}\bar{u}^* = 0$ and $uu^* \in J(FH)$.

Let $h \in H$ with $|h| = p^j$. Then $u = h - 1$ is nilpotent and

$$ uu^* = (h - 1)(h^{-1} - 1) \in J(FH). $$

It follows that $h u u^* = -(h - 1)^2 \in J(FH)$. Using Passman’s result (see Lemma 5 in [8], p.453) we obtain that $h - 1 \in J(FH)$ for all $h \in H$ and $H \cap (1 + J)$ is a normal $p$-subgroup of $H$, which coincides with the $p$-Sylow
subgroup of $H$. Thus the $p$-Sylow subgroup $P$ of $\Delta(G)$ is normal in $G$, so the proof is complete.

\begin{lemma}
Let $FG$ be a semiprime group algebra over the field $F$ with $\text{char}(F) > 2$ such that the set of symmetric units of $FG$ satisfy a group identity $\omega = 1$. Suppose that $|F| > \mathfrak{d}(\omega)$, where $\mathfrak{d}(\omega)$ is an integer which depends only on the word $\omega$. Then one of the following conditions holds:

(i) $t(G)$ is abelian and each idempotent of $Ft(G)$ is central in $FG$.

(ii) $t(G)$ is a Hamiltonian 2-group and each symmetric idempotent of $Ft(G)$ is central in $FG$.
\end{lemma}

\begin{proof}
(i) Let $a \in t(G)$, such that $\langle [a], p \rangle = 1$. Then, by Lemma 2(v), the symmetric idempotent $e = \frac{1}{n}(1 + a + \cdots + a^{[a]^{-1}})$ is central in $FG$, so $\langle a \rangle$ is normal in $G$. Now let $p > 2$ and let $a \in t(G)$ be of order $p$. If $N_G(\langle a \rangle) = G$ then $\langle a \rangle$ is a central nilpotent element of the semiprime algebra $FG$, a contradiction.

Let us prove that each torsion element belongs to $N_G(\langle a \rangle)$. Pick $h \notin N_G(\langle a \rangle)$ such that $|h| = p^t$. The elements $(h - 1)(h^{-1} - 1)$ and $\langle a \rangle$ are symmetric and $(2 - h - h^{-1})^{p^t} = (\langle a \rangle)^2 = 0$. By Lemma 2(i) we get $\langle a \rangle (2 - h - h^{-1}) \langle a \rangle = 0$ and

\begin{equation}
\langle a \rangle h\langle a \rangle + \langle a \rangle h^{-1}\langle a \rangle = 0.
\end{equation}

An element of $\text{Supp}(\langle a \rangle h\langle a \rangle)$ can be written as $a^i h a^j$, where $0 \leq i, j \leq p - 1$. If all the elements in $\text{Supp}(\langle a \rangle h\langle a \rangle)$ and in $\text{Supp}(\langle a \rangle h^{-1}\langle a \rangle)$ are distinct, then on the left-hand side of (1) each element appears at most two times, but this leads to a contradiction if $\text{char}(F) \neq 2$. Therefore, in the subset $\text{Supp}(\langle a \rangle h\langle a \rangle)$ not all elements are different, whence there exist $i, j, k, l$ such that $a^i h a^j = a^k h a^l$ and either $i \neq k$ or $j \neq l$. If, for example, $i > k$, then $h^{-1} a^{i-k} h = a^{l-j}$ and $h \in N_G(\langle a \rangle)$.

Now, let $h \notin N_G(\langle a \rangle)$ be a $p^t$-element. As we have seen before, $\langle h \rangle$ is normal in $G$, so $\langle a, h \rangle$ is a finite subgroup. By Lemma 4 the $p$-Sylow subgroup $P$ of $\langle a, h \rangle$ is normal in $\langle a, h \rangle$ and $(a, h) \in P \cap (h) = \langle 1 \rangle$, a contradiction.

Therefore, each element of finite order belongs to $N_G(\langle a \rangle)$. Moreover, the elements of order $p$ in $G$ form an elementary abelian normal $p$-subgroup $E$ of $G$.

Finally, if $h \notin N_G(\langle a \rangle)$, then $h$ has infinite order and $h$ acts on $E$. The subgroups $\langle a^h \rangle$ and $\langle a \rangle$ are different and we can choose a subgroup $\langle b \rangle \subset E$, which differs from $\langle a \rangle$. Clearly, $\langle a \rangle (h + h^{-1}) \langle a \rangle$ and $\langle b \rangle$ are square-zero symmetric elements and according to Lemma 2(i),

\begin{equation}
\langle b \rangle \langle a \rangle (h + h^{-1}) \langle a \rangle \langle b \rangle = 0.
\end{equation}

Since $hE$ and $h^{-1}E$ are different cosets, from (2) follows that

\begin{equation}
\langle b \rangle \langle a \rangle h \langle a \rangle \langle b \rangle = 0.
\end{equation}

\end{proof}
The subgroup $H = \langle a, b \rangle \subset E$ has order $p^2$ and by (3) we have $\overline{H}h_1\overline{H}h_2 = 0$ for all $h_1, h_2$. Since elements of finite order belong to $N_G(H)$, we get $(\overline{H}FG)^2 = 0$, which is impossible by the semiprimeness of $FG$. Thus $G$ has no $p$-elements and all finite cyclic subgroups of $G$ are normal in $G$. Applying Lemmas 6 and 7 from [8] and the fact that $G$ has no $p$-elements $(p \neq 2)$, we obtain that $t(G)$ is either an abelian group or a Hamiltonian 2-group.

Let $t(G)$ be an abelian group and let $e \in \text{Ft}(G)$ be a noncentral idempotent in $FG$. Set $H = \langle \text{Supp}(e) \rangle$. Since every subgroup of $t(G)$ is normal in $G$, the subgroup $H$ is also normal in $G$ and $FH$ has a primitive idempotent $f$, which does not commute with some $g \in G$ of infinite order. Then $g^{-1}fg \neq f$ is also a primitive idempotent of $FH$ and $(g^{-1}fg)f = 0$, i.e. $(fg)^2 = (gf)^2 = 0$.

Let $g^{-1}fg = \overline{f} \neq f^*$. By Lemma 2(v) we have $f \neq f^*$, so $g^{-1}f + f^*g$ is a square-zero symmetric element and by Lemma 2(iii), we get that

$$f(g^{-1}f + f^*g)fg = 0.$$  

It follows that $f + g(fg^*)gf = f = 0$, a contradiction. Therefore, $g^{-1}fg = f^*$, so $(f^*)^* = (g^{-1}fg)^* = g^{-1}f^*g = f$. Furthermore, $g^{-2}fg^2 = g^{-1}f^*g = f$ and $f^*g^2 = g^2f^*$. Since $f^*g^2 = g^2f^*$, $(g^2)^2 = 0$ and $gf + f^*g^2$ is a square-zero symmetric element, by Lemma 2(iii) we obtain that

$$gf(gf + f^*g)g^2f^* = g^3f^*(g^{-1}fg) g^2f^* + gf^* = g^3f^2f^* + gf^* = 0.$$  

Thus $(g^2 + 1)f^* = 0$, which is impossible, since $g^2H$ and $H$ are different cosets. \hfill \Box

Lemma 6. Let $F$ be a field of characteristic $p$, and suppose that $G$ contains a normal locally finite $p$-subgroup $P$ such that the centralizer of each element of $P$ in every finitely generated subgroup of $G$ is of finite index. Then $\mathfrak{I}(P)$ is a locally nilpotent ideal.

Proof. Clearly, $\{ u(h - 1) \mid u \in T_l(G/P), \ 1 \neq h \in P \}$ is an $F$-basis for the ideal $\mathfrak{I}(P)$. Let us show that the subalgebra $W = \langle u_1(h_1 - 1), \ldots, u_s(h_s - 1) \rangle_F$ is nilpotent. According to our assumption, the centralizers of $h_1, \ldots, h_s$ in the subgroup $H = \langle u_1, \ldots, u_s, h_1, \ldots, h_s \rangle$ have finite index. Since $P$ is normal, its subgroup $L = \langle h_1^u, h_2^u, \ldots, h_s^u \mid u \in H \rangle$ is a finitely generated FC-group and by a Theorem of B.H. Neumann ([1], Theorem 4, p.19) $L$ is a finite $p$-group. Thus the augmentation ideal $A(FL)$ is nilpotent with index, say, $t$. Furthermore, $A(FL) = u^{-1}A(FL)u$ for any $u \in H$ and this implies that $(A(FL) \cdot FH)^n = A^n(FL) \cdot FH$ for any $n > 0$, so $W^t \subseteq A^t(FL) \cdot FH = 0$, because $W \subseteq A(FL) \cdot FH$. Therefore $W$ is a nilpotent subalgebra and $\mathfrak{I}(P)$ is a locally nilpotent ideal. \hfill \Box

Lemma 7. Let $G$ be a group with a nontrivial $p$-Sylow subgroup $P$ and let $\text{char}(F) = p > 2$. If the set of symmetric units of $FG$ satisfy a group identity $\omega = 1$ and $|F| > 3(\omega)$, where $3(\omega)$ is an integer which depends only on the word $\omega$, then $P$ is normal in $G$ and the ideal $\mathfrak{I}(P)$ is nil.
Proof. Let $P$ be a maximal normal $p$-subgroup of $G$ such that the ideal $\mathfrak{I}(P)$ is nil. By Lemma 1 the set of symmetric units of $F[G/P]$ satisfy a group identity. If $F[G/P]$ is semiprime, then by (i) of the Theorem the group $G/P$ has no $p$-elements and $P$ coincides with the $p$-Sylow subgroup of $G$. Now, suppose that $F[G/P]$ is not semiprime. According to Theorem 4.2.13 ([13], p.131) the group $\Delta(G/P)$ has a nontrivial $p$-Sylow subgroup $P_1/P$, which is normal in $G/P$ by Lemma 4. Clearly, $P_1/P$ is an FC-subgroup of $G/P$, so by Lemma 6 the ideal $\mathfrak{I}(P_1/P)$ is nil.

Since $\mathfrak{I}(P_1/P) \cong \mathfrak{I}(P_1)/\mathfrak{I}(P)$ and $P_1$ is normal in $G$, the ideal $\mathfrak{I}(P_1)$ is nil and $P \subset P_1$, a contradiction. Thus $P = Syl_p(G)$ and the proof is done. □

Lemma 8. Let $R$ be an algebra with involution $*$ over a field $F$ of characteristic $p > 2$ and assume that $S_4(R)$ satisfies a group identity and $|F| > \mathfrak{d}(\omega)$. If some nil subring $L$ of $R$ is $*$-stable, then $L$ satisfies a non-matrix polynomial identity.

Proof. Let $A = F(X[[t]]$ be the ring of power series over the polynomial ring $F(X)$ with noncommuting indeterminates $X = \{x_1, x_2\}$. By a result of Magnus, the elements $1 + x_1t, 1 + x_2t$ are units in $A$ and $\langle 1 + x_1t, 1 + x_2t \rangle$ is a free group.

Assume that $S_4(R)$ satisfies the group identity $w$, where $w$ is a reduced word in 2 variables. Then $w(1 + x_1t, 1 + x_2t) \neq 1$ according to result of Magnus and it is well-known that $(1 + x_1t)^{-1} = 1 - x_1t + x_1^2t^2 - \cdots$. If we substitute $(1 + x_1t)^{-1}$ in the expression $w(1 + x_1t, 1 + x_2t) - 1$, then it can be expanded as

$$\sum_{i \geq s} g_i(x_1, x_2)t^i,$$

where $g_i(x_1, x_2) \in F\langle X \rangle$ is a homogeneous polynomial of degree $i$. Obviously there exists a smallest integer $s \geq 1$ such that $g_s(x_1, x_2) \neq 0$.

Let $L$ be a $*$-stable nil subring and let $S(L)$ be the set of the symmetric elements of $L$. Take now $r_1, r_2 \in S(L)$ and let $\lambda \in F$. Obviously, $r_1, r_2$ are nilpotent elements, so each $1 + \lambda r_1$ is a symmetric unit in $R$ and

$$(1 + r_1\lambda)^{-1} = 1 - r_1\lambda + r_1^2\lambda^2 + \cdots + (-1)^{l-1}r_1^{l-1}\lambda^{l-1}$$

for a suitable $t$. By evaluating $w$ on these elements, (4) gives us a finite sum $\sum_{i \geq s} g_i(r_1, r_2) \lambda^i = 0$ for some $l$. Since $|F| > \mathfrak{d}(\omega)$, we can apply the Van dermonde determinant argument to obtain $g_i(r_1, r_2) = 0$ for all $i$. Therefore $g_s(x_1, x_2)$ is a $*$-polynomial identity on $S(L)$. Finally, by [1] it follows that $S(L)$ satisfies an ordinary polynomial identity.

Suppose that the homogeneous polynomial $g(x_1, x_2)$ vanishes on the matrix algebra $M_2(K)$ over a commutative ring $K$. Then

$$g(x_1, x_2) = h(x_1, x_2) + g_{11}(x_1, x_2) + g_{12}(x_1, x_2) + g_{21}(x_1, x_2) + g_{22}(x_1, x_2),$$

where $h(x_1, x_2)$ consists of all monomials which contain $x_1^2$ or $x_2^2$ while the $g_{ij}(x_1, x_2)$ contain all the remaining monomials beginning with $x_1$ and ending with $x_2$ for $i, j \in \{1, 2\}$. If $a$ and $b$ are two square-zero matrices, then $h(a, b) = 0$, because each term of $h$ has $a^2$ or $b^2$ as a factor, so we conclude
that \( ag_{21}(a, b)b = 0 \). Clearly \( x_1 g_{21}(x_1, x_2) x_2 \) is some polynomial \( f(x_1 x_2) \). Then \( f(abx) = 0 \) for each \( \lambda \in F \) and, by the Vandermonde determinant argument, we get \((ab)^d = 0 \) for some \( d \). Take, for instance, the matrix units \( a = e_{12} \) and \( b = e_{21} \), then we obtain a contradiction. \( \square \)

**Lemma 9.** Let \( R \) be an algebra over a field \( F \) of positive characteristic \( p \) satisfying a non-matrix polynomial identity. Then \( R \) satisfies also a polynomial identity of the form \( ([x, y]z)^p \) and \([x, y]^p\).

**Proof.** Let \( g(x_1, x_2, \ldots, x_n) \) be a non-matrix polynomial identity in \( R \). The variety \( W \) determined by the polynomial identity \( g(x_1, x_2, \ldots, x_n) \) contains a relatively free algebra \( K \) of rank 3. Of course, \( K \) is a finitely generated PI-algebra, and the result of Braun and Razmyslov (Theorem 6.3.39, [14]) states that the radical \( J(K) \) of \( K \) is nilpotent. Writing \( K/J(K) \) as a subdirect sum of primitive rings \( \{L_i\} \), we get that every primitive ring \( L_i \) satisfies the non-matrix polynomial identity \( g(x_1, x_2, \ldots, x_n) \), as a homomorphic image of \( K \).

By Theorem 2.1.4 of [9], \( L_i \) is either isomorphic to the matrix ring \( M_m(D) \) over a division ring \( D \), or for any \( m \) the matrix ring \( M_m(D) \) is an epimorphic image of some subring of \( L_i \).

Thus \( M_m(D) \) satisfies a non-matrix polynomial identity \( g \), which is possible only if \( L_i \) is a commutative ring. Consequently, \( K/J(K) \) is a commutative algebra, so \( K \) satisfies a polynomial identity of the form \( ([x, y]z)^p \) such that \( J(K)^p = 0 \). Since \( R \) belongs to the variety \( W \), the algebra \( R \) also satisfies a polynomial identity \( ([x, y]z)^p \). \( \square \)

**Lemma 10.** Let \( FG \) be a non semiprime group algebra over the field \( F \) with \( \text{char}(F) > 2 \), such that the set of symmetric units of \( FG \) satisfy a group identity \( \omega = 1 \) and \( |F| > \mathfrak{d}(\omega) \), where \( \mathfrak{d}(\omega) \) is an integer which depends only on the word \( \omega \). If \( \mathfrak{N}(FG) \) is not nilpotent then \( FG \) is a PI-algebra, where \( \mathfrak{N}(FG) \) is the sum of all nilpotent ideals of \( FG \).

**Proof.** Clearly the non nilpotent ideal \( \mathfrak{N} = \mathfrak{N}(FG) \) is invariant under the involution * and by Lemma 2(iv) the ring \( \mathfrak{N} \) satisfies a polynomial identity \( f(x_1, \ldots, x_n) \). Moreover, by Lemma 2.8 of [12] the algebra \( FG \) satisfies a non-degenerate multilinear generalized polynomial identity and hence, by Theorem 5.3.15 ([13], p.202), \(|G: \Delta(G)| < \infty \) and \( \Delta(G)' \) is finite.

Set \( P = Syl_p(G) \) and \( P_1 = Syl_p(\Delta(G)') \). By Lemma 4, \( P \cap \Delta(G)' = P_1 < G \) and \( P_1 \) is a finite \( p \)-group. Thus \( \Delta(P_1) \) is a nilpotent ideal and by (i) of the Theorem, the set of symmetric units of \( F[\Delta(G)'/P_1] \) satisfy a group identity, so \( \Delta(G)'/P_1 \) is either an abelian \( p \)-group or a Hamiltonian 2-group.

If \( P_1 = \Delta(G)' \), then by Theorem 5.3.9 ([13], p.197) the algebra \( FG \) is a PI-algebra. If \( P_1 \subsetneq \Delta(G)' \) then we can suppose that \( G \) is a group such that \( Syl_p(\Delta(G)') = 1 \) and \( \Delta(G)' \) is either an abelian \( p \)-group or a Hamiltonian 2-group.

Set \( P_2 = Syl_p(\Delta(G)) \). Clearly, \( P_2 = P \cap \Delta(G) \) is normal in \( \Delta(G) \). Since \( [P : P_2] < \infty \) and \( P \) is an infinite group, the group \( P_2 \) is infinite, too. If
$a \in P_2$, $b \in \Delta(G)$, then $(a, b) \in P_2 \cap \Delta(G)' = 1$, so $(a, b) = 1$ and $P_2$ is a central subgroup in $\Delta(G)$.

Let us prove that $F\Delta(G)$ is a $PI$-algebra. If $\Delta(G)$ is a torsion group, then by [8] the statement is trivial.

Since $\mathfrak{N}(F\Delta(G)) \subseteq \mathfrak{N}(FG)$, the ideal $\mathfrak{N}(F\Delta(G))$ also satisfies the same polynomial identity $f(x_1, \ldots, x_n)$. By the standard multilinearization process, we may assume that $f(x_1, \ldots, x_n)$ is multilinear.

Assume that $P_2$ has bounded exponent. Then the maximal elementary abelian $p$-subgroup $E$ of $P_2$ is infinite. Let $f(a_1, \ldots, a_n) = \sum_i a_i y_i$, where $a_1, \ldots, a_n \in F\Delta(G)$, $y_1, \ldots, y_n \in T_i(\Delta(G)/E)$ and $a_i \in FE$. Then there exists a finite subgroup $B$ such that $a_i \in FB$ and $E = B \times \prod_j \langle c_j \rangle$. Since $(c_k - 1)a_k \in \mathfrak{N}(F\Delta(G))$ and $P_2$ is central, we conclude that

$$f((c_1 - 1)a_1, \ldots, (c_n - 1)a_n) = (c_1 - 1) \cdots (c_n - 1)f(a_1, \ldots, a_n) = 0.$$ 

It follows that $f(a_1, \ldots, a_n) = 0$, because $B \cap \prod_j \langle c_j \rangle = \langle 1 \rangle$.

Now let $P_2$ be of unbounded exponent and $c \in P_2$. Then $(c - 1)a_k \in \mathfrak{N}(F\Delta(G))$ and also

$$f((c - 1)a_1, \ldots, (c - 1)a_n) = (c - 1)^n f(a_1, \ldots, a_n) = 0$$

for all $c \in P_2$. Then $f(a_1, \ldots, a_n)$ belongs to the annihilator of the augmentation ideal $A(FP_2^{p'})$, where $n \leq p'$. Since $P_2^{p'}$ is infinite, we have

$$\text{Ann}_I(A(FP_2^{p'})) = 0.$$

It follows that $f(a_1, \ldots, a_n) = 0$, so $f(x_1, \ldots, x_n)$ is a polynomial identity for $F\Delta(G)$. Since $F\Delta(G)$ is a $PI$-algebra and $[G : \Delta(G)] < \infty$, the algebra $FG$ is $PI$, too.

\begin{proof}
Let $FG$ be a group algebra of a non-torsion group $G$ over a field of positive characteristic $p$. By Lemma 7 the $p$-Sylow subgroup $P$ is normal in $G$ and $F[G/P] \cong FG/J(P)$, so the symmetric units of semiprime algebra $F[G/P]$ satisfy a group identity. By Lemma 5 $B = t(G/P)$ is a subgroup of $G/P$ and $B$ is either an abelian $p'$-group or a Hamiltonian 2-group. If $B$ is a Hamiltonian 2-group, then $Q_8$ is a subgroup of $B$. Choose an element $c \in G/P$ of infinite order. Since every subgroup of $t(G)/P$ is normal in $G/P$ and $|\text{Aut}(Q_8)| < \infty$, there exists a $t \in \mathbb{N}$ such that $c^t \in C_{G/P}(Q_8)$ and $Q_8 \times \langle c^t \rangle \subseteq G/P$. Then Lemma 2(vii) asserts that $c$ has finite order, a contradiction. So $B$ is an abelian $p'$-group and by Lemma 5 every idempotent of $FB$ is central in $F[G/P]$. Moreover, if $B$ is noncentral, then according to [7] the group $B$ satisfy (i.b) of our Theorem.

Now, let $P$ be infinite. By Corollary 8.1.14 ([13], p.312) the ideal $\mathfrak{N}(FG)$ is non-nilpotent, so by Lemma 10, the algebra $FG$ is a $PI$-algebra, i.e. $G$ has a subgroup $A$ with finite index such that $A'$ is a finite $p$-group. According to Lemma 1, it can be assumed that $G$ has an abelian subgroup $A$ of finite index.
We claim that the commutator subgroup of \( H = P \cdot A \) is a bounded \( p \)-group.
Clearly \( S_z(FP) \) satisfies a group identity and according to Lemma 3 \( P' \) is a bounded \( p \)-group. The normal abelian \( p \)-subgroup \( P' \cap A \) has finite exponent and according to Lemma 6 the ideal \( \mathcal{I}(P' \cap A) \) is locally nilpotent of bounded degree. The subgroup \( P' \cap A \) of \( P' \) has finite index in \( P' \) and
\[
\mathcal{I}(P') / \mathcal{I}(P' \cap A) \cong \mathcal{I}(P' / (P' \cap A)).
\]

Therefore \( \mathcal{I}(P') \) is a locally nilpotent ideal of bounded degree \( p^t \) for some \( t \). Clearly \( FG / \mathcal{I}(P') \cong F[G / P'] \) and put \( P' = \langle 1 \rangle \). Since \( A \) has a finite index in \( H = P \cdot A \), Lemma 3 ensures that \( H' \) is a \( p \)-group of bounded exponent and according to Lemma 1, we can put \( H' = \langle 1 \rangle \) again.

The \( p \)-Sylow subgroup \( P \) of \( G \) is abelian and by Lemma 8 the ideal \( \mathcal{I}(P) \) satisfies a non-matrix polynomial identity. Moreover, by Lemma 9 the ideal \( \mathcal{I}(P) \) satisfies polynomial identities of the following forms: \( [x, y]^{p^t} \) and \( (x, y)^{p^t} \).

Let \( h \in G \) and \( a \in P \). Clearly \((a - 1)h, h^{-1}(a^{-1} - 1) \in \mathcal{I}(P) \) and
\[
[(a - 1)h, h^{-1}(a^{-1} - 1)]^{p^t} = (a^h)^{p^t} + (a^h)^{-p^t} - a^{p^t} - a^{-p^t} = 0
\]
which implies that either \( (h, a)^{p^t} = 1 \) or \( h^{-1}a^{p^t}h = a^{-p^t} \).

Put \( z = a^{p^t} \). From \( h^{-1}a^{p^t}h = a^{-p^t} \) it follows that \( h^{-1}zh = z^{-1} \) and \((z - 1, (z^{-1} - 1)h)^{p^t} = 0 \). Clearly \((z - 1, (z^{-1} - 1)h) = -z^{-2}(z + 1)(z - 1)^2h\) so
\[
0 = ([z - 1, (z^{-1} - 1)h])^{p^t} = -((z + 1)(z - 1)^2(z^{-1} - 1)(z^{-1} - 1)^2h^2)^{\frac{p^t - 1}{2}} (z^{-2}(z + 1)(z - 1)^2h) = -z^{-\frac{3p^t - 1}{2}} \cdot (z + 1)^{p^t} \cdot (z - 1)^{2p^t} \cdot h^{p^t}.
\]

Since \( \text{char}(K) > 2 \), the element \( z + 1 \) is a unit and \((z - 1)^{2p^t} = (a - 1)^{2p^t} = 0 \) and the order of \( a \) at most \( 2p^t \). Therefore \((h, a)^{p^{2t+1}} = 1 \) for all \( h \in G \), \( a \in P \) and \( 2t + 1 \) depends on only the group identity. Since \( (G, P) \) is a \( p \)-group of bounded exponent, we can again make a reduction, so we can assume that \( (G, P) = 1 \) and \( P \) is central.

Let \( P \) be a central subgroup of unbounded exponent and \( h_1, h_2 \in G \). Obviously
\[
((h_1, h_2)^{p^t} - 1)(a - 1)^{p^{3t}} = ((h_1, h_2) - 1)^{p^t} (a - 1)^{p^{3t}} = ([h_1^{-1}(a - 1), h_2^{-1}(a - 1)]h_1h_2(a - 1))^{p^t} = 0
\]
for \( a \in P \). Since there are infinitely many element of the form \( a^{p^{3t}} \) we conclude that \((h_1, h_2)^{p^t} = 1 \) and the proof is complete. \( \square \)

References


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INSTITUTE OF MATHEMATICS,
UNIVERSITY OF DEBRECEN,
4010 Debrecen, Pf. 12,
HUNGARY

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE,
COLLEGE OF NYÍREGYHÁZA
SÓSTÓI ÚT 31/B,
H-4410 NYÍREGYHÁZA, HUNGARY
E-mail address: vbovdi@math.klte.hu