A STUDY ON THE GENERALIZATION OF JANOWSKI
FUNCTIONS IN THE UNIT DISC

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Abstract. Let $\Omega$ be the class of functions $w(z)$, $w(0) = 0$, $|w(z)| < 1$ regular in the unit disc $D = \{z : |z| < 1\}$. For arbitrarily fixed numbers $A \in (-1, 1]$, $B \in [-1, A)$, $0 \leq \alpha < 1$ let $P(A, B, \alpha)$ be the class of regular functions $p(z)$ in $D$ such that $p(0) = 1$, and which is $p(z) \in P(A, B, \alpha)$ if and only if

$$p(z) = 1 + \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bw(z)}$$

for some function $w(z) \in \Omega$ and every $z \in D$.

In the present paper we apply the principle of subordination ([1], [3], [4], [5]) to give new proofs for some classical results concerning the class $S^*(A, B, \alpha)$ of functions $f(z)$ with $f(0) = 0$, $f'(0) = 1$, which are regular in $D$ satisfying the condition: $f(z) \in S^*(A, B, \alpha)$ if and only if $z \frac{f'(z)}{f(z)} = p(z)$ for some $p(z) \in P(A, B, \alpha)$ and for all $z \in D$.

1. Introduction

Let $\Omega$ be the family of functions $w(z)$ regular in the unit disc $D$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$, for $z \in D$.

For arbitrary fixed numbers $A, B, \alpha$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, let $P(A, B, \alpha)$ denote the family of functions

(1) $$p(z) = 1 + p_1z + p_2z^2 + \cdots + p_nz^n + \cdots$$

regular in $D$ and such that $p(z)$ is in $P(A, B, \alpha)$ if and only if

(2) $$p(z) \prec \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bw(z)} \iff p(z) = \frac{1 + [(1 - \alpha)A + \alpha B]w(z)}{1 + Bw(z)}$$

for some function $w(z) \in \Omega$ and every $z \in D$.

Furthermore, let $S^*(A, B, \alpha)$ denote the family of functions

(3) $$f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

regular in $D$ and such that $f(z)$ is in $S^*(A, B, \alpha)$ if and only if

(4) $$z \frac{f'(z)}{f(z)} = p(z)$$

for some $p(z) \in P(A, B, \alpha)$ and for all $z \in D$.

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2. New Results On The Class $S^*(A, B, \alpha)$

In this section we shall give representation theorems, distortion theorems and establish the radius of starlikeness for the class $S^*(A, B, \alpha)$. Our proofs are based on I.S. Jack’s Lemma \[2\].

**Lemma 1.** Let $w(z)$ be a non-constant and analytic function in the unit disc $D$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_1$, then $z_1 w'(z_1) = kw(z_1)$ and $k \geq 1$.

From the definition of the class $P(1,-1,0)$ called the Caratheodory class and $P(A, B, \alpha)$ we easily obtain the following lemma.

**Lemma 2.** If $p(z) \in P(A, B, \alpha)$ if and only if
\[
(5) \quad p(z) = \left[ 1 + \frac{(1 - \alpha)(A + \alpha B)q(z) + [(1 - \alpha)(A + B)]}{[1 + B]q(z) + [1 - B]} \right]
\]
for some $q(z) \in P(1,-1,0)$.

Let $\zeta$ be an arbitrary fixed point of $D$. We consider the functional
\[
(6) \quad F(p) = p(\zeta), p(z) \in P(A, B, \alpha).
\]

**Lemma 3.** The set of the values of the functional (6) is the closed disc with centered at $C(r)$ and having the radius $\rho(r)$, where
\[
\begin{cases}
C(r) = \left( \frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-Br^2}, 0 \right), & \rho(r) = \frac{(1-\alpha)(A-B)r}{1-Br^2}, \quad B \neq 0, \\
C(r) = (1,0), & \rho(r) = (1 - \alpha) |A| r, \quad B = 0.
\end{cases}
\]

*Proof.* Every boundary function $p_0(z)$ of $P(A, B, \alpha)$ with respect to the functional (6) can be written in the form (5), where
\[
q(z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}, \quad |\varepsilon| = 1.
\]
Hence
\[
(7) \quad p_0(z) = \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bz}.
\]
Since $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$,
\[
p_0(z) = C(r) + \rho \eta,
\]
\[
\eta = \varepsilon e^{i\theta} \frac{1 + Br e^{i\theta} - \varepsilon e^{-i\theta}}{1 + Br e^{i\theta}},
\]
which completes the proof. $\square$

**Lemma 4.** The function
\[
w = w(z) = \left\{ \begin{aligned}
\frac{(1-\alpha)(A-B)z}{1+Bz}, & \quad B \neq 0, \\
\frac{(1-\alpha)Az}{(1-\alpha)A}, & \quad B = 0,
\end{aligned} \right.
\]
maps $|z| = r$ onto the disc centered at $C(r)$, and having the radius $\rho(r)$
\[
\begin{cases}
C(r) = \left( -\frac{B[(1-\alpha)(A-B)r^2]}{1-Br^2}, 0 \right), & \rho(r) = \frac{(1-\alpha)(A-B)r}{1-Br^2}, \quad B \neq 0, \\
C(r) = (0,0), & \rho(r) = (1 - \alpha) |A| r, \quad B = 0.
\end{cases}
\]
Proof. This is immediate from
\[ w = \frac{(1 - \alpha)(A - B)z}{1 + Bz} \Rightarrow \]
\[ u^2 + v^2 + \frac{2B(1 - \alpha)(A - B)v^2}{1 - B^2r^2} = 0, \quad B \neq 0, \]
\[ w = (1 - \alpha)Az \Rightarrow u^2 + v^2 - (1 - \alpha)^2A^2r^2 = 0, \quad B = 0. \]
\[ \square \]

**Theorem 1.** Let \( f(z) = z + a_2z^2 + \cdots \) be an analytic function in the unit disc \( D \).

If \( f(z) \) satisfying
\[ (10) \quad \left( z \frac{f'(z)}{f(z)} - 1 \right) = \begin{cases} \frac{(1 - \alpha)(A - B)z}{1 + Bz}, & B \neq 0, \\ (1 - \alpha)Az = F_2(z), & B = 0, \end{cases} \]
then \( f(z) \in S^*(A, B, \alpha) \) and this result is as sharp as the function
\[ \left( \frac{1 + [(1 - \alpha)A + B\alpha]z}{1 + Bz} \right). \]

**Proof.** We define the function \( w(z) \) by
\[ (11) \quad \frac{f(z)}{z} = \begin{cases} (1 + Bw(z))^{(1 - \alpha)(A - B)z}, & B \neq 0, \\ e^{(1 - \alpha)Aw(z)}, & B = 0, \end{cases} \]
where \((1 + Bw(z))^{(1 - \alpha)(A - B)z}\) and \(e^{(1 - \alpha)Aw(z)}\) have the value 1 at the origin. Then \( w(z) \) is analytic in \( D \) and \( w(0) = 0 \). If we take the logarithmic derivate of equality \((11)\), simple calculations yield
\[ (12) \quad \left( z \frac{f'(z)}{f(z)} - 1 \right) = \begin{cases} \frac{(1 - \alpha)(A - B)zw'(z)}{1 + Bw(z)}, & B \neq 0, \\ (1 - \alpha)Azw'(z), & B = 0. \end{cases} \]

Now it is easy to realize that the subordination \((10)\) is equivalent to \(|w(z)| < 1\) for all \( z \in D \) indeed assume the contrary. There exist \( z_1 \in D \) such that \(|w(z_1)| = 1\). Then by I.S. Jack’s Lemma \( z_1w'(z_1) = kw(z_1) \) and \( k \geq 1 \), for such \( z_1 \in D \) and using Lemma 4 we have
\[ (13) \quad \left( z_1 \frac{f'(z)}{f(z)} - 1 \right) = \begin{cases} \frac{(1 - \alpha)(A - B)kw(z_1)}{1 + Bw(z_1)}, & B \neq 0, \\ (1 - \alpha)Aw(z_1) = F_2(w(z_1)) \neq F_2(D), & B = 0, \end{cases} \]
because \(|w(z_1)| = 1\) and \( k \geq 1 \). But this contradicts condition \((10)\) of this theorem and so \(|w(z)| < 1\) for all \( z \in D \). By using condition \((10)\) we get
\[ \frac{f'(z)}{f(z)} = \begin{cases} \frac{1 + [(1 - \alpha)A + B\alpha]w(z)}{1 + Bw(z)}, & B \neq 0, \\ (1 - \alpha)Aw(z), & B = 0, \end{cases} \]
which ends the proof. \( \square \)

**Corollary 1.** Let \( f(z) \in S^*(A, B, \alpha) \). Then \( f(z) \) can be written in the form
\[ f(z) = \begin{cases} z(1 + Bw(z))^{(1 - \alpha)(A - B)z}, & B \neq 0, \\ ze^{(1 - \alpha)Aw(z)}, & B = 0. \end{cases} \]

**Theorem 2.** If \( f(z) \in S^*(A, B, \alpha) \), then
\[ (14) \quad \begin{cases} r(1 - Br)^{(1 - \alpha)(A - B)} \leq |f(z)| \leq r(1 + Br)^{(1 - \alpha)(A - B)}, & B \neq 0, \\ rD(1 - \alpha)|A|r \leq |f(z)| \leq rD(1 - \alpha)|A|r, & B = 0. \end{cases} \]
These bounds are sharp with the extremal function
\begin{equation}
(15)
f_s(z) = \begin{cases} 
  
  \frac{z(1+Bz)^{(1-\alpha)(A-B)}}{z^{(1-\alpha)A}}, & B \neq 0, \\
  
  \frac{1}{z}, & B = 0.
\end{cases}
\end{equation}

Proof. The set of the values of \( \frac{z'f(z)}{f(z)} \) is the closed disc with centered at \( C(r) = \frac{1-B[1-(\alpha + B)]r^2}{1-B^2r^2} \) and having the radius \( \rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2} \) by using Lemma 3, that is
\begin{equation}
(16) \quad \left| \frac{zf'(z)}{f(z)} - \frac{1-B[(1-\alpha)A + \alpha B]r^2}{1-B^2r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1-B^2r^2}.
\end{equation}

After simple calculations from (16) we get
\begin{equation}
(17) \quad \begin{cases} 
  \frac{1- (1-\alpha)(A-B)r-B[(1-\alpha)A + \alpha B]r^2}{1-B^2r^2} \leq Re \left( \frac{zf'(z)}{f(z)} \right), & B \neq 0, \\
  \frac{1}{1+B^2r^2} \leq Re \left( \frac{zf'(z)}{f(z)} \right) \leq 1 + (1-\alpha)|A|, & B = 0.
\end{cases}
\end{equation}

On the other hand we have
\begin{equation}
(18) \quad Re \left( \frac{zf'(z)}{f(z)} \right) = \frac{\partial}{\partial r} \log |f(z)|, |z| = r.
\end{equation}

If we substitute (18) into the (17) we get
\begin{equation}
(19) \quad \begin{cases} 
  \frac{1}{r} - \frac{(1-\alpha)(A-B)}{1+B^2r^2} \leq \frac{\partial}{\partial r} \log |f(z)| \leq \frac{1}{r} + \frac{(1-\alpha)(A-B)}{1+B^2r^2}, & B \neq 0, \\
  \frac{1}{r} - (1-\alpha)|A| \leq \frac{\partial}{\partial r} \log |f(z)| \leq \frac{1}{r} + (1-\alpha)|A|, & B = 0.
\end{cases}
\end{equation}

Integrating both sides (19) we obtain (14). \( \Box \)

Corollary 2. The radius of starlikeness of the class \( S^*(A, B, \alpha) \) is
\begin{equation}
(20) \quad r_s = \frac{2}{(1-\alpha)(A-B) + \sqrt{(1-\alpha)^2(A-B)^2 + 4B[(1-\alpha)A + \alpha B]}}.
\end{equation}

This radius is sharp because the extremal function is given in (15).

Proof. From (17) we have
\begin{equation}
(21) \quad Re \left( \frac{zf'(z)}{f(z)} \right) \geq \frac{1- (1-\alpha)(A-B)r-B[(1-\alpha)A + \alpha B]r^2}{1-B^2r^2}.
\end{equation}

Hence for \( r < r_s \) the first hand side of the preceding inequality is positive this implies that
\begin{equation} 
(22) \quad r_s = \frac{2}{(1-\alpha)(A-B) + \sqrt{(1-\alpha)^2(A-B)^2 + 4B[(1-\alpha)A + \alpha B]}}.
\end{equation}

Also note that the inequality (20) becomes an equality for the function which is given in (15). It follows that
\begin{equation} 
(23) \quad r_s = \frac{2}{(1-\alpha)(A-B) + \sqrt{(1-\alpha)^2(A-B)^2 + 4B[(1-\alpha)A + \alpha B]}}.
\end{equation}

and the proof is complete. \( \Box \)

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