CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS II

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Abstract. Let \( f \) be analytic in \( D = \{ z : |z| < 1 \} \) with \( f(0) = f'(0) - 1 = 0 \) and \( \frac{f(z)}{f'(z)} \neq 0 \). Suppose \( \delta \geq 0 \) and \( \gamma > 0 \). For \( 0 < \beta < 1 \), the largest \( \alpha(\beta, \delta, \gamma) \) is found such that
\[
\left( \delta \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (\gamma - \delta) \left( \frac{zf'(z)}{f(z)} \right) \right) \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta, \delta, \gamma)}.
\]
The result solves the inclusion problem for certain subclass of analytic functions involving starlike and convex functions defined in a sector. Further we investigate the inclusion problem involving addition of powers of convex and starlike functions.

1. Introduction

Let \( S \) denote the class of normalised analytic univalent functions \( f \) defined by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) for \( z \in D = \{ z : |z| < 1 \} \). It is well-known \([7], [2]\) that \( f \in C(\alpha) \) implies \( f \in S^*(\beta) \) where
\[
\beta = \begin{cases} 
\frac{1 - 2\alpha}{2^{2-2\alpha}(1 - 2^{2\alpha-1})} & \text{if } \alpha \neq \frac{1}{2} \\
\frac{1}{2 \log 2} & \text{if } \alpha = \frac{1}{2}
\end{cases}
\]
and \( C(\alpha) \) denotes the class of analytic convex functions satisfying
\[
\Re \left( 1 + \frac{z^n}{f'(z)} \right) > \alpha
\]
for \( 0 \leq \alpha < 1 \) and \( S(\beta) \) denotes the class of analytic starlike functions satisfying
\[
\Re \left( \frac{zf'(z)}{f'(z)} \right) > \beta
\]
for \( 0 \leq \beta < 1 \), and that this result is best possible. Nunokawa and Thomas \([5]\) proved the analogue of this result for function defined via a sector as follows:

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Theorem 1.1. Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) = 0 \). Then for \( 0 < \beta \leq 1 \) and \( z \in D \),

\[
1 + \frac{zf''(z)}{f'(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta)}
\]

implies

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]

where

\[
\alpha(\beta) = \frac{2}{\pi} \arctan \left( \tan \frac{\beta \pi}{2} + \frac{\beta}{1 - \beta} \frac{1 + \beta}{2} \right),
\]

and \( \alpha(\beta) \) given by (1.2) is the largest number such that (1.1) holds.

Subsequently, Marjono and Thomas [3] extended this and proved:

Theorem 1.2. Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) - 1 = 0 \) and \( \frac{f(z)}{z} f'(z) \neq 0 \), and, \( 0 < \beta \leq 1 \) be given. Then for \( \delta > 0 \) and \( z \in D \),

\[
\delta \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \delta) \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta, \delta)}
\]

implies

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]

where

\[
\alpha(\beta, \delta) = \frac{2}{\pi} \arctan \left( \tan \frac{\beta \pi}{2} + \frac{\beta \delta}{1 - \beta} \frac{1 + \beta}{2} \cos \frac{\beta \pi}{2} \right),
\]

and \( \alpha(\beta, \delta) \) given by (1.4) is the largest number such that (1.3) holds.

Recently, Darus [1] gave the following:

Theorem 1.3. Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) - 1 = 0 \) and \( \frac{f(z)}{z} f'(z) \neq 0 \). Suppose \( \lambda \geq 0 \) and \( \lambda + \mu > 0 \). Then for \( 0 < \beta \leq 1 \),

\[
\lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \mu \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta, \lambda, \mu)}
\]

implies

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]
for \( z \in D \), where

\[
\alpha(\beta, \lambda, \mu) = \frac{2}{\pi} \arctan \left( \frac{\beta \pi}{2} + \frac{\beta \lambda}{1 - \beta} \frac{1 + \beta}{2} \frac{1 + \beta}{2} \cos \frac{\beta \pi}{2} \right),
\]

and \( \alpha(\beta, \lambda, \mu) \) given by (1.6) is the largest number such that (1.5) holds.

Next we consider a more general case involves convex and starlike functions.

2. Result

**Theorem 2.1.** Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) = 0 \) and \( f(z)/z \neq 0 \). Suppose \( \delta \geq 0 \) and \( \gamma > 0 \). For \( 0 < \beta < 1 \),

\[
\left( \delta \left( 1 + \frac{zf'(z)}{f(z)} \right) + \left( \gamma - \delta \right) \left( \frac{zf'(z)}{f(z)} \right) \right) \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta, \delta, \gamma)}
\]

implies

\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]

for \( z \in D \), where

\[
(2.2) \quad \alpha(\beta, \delta, \gamma) = \frac{2}{\pi} \arctan \left( \frac{\beta \pi}{2} + \frac{\beta \delta}{1 - \beta} \frac{1 + \beta}{2} \frac{1 + \beta}{2} \cos \frac{\beta \pi}{2} \right) + \beta,
\]

and \( \alpha(\beta, \delta, \gamma) \) given by (2.2) is the largest number such that (2.1) holds.

We shall need the following lemma.

**Lemma 1 ([4]).** Let \( F \) be analytic in \( D \) and \( G \) be analytic and univalent in \( D \), with \( F(0) = G(0) \). If \( F \neq G \), then there is a point \( z_0 \in D \) and \( \zeta_0 \in \delta D \) such that \( F(|z| < |z_0|) \subset G(D) \), \( F(z_0) = G(\zeta_0) \) and \( z_0F'(z_0) = m\zeta_0G'(\zeta_0) \) for \( m \geq 1 \).

**Proof of Theorem 2.1.** Write \( p(z) = \frac{zf'(z)}{f(z)} \), so that \( p \) is analytic in \( D \) and \( p(0) = 1 \). Thus we need to show that

\[
\delta zp'(z) + \gamma p(z)^2 < \left( \frac{1 + z}{1 - z} \right)^{\alpha}
\]

implies

\[
p(z) < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]

whenever \( \alpha = \alpha(\beta, \delta, \gamma) \).

Let \( h(z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta)} \) and \( q(z) = \left( \frac{1 + z}{1 - z} \right)^{\beta} \) so that \( |\arg h(z)| < \frac{\alpha(\beta) \pi}{2} \) and \( |\arg q(z)| < \frac{\beta \pi}{2} \). Suppose that \( p \neq q \), then from Lemma 2.1, there exists \( z_0 \in D \) and
\( \zeta_0 \in \delta D \) such that \( p(z_0) = q(\zeta_0) \) and \( p(|z| < |z_0|) \subset q(D) \). Since \( p(z_0) = q(\zeta_0) \neq 0 \), it follows that \( \zeta_0 \neq \pm 1 \). Thus we can write \( ri = \left(\frac{1 + \zeta_0}{1 - \zeta_0}\right) \) for \( r \neq 0 \). Next assume that \( r > 0 \), (if \( r < 0 \), the proof is similar) and Lemma 2.1 gives
\[
\delta z p'(z) + \gamma p(z)^2 = m \delta \zeta_0 q'(\zeta_0) + \gamma q(\zeta_0)^2
\]
\[(2.3) = \left(\frac{\gamma(r)^\beta + m \delta (1 + r^2)i}{2r}\right) (ri)^\beta.
\]
Since \( m \geq 1 \), taking the arguments, we obtain
\[
\arg \left( \delta z_0 p'(z_0) + \gamma p(z_0)^2 \right) = \arg \left( \tan \frac{\beta \pi}{2} + \frac{m \delta (1 + r^2)}{\gamma r^1 + \beta \cos \frac{\beta \pi}{2}} \right) + \frac{\beta \pi}{2},
\]
\[\geq \arg \left( \tan \frac{\beta \pi}{2} + \frac{\beta \delta (1 + r^2)}{\gamma r^1 + \beta \cos \frac{\beta \pi}{2}} \right) + \frac{\beta \pi}{2},
\]
\[\geq \arg \left( \tan \frac{\beta \pi}{2} + \frac{\beta \delta}{1 - \beta} \frac{1 + \beta}{2} \cos \frac{\beta \pi}{2} \right) + \frac{\beta \pi}{2},
\]
\[= \frac{\alpha(\beta, \delta, \gamma) \pi}{2},
\]
where a minimum is attained when \( r = \left(\frac{1 + \beta}{1 - \beta}\right)^{\frac{1}{2}} \).

Hence combining the cases \( r > 0 \) and \( r < 0 \) we obtain
\[
\frac{\alpha(\beta, \delta, \gamma) \pi}{2} \leq |\arg \left( \delta z_0 p'(z_0) + \gamma p(z_0)^2 \right)| \leq \pi,
\]
which contradicts the fact that \( |\arg h(z)| < \frac{\alpha(\beta, \delta, \gamma) \pi}{2} \), provided that (2.2) holds.

To show that \( \alpha(\beta, \delta, \gamma) \) is exact, take \( \alpha(\beta, \delta, \gamma) < \sigma < 1 \) so that for some \( \beta_0 > \beta \) we can write \( \sigma = \alpha(\beta_0, \delta, \gamma) \). Now let \( p(z) = \left(\frac{1 + \beta}{1 - \beta}\right)^{\beta_0} \). Then from the minimum principle for harmonic functions, it follows that
\[
\inf_{|z| < 1} \arg \left( \delta z p'(z) + \gamma p(z)^2 \right)
\]
is attained at some point \( z = e^{i \theta} \) for \( 0 < \theta < 2\pi \). Thus
\[
\delta z p'(z) + \gamma p(z)^2 = \left( \gamma \left(\frac{\sin \theta}{1 - \cos \theta}\right)^{\beta_0} \frac{\beta_0 \pi i}{2} + \frac{i \delta \beta_0}{\sin \theta} \right) \left( \frac{\sin \theta}{1 - \cos \theta}\right)^{\beta_0} \frac{\beta_0 \pi i}{2},
\]
and so taking $t = \cos \theta$, we obtain
\[
\arg (\delta z p'(z) + \gamma p(z)^2) = \arctan \left( \frac{\tan \frac{\beta_0 \pi}{2} + \frac{1}{2} - \frac{1}{2} \frac{\beta_0 \delta}{(1 + t) \cos \frac{\beta_0 \pi}{2}}}{\gamma (1 - t) \frac{1}{2} (1 + \frac{1}{2} \beta_0 \cos \frac{\beta_0 \pi}{2})} \right) + \frac{\beta_0 \pi}{2},
\]
and elementary calculation shows that the minimum of this expression is attained when $t = \beta_0$. Thus completes the proof of Theorem 2.1.

Particular choices for $\delta$ and $\gamma$ give the following interesting corollaries. First when $\delta = 1$ we have

**Corollary 2.1.** Let $f$ be analytic in $D$, with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{z} f'(z) \neq 0$. Then for $\gamma > 0$ and $0 < \beta < 1$,
\[
\left(1 + \frac{z f''(z)}{f'(z)} + (\gamma - 1) \left( \frac{z f'(z)}{f(z)} \right) \right) \frac{z f'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta,1,\gamma)}
\]
implies
\[
\frac{z f'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]
for $z \in D$, where
\[
\alpha(\beta,1,\gamma) = \frac{2}{\pi} \arctan \left( \frac{\tan \frac{\beta \pi}{2} + \frac{1}{2} - \frac{1}{2} \frac{\beta}{(1 + \frac{1}{2} \beta \cos \frac{\beta \pi}{2})}}{\gamma (1 - \beta) \frac{1}{2} (1 + \frac{1}{2} \beta \cos \frac{\beta \pi}{2})} \right) + \beta.
\]
Similarly, when $\gamma = 1$, we obtain

**Corollary 2.2.** Let $f$ be analytic in $D$, with $f(0) = f'(0) - 1 = 0$ and $\frac{f(z)}{z} f'(z) \neq 0$. Then for $\delta > 0$ and $0 < \beta < 1$,
\[
\left( \delta \left(1 + \frac{z f''(z)}{f'(z)} + (1 - \delta) \left( \frac{z f'(z)}{f(z)} \right) \right) \frac{z f'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta,\delta,1)}
\]
implies
\[
\frac{z f'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]
for $z \in D$, where
\[
\alpha(\beta,\delta,1) = \frac{2}{\pi} \arctan \left( \frac{\tan \frac{\beta \pi}{2} + \frac{1}{2} - \frac{1}{2} \frac{\beta \delta}{(1 - \beta) \frac{1}{2} (1 + \frac{1}{2} \beta \cos \frac{\beta \pi}{2})}}{} \right) + \beta.
\]
Finally, when $\delta = \gamma = 1$, we have the following interesting result.
Corollary 2.3. Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) - 1 = 0 \) and \( \frac{f(z)}{z} f'(z) \neq 0 \). Then for \( 0 < \beta < 1 \),
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta,1,1)}
\]
implies
\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]
for \( z \in D \), where
\[
\alpha(\beta,1,1) = \frac{2}{\pi} \arctan \left( \frac{\tan \frac{\beta\pi}{2} + \beta}{(1 - \beta) \frac{1 - \beta}{2} (1 + \beta) \frac{1 + \beta}{2} \cos \frac{\beta\pi}{2} + \beta} \right).
\]

Remark 2.1. We note that \( \lim_{\beta \to 1} \alpha(\beta,1,1) = 2 \), which suggests from Theorem 2.1 that
\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right) \frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{2}
\]
implies
\[
\frac{zf''(z)}{f'(z)} < \frac{1 + z}{1 - z}.
\]
However if we let \( \beta = 1 \), the right hand side in (2.3) is real and the method of proof in Theorem 2.1 breaks down.

Next we give the following:

Theorem 2.2. Let \( f \) be analytic in \( D \), with \( f(0) = f'(0) - 1 = 0 \) and \( \frac{f(z)}{z} f'(z) \neq 0 \). Suppose \( \lambda < \beta \mu \) and \( 0 < \lambda \leq 1 \). Then for \( 0 < \beta < 1 \),
\[
\left( \frac{zf'(z)}{f(z)} \right)^{\mu} + \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right)^{\lambda} \times \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta,\lambda,\mu)}
\]
implies
\[
\frac{zf'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta},
\]
for \( z \in D \), where
\[
(2.4)
\]
and
\[
\alpha(\beta,\lambda,\mu) = \frac{2}{\pi} \arctan \left( \frac{\tan \frac{\beta\mu\pi}{2} + (\lambda\beta)^{\frac{\lambda}{2}} \sin \frac{\lambda\pi}{2}}{(\lambda - \beta\mu) \frac{(\lambda - \beta\mu)^{\frac{\lambda}{2}}}{2} (\lambda + \beta\mu) \frac{(\lambda + \beta\mu)}{2} \cos \frac{\beta\mu\pi}{2} + \beta} \right)
\]

and \( \alpha(\beta,\lambda,\mu) \) given by (2.5) is the largest number such that (2.4) holds.
Proof. Write \( p(z) = \frac{zf'(z)}{f(z)} \), so that \( p \) is analytic in \( D \) and \( p(0) = 1 \). Thus we need to show that

\[
p(z)^\mu + \left( \frac{zp'(z)}{p(z)} \right)^\lambda < \left( \frac{1 + z}{1 - z} \right)^\alpha
\]

implies

\[
p(z) < \left( \frac{1 + z}{1 - z} \right)^\beta,
\]

whenever \( \alpha = \alpha(\beta, \lambda, \mu) \).

As before, let \( h(z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha(\beta)} \) and \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\beta \) so that

\[
|\arg h(z)| < \frac{\alpha(\beta)\pi}{2}
\]

and \( |\arg q(z)| < \frac{\beta\pi}{2} \). Suppose that \( p \neq q \), then from Lemma 2.1, there exists \( z_0 \in D \) and \( \zeta_0 \in \delta D \) such that \( p(z_0) = q(\zeta_0) \) and \( p(|z| < |z_0|) < q(D) \). Since \( p(z_0) = q(\zeta_0) \neq 0 \), it follows that \( \zeta_0 \neq \pm 1 \). Thus we can write \( ri = \frac{1 + \zeta_0}{1 - \zeta_0} \) for \( r \neq 0 \). Next assume that \( r > 0 \), (if \( r < 0 \), the proof is similar) and Lemma 2.1 gives

\[
p(z_0)^\mu + \left( \frac{z_0 p'(z_0)}{p(z_0)} \right)^\lambda = q(\zeta_0)^\beta + \left( \frac{m \zeta_0 q'(\zeta_0)}{q(\zeta_0)} \right)^\lambda,
\]

\[
= (ri)^\beta + \left( \frac{m \beta (1 + r^2)i}{2r} \right)^\lambda.
\]

The result now follows by using the same arguments as before.

To show that \( \alpha(\beta, \lambda, \mu) \) is exact, we argue as in the proof of Theorem 2.1 so that for some \( \beta_0 \), again choose \( p(z) = \frac{zf'(z)}{f(z)} = \left( \frac{1 + z}{1 - z} \right)^{\beta_0} \) with \( z = e^{i\theta} \) for \( 0 < \theta < 2\pi \). Thus with \( t = \cos \theta \), we obtain

\[
p(z)^\mu + \left( \frac{zp'(z)}{p(z)} \right)^\lambda = \left( 1 + t \right)^{\beta_0} e^{\frac{\mu \pi}{2}} + \left( \frac{\beta_0}{\sqrt{1 - t^2}} \right)^\lambda e^{\frac{\lambda \pi}{2}}.
\]

and taking arguments, we have

\[
\arg \left( \left( \frac{zf'(z)}{f(z)} \right)^\mu + \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right)^\lambda \right) = \arctan \left( \frac{\tan \frac{\beta_0 \mu \pi}{2} + \frac{\beta_0^\lambda \sin \frac{\lambda \pi}{2}}{(\lambda - \beta_0 \mu)^\frac{(\lambda - \beta_0 \mu)}{2} (\lambda + \beta_0 \mu)^\frac{(\lambda + \beta_0 \mu)}{2} \cos \frac{\lambda \mu \pi}{2}}}{1 + \frac{\beta_0^\lambda \cos \frac{\lambda \pi}{2}}{(\lambda - \beta_0 \mu)^\frac{(\lambda - \beta_0 \mu)}{2} (\lambda + \beta_0 \mu)^\frac{(\lambda + \beta_0 \mu)}{2} \cos \frac{\lambda \mu \pi}{2}}} \right).
\]

and elementary calculation shows that the minimum of this expression is attained when \( t = \frac{\beta_0 \mu}{2} \). Thus the proof of Theorem 2.2 is complete. \( \square \)

Remark 2.2. When \( \lambda = \mu = 1 \) we obtain Theorem 1.1.
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