COEFFICIENT PROBLEMS FOR A CLASS OF ANALYTIC FUNCTIONS INVOLVING HADAMARD PRODUCTS

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Abstract. For $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and $\beta > 0$, let $M_{S}(\Phi, \Psi; \lambda, \alpha, \beta)$ be the class of functions defined in the open unit disc $D$ by

$$\left| \frac{\lambda z^{2} f''(z) + zf'(z)}{\lambda z g'(z) + (1 - \lambda)g(z)} \right| < \frac{\pi \alpha}{2}, \quad z \in D$$

where $g(z) = z + b_{2}z^{2} + b_{3}z^{3} + \cdots$ is analytic function and satisfies

$$\left| \frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} \right| < \frac{\pi \beta}{2}, \quad z \in D$$

for some $\Phi(z) = z + \sum_{n=2}^{\infty} \gamma_{n}z^{n}$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_{n}z^{n}$ analytic in $D$ such that $g(z) * \Phi(z) \neq 0$, $\gamma_{n} \geq 0$, $\gamma_{n} \geq 0$, and $\gamma_{n} > \gamma_{n}$ ($n \geq 2$). For $f \in M_{S}(\Phi, \Psi; \lambda, \alpha, \beta)$ and given by $f(z) = z + a_{2}z^{2} + a_{3}z^{3} + \cdots$, a sharp upper bound is obtained for $|a_{3} - \mu a_{2}^{2}|$ when $\mu \geq 1$.

1. Introduction

Let $A$ denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n}$$

which are in the open unit disk $D = \{z : |z| < 1\}$. Further, let $S$ denote the class of functions which are univalent in $D$.

Let the function $f(z)$ be defined by (1.1). Also let the function $g(z)$ be defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_{n}z^{n}.$$

Then the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ is defined by

$$(1.3) \quad f(z) * g(z) = z + \sum_{n=2}^{\infty} a_{n}b_{n}z^{n}.$$
Fekete and Szegő [8] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ when $\mu$ is real. For various subclasses of $S$, sharp upper bound for functional $|a_3 - \mu a_2^2|$ has been studied by many different authors including [1–7, 9–11, 13–17, 19–20].

In this paper we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when $f$ belonging to the class of functions defined as follows:

**Definition 1.1.** Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and $\beta > 0$, and let $f \in A$. Then $f \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$ if and only if

$$\left| \arg \left( \frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z g'(z) + (1-\lambda)g(z)} \right) \right| < \frac{\pi \alpha}{2},$$

with $g \in A$ and satisfies

$$\left| \arg \left( \frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} \right) \right| < \frac{\pi \beta}{2}, \quad z \in D$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} y_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in $D$ such that $g(z) * \Phi(z) \neq 0$, $y_n \geq 0$, $\gamma_n \geq 0$ and $y_n > \gamma_n$ ($n \geq 2$).

Note that $MS\left( \frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, \beta \right) = K(\beta)$ the class of close-to-convex functions defined in [3] and $MS\left( \frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, 1 \right) = K(1)$ is the class of normalised close-to-convex functions defined by Kaplan [12]. Whereas,

$$MS\left( \frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, \alpha, \beta \right) = K(\alpha, \beta)$$

is the class of normalised close-to-convex functions defined in [7].

2. **Main Result**

In order to derive our main results, we have to recall here the following lemma.

**Lemma 2.1** ([18]). Let $h \in P$ that is, $h$ be analytic in $D$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$, and $\text{Re } h(z) > 0$ for $z \in D$, then

$$|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}$$

**Theorem 2.2.** Let $f(z)$ be given by (1.1). If $f(z) \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$; $0 < \alpha \leq 1$, $\beta \geq 1$, $0 \leq \lambda \leq 1$ and $3\mu \lambda \geq 2\delta^2 + 4\delta \gamma_2$ where $\delta = \lambda_2 - \gamma_2$, $\eta = \lambda_3 - \gamma_3$ and $\mu \geq 1$, then we have the sharp inequality

$$|a_3 - \mu a_2^2| \leq \frac{\beta^2}{3 \eta \delta^2} [3\mu \eta - 2\delta (\delta + 2 \gamma_2)]$$

$$+ \frac{\alpha (\alpha \delta + 2\delta (1 + \lambda))(3\mu(1 + 2\lambda) - 2(1 + \lambda)^2)}{3\delta (1 + \lambda)^2(1 + 2\lambda)}.$$

**Proof.** Let $f(z) \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$. It follows from (1.4) that

$$\lambda z^2 f''(z) + zf'(z) = q(z) \left[ \lambda z g'(z) + (1-\lambda)g(z) \right],$$

for $z \in D$, with $q \in P$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$. Equating coefficients, we obtain

$$2a_2 (1 + \lambda) = b_2 (1 + \lambda) + a q_1$$
and

\[(2.5) \quad 3(1 + 2\lambda) a_3 = \alpha q_2 + \alpha q_1 b_2 (1 + \lambda) + \frac{\alpha (\alpha - 1)}{2} q_1^2 + b_3 (1 + 2\lambda).\]

Also, it follows from (1.5) that

\[(2.6) \quad g(z) * \Phi(z) = (g(z) * \Psi(z)) p^3(z),\]

where \( p \in P \) with \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) for \( z \in D \). Thus equating coefficients, we obtain

\[(2.7) \quad \delta b_2 = \beta p_1\]

and

\[(2.8) \quad \eta b_3 = \beta \left( p_2 + \frac{\beta (\delta + 2\gamma_2) - \delta}{2\eta} p_1^2 \right).\]

From (2.4), (2.5), (2.7) and (2.8) we have

\[(2.9) \quad a_3 - \mu a_2^2 = \frac{\alpha}{3(1 + 2\lambda)} \left( q_2 - \frac{q_1^2}{2} \right) + \frac{\alpha^2 q_1^2 [2(1 + \lambda)^2 - 3\mu(1 + 2\lambda)]}{12(1 + \lambda)^2(1 + 2\lambda)} \]

\[+ \frac{\alpha \beta p_1 q_1 [2(1 + \lambda)^2 - 3\mu(1 + 2\lambda)]}{6\delta (1 + \lambda)(1 + 2\lambda)} \]

\[+ \frac{\beta^2 p_1^2 [2\delta^2 + 4\gamma_2 \delta - 3\mu \eta]}{12\eta \delta^2}.\]

Assume that \( a_3 - \mu a_2^2 \) positive. Thus we now estimate \( \text{Re}(a_3 - \mu a_2^2) \) by applying the same technique done by London [16]. And so from (2.9) and by using lemma 2.1 and letting \( p_1 = 2re^{i\theta}, q_1 = 2Re^{i\theta}, 0 \leq r \leq 1, 0 \leq R \leq 1, 0 \leq \theta \leq 2\pi \), and \( 0 \leq \phi \leq 2\pi \), we obtain

\[
\text{Re}(a_3 - \mu a_2^2) = \frac{\alpha}{3(1 + 2\lambda)} \text{Re} \left( q_2 - \frac{q_1^2}{2} \right) + \frac{\alpha^2 [2(1 + \lambda)^2 - 3\mu(1 + 2\lambda)]}{12(1 + \lambda)^2(1 + 2\lambda)} \text{Re} q_1^2 \]

\[+ \frac{\beta \text{Re} \left( p_2 - \frac{p_1^2}{2} \right)}{3\eta} \]

\[+ \frac{\alpha \beta (2 + \lambda)^2 - 3\mu(1 + 2\lambda))}{6\delta (1 + \lambda)(1 + 2\lambda)} \text{Re} p_1 q_1 \]

\[+ \frac{\beta^2 (2\delta^2 + 4\gamma_2 \delta - 3\mu \eta)}{12\eta \delta^2} \text{Re} p_1^2 \]

\[\leq \frac{2\alpha}{3(1 + 2\lambda)} (1 - R^2) + \frac{\alpha^2 [2(1 + \lambda)^2 - 3\mu(1 + 2\lambda)]}{3(1 + \lambda)^2(1 + 2\lambda)} R^2 \cos 2\phi \]

\[+ \frac{2\beta}{3\eta} (1 - r^2) + \frac{2\alpha \beta (2 + \lambda)^2 - 3\mu(1 + 2\lambda)) r R \cos(\theta + \phi)}{3\delta (1 + \lambda)(1 + 2\lambda)} \]

\[+ \frac{\beta^2 (2\delta^2 + 4\gamma_2 \delta - 3\mu \eta) r^2 \cos 2\theta}{3\eta \delta^2} \]

\[\leq \frac{2\alpha}{3(1 + 2\lambda)} (1 - R^2) + \frac{\alpha^2 [3\mu(1 + 2\lambda) - 2(1 + \lambda)^2]}{3(1 + \lambda)^2(1 + 2\lambda)} R^2 \]

\[+ \frac{2\beta}{3\eta} (1 - r^2) + \frac{2\alpha \beta [3\mu(1 + 2\lambda) - 2(1 + \lambda)^2] r R}{3\delta (1 + \lambda)(1 + 2\lambda)} \]

Thus we have
\[
\frac{\beta^2(3\mu \eta - (2\delta^2 + 4\gamma_2 \delta))}{3\eta \delta^2} = G(r, R).
\]

Letting \(\alpha, \beta\) and \(\mu\) fixed and differentiating \(G(r, R)\) partially when \(0 < \alpha \leq 1, \beta \geq 1,\) and \(\mu \geq 1\) we observe that
\[
G_{rr}G_{RR} - (G_r R)^2 = \frac{16\alpha \beta}{9\eta(1 + 2\lambda)} + \frac{4\alpha^2 \beta^2(3\mu \eta - (2\delta^2 + 4\gamma_2 \delta))(3\mu(1 + 2\lambda) - 2(1 + \lambda)^2)}{9\delta^2 \eta(1 + \lambda)^2(1 + 2\lambda)} - \frac{8\alpha^2 \beta(3\mu(1 + 2\lambda) - 2(1 + \lambda)^2)}{9\eta(1 + \lambda)^2(1 + 2\lambda)} - \frac{8\alpha \beta^2(3\mu \eta - (2\delta^2 + 4\gamma_2 \delta))}{9\delta^2 \eta(1 + 2\lambda)} - \frac{4\alpha^2 \beta^2(3\mu(1 + 2\lambda) - 2(1 + \lambda)^2)^2}{9\delta^2(1 + \lambda)^2(1 + 2\lambda)^2} < 0.
\]

Therefore, the maximum of \(G(r, R)\) occurs on the boundaries. Thus the desired inequality follows by observing that \(G(r, R) \leq G(1, 1) = \frac{\beta^2}{3\eta \delta^2}[3\mu \eta - 2\delta(\delta + 2\gamma_2)]\)
\[
(2.10) + \frac{\alpha(\alpha \delta + 2\beta(1 + \lambda))(3\mu(1 + 2\lambda) - 2(1 + \lambda)^2)}{3\delta(1 + \lambda)^2(1 + 2\lambda)}.
\]

The equality is attained when choosing \(p_1 = q_1 = 2i\) and \(p_2 = q_2 = -2\) in (2.9).

Remark 1. Letting \(\lambda = 0\) in Theorem 2.2, we have the result given by Darus and Thomas [7].

Remark 2. Letting \(\Phi(z) = \frac{z}{(1 - z)^2}, \ U(z) = \frac{z}{1 - z}, \ \lambda = 0\) and \(\alpha = 1\) in Theorem 2.2, we have the result given by Jahangiri [11].

References

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