THE BECKMAN-QUARLES THEOREM FOR MAPPINGS FROM 
\( \mathbb{C}^2 \) TO \( \mathbb{C}^2 \)

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Abstract. Let \( \varphi : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \), \( \varphi((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \). We say that \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) preserves distance \( d \geq 0 \), if for each \( X, Y \in \mathbb{C}^2 \) \( \varphi(X, Y) = d^2 \) implies \( \varphi(f(X), f(Y)) = d^2 \). If \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) preserves distance \( d \geq 0 \), if for each \( X, Y \in \mathbb{C}^2 \) \( \varphi(X, Y) = d^2 \) implies \( \varphi(f(X), f(Y)) = d^2 \). We prove that each unit-distance preserving mapping \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) has a form \( I \circ \gamma \), where \( \gamma : \mathbb{C} \to \mathbb{C} \) is a field homomorphism and \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) is an affine mapping with orthogonal linear part; it follows from a general theorem proved in \( [8] \) that such a mapping from \( \mathbb{K}^2 \) to \( \mathbb{K}^2 \), where \( \mathbb{K} \) is a field such that \( \text{char}(\mathbb{K}) \notin \{2, 5\} \) and \( -1 \) is a square.

The classical Beckman-Quarles theorem states that each unit-distance preserving mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) \( (n \geq 2) \) is an isometry, see \([1]\)–\([5]\). Let \( \varphi : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \), \( \varphi((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \). We say that \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) preserves distance \( d \geq 0 \), if for each \( X, Y \in \mathbb{C}^2 \) \( \varphi(X, Y) = d^2 \) implies \( \varphi(f(X), f(Y)) = d^2 \). If \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) and for each \( X, Y \in \mathbb{C}^2 \) \( \varphi(X, Y) = \varphi(f(X), f(Y)) \), then \( f \) is an affine mapping with orthogonal linear part; it follows from a general theorem proved in \([3, 58 \text{ ff.}]\), see also \([4, \text{ p. 30}]\). The author proved in \([9]\): each unit-distance preserving mapping \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) satisfies

\[ \varphi(X, Y) = \varphi(f(X), f(Y)) \]

for all \( X, Y \in \mathbb{C}^2 \) with rational \( \varphi(X, Y) \).

Theorem 1. If \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) preserves unit distance, \( f((0, 0)) = (0, 0), f((1, 0)) = (1, 0) \text{ and } f((0, 1)) = (0, 1) \), then there exists a field homomorphism \( \rho : \mathbb{R} \to \mathbb{C} \) satisfying \( \forall x_1, x_2 \in \mathbb{C} : \)

\[ f((x_1, x_2)) \in \{(\rho(\text{Re}(x_1)) + \rho(\text{Im}(x_1)) \cdot i, \rho(\text{Re}(x_2)) + \rho(\text{Im}(x_2)) \cdot i), \]

\[ (\rho(\text{Re}(x_1)) - \rho(\text{Im}(x_1)) \cdot i, \rho(\text{Re}(x_2)) - \rho(\text{Im}(x_2)) \cdot i) \} . \]

Proof. Obviously, \( g = f_{|\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{C}^2 \) preserves unit distance. The author proved in \([8]\) that such a \( g \) has a form \( I \circ \rho \), where \( \rho : \mathbb{R} \to \mathbb{C} \) is a field homomorphism and \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) is an affine mapping with orthogonal linear part. Since \( f((0, 0)) = (0, 0), f((1, 0)) = (1, 0), f((0, 1)) = (0, 1) \), we conclude that \( f_{|\mathbb{R}^2} = (\rho, \rho) \). From this, condition (2) holds true if \( x_1, x_2 \in \mathbb{R}^2 \). Assume now that \( x_1, x_2 \in \mathbb{C}^2 \setminus \mathbb{R}^2 \).
Let \( x_1 = a_1 + b_1 \cdot i, \ x_2 = a_2 + b_2 \cdot i, \) where \( a_1, b_1, a_2, b_2 \in \mathbb{R}, \) and, for example \( b_1 \neq 0. \) For each \( t \in \mathbb{R} \)

\[
\varphi((a_1 + b_1 \cdot i, \ a_2 + b_2 \cdot i), \ (a_1 + tb_2, \ a_2 - tb_1)) = (t^2 - 1)(b_1^2 + b_2^2).
\]

By this and (1): for each \( t \in \mathbb{R} \) with rational \((t^2 - 1)(b_1^2 + b_2^2)\) we have:

(3) \quad \varphi(f((a_1 + b_1 \cdot i, \ a_2 + b_2 \cdot i)), \ f((a_1 + tb_2, \ a_2 - tb_1))) = (t^2 - 1)(b_1^2 + b_2^2).

Let \( f((a_1 + b_1 \cdot i, \ a_2 + b_2 \cdot i)) = (y_1, y_2). \) From (3) and \( f_{\mathbb{R}^2} = (\rho, \rho) \) we obtain: for each \( t \in \mathbb{R} \) with rational \((t^2 - 1)(b_1^2 + b_2^2)\) we have:

(4) \quad (y_1 - \rho(a_1) - \rho(t)\rho(b_2))^2 + (y_2 - \rho(a_2) + \rho(t)\rho(b_1))^2 = (t^2 - 1)(b_1^2 + b_2^2).

For each \( t \in \mathbb{R} \) with rational \((t^2 - 1)(b_1^2 + b_2^2)\) we have:

\[(t^2 - 1)(b_1^2 + b_2^2) = \rho((t^2 - 1)(b_1^2 + b_2^2)) = (\rho(t)^2 - 1)(\rho(b_1)^2 + \rho(b_2)^2).\]

By this and (4): for each \( t \in \mathbb{R} \) with rational \((t^2 - 1)(b_1^2 + b_2^2)\) we have:

(5) \quad (y_1 - \rho(a_1))^2 + (y_2 - \rho(a_2))^2 + \rho(b_1)^2 + \rho(b_2)^2 + 2\rho(t) \times (\rho(b_1)(y_2 - \rho(a_2)) - \rho(b_2)(y_1 - \rho(a_1))) = 0.

There are infinitely many \( t \in \mathbb{R} \) with rational \((t^2 - 1)(b_1^2 + b_2^2)\) and \( \rho \) is injective. From these two facts and (5), we obtain:

(6) \quad \rho(b_1)(y_2 - \rho(a_2)) - \rho(b_2)(y_1 - \rho(a_1)) = 0

and

(7) \quad (y_1 - \rho(a_1))^2 + (y_2 - \rho(a_2))^2 + \rho(b_1)^2 + \rho(b_2)^2 = 0.

By (6):

(8) \quad y_2 - \rho(a_2) = \frac{\rho(b_2)}{\rho(b_1)} (y_1 - \rho(a_1)).

Applying (8) to (7) we get:

\[
(y_1 - \rho(a_1))^2 + \frac{\rho(b_2)^2}{\rho(b_1)^2} \cdot (y_1 - \rho(a_1))^2 + \rho(b_1)^2 + \rho(b_2)^2 = 0.
\]

It gives \( \left( \frac{(y_1 - \rho(a_1))^2}{\rho(b_1)^2} + 1 \right) \cdot (\rho(b_1)^2 + \rho(b_2)^2) = 0. \) Since \( \rho(b_1)^2 + \rho(b_2)^2 \neq 0, \) we get

\[
y_1 = \rho(a_1) + \rho(b_1) \cdot i \quad \text{or} \quad y_1 = \rho(a_1) - \rho(b_1) \cdot i.
\]

In case 1, by (8)

\[
y_2 = \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1))
\]

\[
= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (\rho(a_1) + \rho(b_1) \cdot i - \rho(a_1))
\]

\[
= \rho(a_2) + \rho(b_2) \cdot i.
\]
In case 2, by (8)
\[ y_2 = \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)) \]
\[ = \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (\rho(a_1) - \rho(b_1) \cdot i - \rho(a_1)) \]
\[ = \rho(a_2) - \rho(b_2) \cdot i. \]

The proof is completed. \( \square \)

Let \( f: \mathbb{C}^2 \to \mathbb{C}^2 \) preserves unit distance, \( f((0, 0)) = (0, 0), f((1, 0)) = (1, 0) \) and \( f((0, 1)) = (0, 1) \). Theorem 1 provides a field homomorphism \( \rho: \mathbb{R} \to \mathbb{C} \) satisfying (2). By Theorem 1 the sets
\[ \mathbf{A} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\rho(\text{Re}(x_1)) + \rho(\text{Im}(x_1)) \cdot i, \rho(\text{Re}(x_2)) + \rho(\text{Im}(x_2)) \cdot i) \} \]
and
\[ \mathbf{B} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\rho(\text{Re}(x_1)) - \rho(\text{Im}(x_1)) \cdot i, \rho(\text{Re}(x_2)) - \rho(\text{Im}(x_2)) \cdot i) \} \]

satisfy \( \mathbf{A} \cup \mathbf{B} = \mathbb{C}^2 \). The mapping
\[ \mathbb{C} \ni x \xrightarrow{\theta} \rho(\text{Re}(x)) + \rho(\text{Im}(x)) \cdot i \in \mathbb{C} \]
is a field homomorphism, \( \theta \) extends \( \rho \),
\[ \mathbf{A} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\theta(x_1), \theta(x_2)) \}. \]
The mapping
\[ \mathbb{C} \ni x \xrightarrow{\zeta} \rho(\text{Re}(x)) - \rho(\text{Im}(x)) \cdot i \in \mathbb{C} \]
is a field homomorphism, \( \zeta \) extends \( \rho \),
\[ \mathbf{B} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\zeta(x_1), \zeta(x_2)) \}. \]

We would like to prove \( f = (\theta, \theta) \) or \( f = (\zeta, \zeta) \); we will prove it later in Theorem 2.

Let \( \psi: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}, \psi((x_1, x_2), (y_1, y_2)) = \text{Im}(x_1) \cdot \text{Im}(y_1) + \text{Im}(x_2) \cdot \text{Im}(y_2). \)

**Lemma 1.** If \( x_1, x_2, y_1, y_2 \in \mathbb{C}, \varphi((x_1, x_2), (y_1, y_2)) \in \mathbb{Q} \) and \( \psi((x_1, x_2), (y_1, y_2)) \neq 0, \) then
\[ (y_1, y_2) \in \mathbf{A} \implies (x_1, x_2) \in \mathbf{A} \]
and
\[ (y_1, y_2) \in \mathbf{B} \implies (x_1, x_2) \in \mathbf{B}. \]

**Proof.** We prove only (9), the proof of (10) follows analogically.

Let \( \varphi((x_1, x_2), (y_1, y_2)) = r \in \mathbb{Q}. \) Assume, on the contrary, that \( (y_1, y_2) \in \mathbf{A} \) and \( (x_1, x_2) \notin \mathbf{A}. \) Since \( \mathbf{A} \cup \mathbf{B} = \mathbb{C}^2 \), \( (x_1, x_2) \in \mathbf{B}. \) Let \( x_1 = a_1 + b_1 \cdot i, x_2 = a_2 + b_2 \cdot i, \)
\[ y_1 = \tilde{a}_1 + \tilde{b}_1 \cdot i, y_2 = \tilde{a}_2 + \tilde{b}_2 \cdot i, \]
where \( a_1, b_1, a_2, b_2, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2 \in \mathbb{R}. \) By (1):
\[ r = \varphi((x_1, x_2), (y_1, y_2)) = \varphi(f((x_1, x_2)), f((y_1, y_2))) \]
\[ = (\rho(a_1) - \rho(b_1) \cdot i - \rho(\tilde{a}_1) - \rho(\tilde{b}_1) \cdot i)^2 \]
\[ + (\rho(a_2) - \rho(b_2) \cdot i - \rho(\tilde{a}_2) - \rho(\tilde{b}_2) \cdot i)^2. \]
Since \( r \in \mathbb{Q} \),
\[
   r = \theta(r) = \theta((a_1 + b_1 \cdot i - \tilde{a}_1 - \tilde{b}_1 \cdot i)^2 + (a_2 + b_2 \cdot i - \tilde{a}_2 - \tilde{b}_2 \cdot i)^2)
\]
(12)
\[
   = (\rho(a_1) + \rho(b_1) \cdot i - \rho(\tilde{a}_1) - \rho(\tilde{b}_1) \cdot i)^2
   + (\rho(a_2) + \rho(b_2) \cdot i - \rho(\tilde{a}_2) - \rho(\tilde{b}_2) \cdot i)^2.
\]

Subtracting (11) and (12) by sides we obtain:
\[
   2\rho(b_1) \cdot i \cdot (2\rho(\tilde{b}_1) \cdot i - 2\rho(a_1) + 2\rho(\tilde{a}_1)) + 2\rho(b_2) \cdot i \cdot (2\rho(\tilde{b}_2) \cdot i - 2\rho(a_2) + 2\rho(\tilde{a}_2)) = 0.
\]
Thus
\[
   -\rho(b_1 \tilde{b}_1 + b_2 \tilde{b}_2) = \rho(b_1(a_1 - \tilde{a}_1) + b_2(a_2 - \tilde{a}_2)) \cdot i.
\]

Squaring both sides of (13) we get:
\[
   \rho((b_1 \tilde{b}_1 + b_2 \tilde{b}_2)^2 + (b_1(a_1 - \tilde{a}_1) + b_2(a_2 - \tilde{a}_2))^2) = 0,
\]
so in particular \( \psi((x_1, x_2), (y_1, y_2)) = b_1 \tilde{b}_1 + b_2 \tilde{b}_2 = 0 \), a contradiction. \( \square \)

The next lemma is obvious.

**Lemma 2.** For each \( S, T \in \mathbb{R}^2 \) there exist \( n \in \{1, 2, 3, \ldots\} \) and \( P_1, \ldots, P_n \in \mathbb{R}^2 \) such that \( ||S - P_1|| = ||P_1 - P_2|| = \ldots = ||P_{n-1} - P_n|| = ||P_n - T|| = 1 \).

**Lemma 3.** For each \( X \in \mathbb{C}^2 \setminus \mathbb{R}^2 \)
\[
   (i, i) \in A \text{ implies } X \in A
\]
and
\[
   (i, i) \in B \text{ implies } X \in B.
\]

**Proof.** Let \( X = (a_1 + b_1 \cdot i, a_2 + b_2 \cdot i) \), where \( a_1, b_1, a_2, b_2 \in \mathbb{R} \). Since \( X \in \mathbb{C}^2 \setminus \mathbb{R}^2 \), \( b_1 \neq 0 \) or \( b_2 \neq 0 \). Assume that \( b_1 \neq 0 \), when \( b_2 \neq 0 \) the proof is analogous. The points \( S = (a_1 + \sqrt{1 + b_2^2}, a_2 + \sqrt{1 + (b_1 - 1)^2}) \) and \( T = (\sqrt{2}, 0) \) belong to \( \mathbb{R}^2 \). Applying Lemma 2 we find \( P_1, \ldots, P_n \in \mathbb{R}^2 \) satisfying \( ||S - P_1|| = ||P_1 - P_2|| = \ldots = ||P_{n-1} - P_n|| = ||P_n - T|| = 1 \). The points
\[
   X_1 = X,
   X_2 = \left( a_1 + \sqrt{1 + b_2^2}, a_2 + b_2 \cdot i, a_2 \right),
   X_3 = S + (i, 0) = \left( a_1 + \sqrt{1 + b_2^2} + b_1 \cdot i, a_2 + \sqrt{1 + (b_1 - 1)^2} \right),
   X_4 = P_1 + (i, 0),
   X_5 = P_2 + (i, 0),
   \ldots
   X_{n+3} = P_n + (i, 0),
   X_{n+4} = T + (i, 0) = (\sqrt{2} + i, 0),
   X_{n+5} = (i, i)
\]
satisfy \( \varphi(X_{k-1}, X_k) = 1 \) for \( k \in \{2, 3, \ldots, n+5\} \); \( \psi(X_1, X_2) = b_2^2 \neq 0 \), \( \psi(X_2, X_3) = b_1 \neq 0 \) and \( \psi(X_{k-1}, X_k) = 1 \) for \( k \in \{4, 5, \ldots, n+5\} \).
By Lemma 1 for each $k \in \{2, 3, \ldots, n+5\}$

$$X_k \in A \text{ implies } X_{k-1} \in A$$

and

$$X_k \in B \text{ implies } X_{k-1} \in B.$$ 

Therefore, $(i, i) = X_{n+5} \in A$ implies $X = X_1 \in A$, and also, $(i, i) = X_{n+5} \in B$ implies $X = X_1 \in B$. \hfill \Box

**Theorem 2.** If $f: \mathbb{C}^2 \to \mathbb{C}^2$ preserves unit distance, $f(0, 0) = (0, 0)$, $f(1, 0) = (1, 0)$ and $f(0, 1) = (0, 1)$, then there exists a field homomorphism $\gamma: \mathbb{C} \to \mathbb{C}$ satisfying $f = (\gamma, \gamma)$.

**Proof.** By (1):

$$(i, i) \in A \text{ implies } \mathbb{C}^2 \setminus \mathbb{R}^2 \subseteq A$$

and

$$(i, i) \in B \text{ implies } \mathbb{C}^2 \setminus \mathbb{R}^2 \subseteq B.$$ 

Obviously, $\mathbb{R}^2 \subseteq A$ and $\mathbb{R}^2 \subseteq B$. Therefore,

$$A = \mathbb{C}^2 \text{ and } f = (\theta, \theta), \text{ if } (i, i) \in A,$$

and also,

$$B = \mathbb{C}^2 \text{ and } f = (\zeta, \zeta), \text{ if } (i, i) \in B.$$

\hfill \Box

As a corollary of Theorem 2 we get:

**Theorem 3.** Each unit-distance preserving mapping $f: \mathbb{C}^2 \to \mathbb{C}^2$ has a form $I \circ (\gamma, \gamma)$, where $\gamma: \mathbb{C} \to \mathbb{C}$ is a field homomorphism and $I: \mathbb{C}^2 \to \mathbb{C}^2$ is an affine mapping with orthogonal linear part.

**Proof.** By (1):

$$1 = \varphi((0, 0), (1, 0)) = \varphi(f((0, 0)), f((1, 0))),$$

$$1 = \varphi((0, 0), (0, 1)) = \varphi(f((0, 0)), f((0, 1))),$$

$$2 = \varphi((1, 0), (0, 1)) = \varphi(f((1, 0)), f((0, 1))).$$

By the above equalities there exists an affine mapping $J: \mathbb{C}^2 \to \mathbb{C}^2$ with orthogonal linear part such that

$$J(f((0, 0))) = (0, 0), \quad J(f((1, 0))) = (1, 0), \quad J(f((0, 1))) = (0, 1).$$

By Theorem 2 there exists a field homomorphism $\gamma: \mathbb{C} \to \mathbb{C}$ satisfying $J \circ f = (\gamma, \gamma)$, so $f = J^{-1} \circ (\gamma, \gamma)$. \hfill \Box

Obviously, Theorem 3 implies (1). The author proved in [10]:

(14) if $n \geq 2$ and a continuous $f: \mathbb{C}^n \to \mathbb{C}^n$ preserves unit distance, then $f$ has a form $I \circ (\rho, \ldots, \rho)$, where $I: \mathbb{C}^n \to \mathbb{C}^n$ is an affine mapping with orthogonal linear part and $\rho: \mathbb{C} \to \mathbb{C}$ is the identity or the complex conjugation.

The only continuous endomorphisms of $\mathbb{C}$ are the identity and the complex conjugation, see [6, Lemma 1, p. 356]. Therefore, Theorem 3 implies (14) restricted to $n = 2$.

Let $K$ be a field, $\text{char}(K) \not\in \{2, 3, 5\}$. Let $d: K^2 \times K^2 \to K$ denote the Lorentz-Minkowski distance defined by $d((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)(x_2 - y_2)$. H. Schaeffer proved in [7, Satz 1, Satz 2, Satz 3]:
(15) if \( f : K^2 \rightarrow K^2 \) preserves the Lorentz-Minkowski distance 1, \( f((0, 0)) = (0, 0) \) and \( f((1, 1)) = (1, 1) \), then there exists a field homomorphism \( \sigma : K \rightarrow K \) satisfying \( \forall x_1, x_2 \in K \ f((x_1, x_2)) = (\sigma(x_1), \sigma(x_2)) \) or \( \forall x_1, x_2 \in K \ f((x_1, x_2)) = (\sigma(x_1), \sigma(x_2)) \).

Unfortunately, the proof of Satz 3 in [7] is complicated, the main part of this proof was constructed using computer software.

Let \( \varphi_K : K^2 \times K^2 \rightarrow K \), \( \varphi_K((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \). Theorem 4 generalizes Theorem 3.

**Theorem 4.** Let there exists \( i \in K \) such that \( i^2 + 1 = 0 \). Let \( f : K^2 \rightarrow K^2 \) preserves unit distance defined by \( \varphi_K \). We claim that \( f \) has a form \( I \circ \sigma, \) where \( \sigma : K \rightarrow K \) is a field homomorphism and \( I : K^2 \rightarrow K^2 \) is an affine mapping with orthogonal linear part.

**Proof.** Assume that \( f((0, 0)) = (0, 0) \). The mappings
\[
K^2 \ni (x_1, x_2) \xrightarrow{\xi} (x_1 + i \cdot x_2, x_1 - i \cdot x_2) \in K^2
\]
and
\[
K^2 \ni (x_1, x_2) \xrightarrow{\eta} \left( \frac{1}{2}x_1 + \frac{1}{2}x_2, -\frac{i}{2}x_1 + \frac{i}{2}x_2 \right) \in K^2
\]
satisfy:
\[
\eta \circ \xi = \xi \circ \eta = \text{id}(K^2),
\]
\[
\forall x_1, x_2, y_1, y_2 \in K, \varphi_K((x_1, x_2), (y_1, y_2)) = d(\xi((x_1, x_2)), \xi((y_1, y_2))),
\]
\[
\forall x_1, x_2, y_1, y_2 \in K, d((x_1, x_2), (y_1, y_2)) = \varphi_K((\eta((x_1, x_2)), \eta((y_1, y_2))).
\]

Therefore, \( \xi \circ f \circ \eta : K^2 \rightarrow K^2 \) preserves the Lorentz-Minkowski distance 1. Obviously, \( (\xi \circ f \circ \eta)((0, 0)) = (0, 0) \). Let \( (\xi \circ f \circ \eta)((1, 1)) = (a, b) \in K^2 \). We have:
\[
i = d((1, 1), (0, 0)) = d((\xi \circ f \circ \eta)((1, 1)), (\xi \circ f \circ \eta)((0, 0))) = d((a, b), (0, 0)) = a \cdot b.
\]
Hence \( b = \frac{1}{a} \). For each \( z \in K \setminus \{0\} \) the mapping
\[
K^2 \ni (x, y) \xrightarrow{\lambda(z)} \left( \frac{x}{z}, z \cdot y \right) \in K^2
\]
preserves all Lorentz-Minkowski distances, \( \lambda(\frac{1}{z}) \circ \lambda(z) = \lambda(z) \circ \lambda(\frac{1}{z}) = \text{id}(K^2) \). The mapping \( \lambda(a) \circ \xi \circ f \circ \eta : K^2 \rightarrow K^2 \) preserves the Lorentz-Minkowski distance 1, \( (\lambda(a) \circ \xi \circ f \circ \eta)((0, 0)) = (0, 0) \) and \( (\lambda(a) \circ \xi \circ f \circ \eta)((1, 1)) = (1, 1) \). By (15) there exists a field homomorphism \( \sigma : K \rightarrow K \) satisfying
\[
\lambda(a) \circ \xi \circ f \circ \eta = h \circ (\sigma, \sigma),
\]
where \( h : K^2 \rightarrow K^2, h((x_1, x_2)) = (x_2, x_1) \).

In case 1: \( f = \eta \circ \lambda(\frac{1}{z}) \circ (\sigma, \sigma) \circ \xi = f_1 \circ (\sigma, \sigma), \) where \( f_1 : K^2 \rightarrow K^2, \)
\[
f_1((x_1, x_2)) = \left( \left( \frac{a}{2} + \frac{1}{2a} \right) \cdot x_1 + \left( \frac{a}{2} - \frac{1}{2a} \right) i \cdot x_2, \right. \]
\[
\left. - \left( \frac{a}{2} + \frac{1}{2a} \right) i \cdot x_1 - \left( \frac{a}{2} + \frac{1}{2a} \right) i \cdot x_2 \right). \]
In case 2: \( f = \eta \circ \lambda(\frac{1}{2}) \circ h \circ (\sigma, \sigma) \circ \xi = f_2 \circ (\sigma, \sigma) \), where \( f_2 : \mathbb{K}^2 \to \mathbb{K}^2 \),

\[
f_2((x_1, x_2)) = \left( \left( \frac{a}{2} + \frac{1}{2a} \right) x_1 - \left( \frac{a}{2} - \frac{1}{2a} \right) \sigma(i) \cdot x_2, \right.
\]

\[
- \left( \frac{a}{2} - \frac{1}{2a} \right) i \cdot x_1 + \left( \frac{a}{2} + \frac{1}{2a} \right) i \sigma(i) \cdot x_2. \]

The mappings \( f_1 \) and \( f_2 \) are linear and orthogonal. The proof is completed. \( \square \)

References

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