MODIFIED DYADIC DERIVATIVES AND INTEGRALS OF FRACTIONAL ORDER ON $R_+$

B.I. GOLUBOV

Dedicated to the 60th birthday of Professor W.R. Wade

Abstract. We give a brief review of the known results on pointwise and strong dyadic differentiation and integration of real functions. In section 3 some new results on modified dyadic fractional differentiation and integration are formulated.

Introduction

Following the concept of J.E. Gibbs [1] P.L. Butzer and H.J. Wagner [2] defined dyadic strong derivative $D$. After that they introduced dyadic pointwise derivative $d$ and dyadic strong integral $I$ (see [3]–[5]). Their definitions concern to functions defined on dyadic group $G$ or dyadic field $K$. Dyadic group $G$ and dyadic field $K$ are isomorphic to modified segment $[0,1]^*$ and modified positive half-line $R^*_+ = [0, +\infty)^*$ respectively. The characters of dyadic group $G$ and dyadic field $K$ are Walsh-Paley functions $w_n(\cdot)$, $n \in Z_+ = \{0, 1, 2, \ldots\}$ and generalized Walsh functions $\psi_y(\cdot)$, $y \in R_+$ respectively. P.L. Butzer and H.J. Wagner proved the equalities $D w_n = n w_n$ and $d w_n(x) = n w_n(x)$ for $n \in Z_+$, $x \in G$ and $d \psi_y(x) = |y| \psi_y(x)$ for $x, y \in K$.

C.W. Onneweer [6] introduced modified pointwise and strong dyadic derivatives for functions defined on dyadic group $G$ or dyadic field $K$. He proved that the characters of dyadic group $G$ or dyadic field $K$ are differentiable in his sense and they are eigenfunctions of modified differential operator $\delta$. For example, he proved the equalities

$$\delta(w_0)(x) \equiv 0, \quad \delta(w_n)(x) = 2^k w_n(x), \quad 2^k \leq n < 2^k+1, \quad k \in Z_+, \quad x \in D.$$  

In another article [7] C.W. Onneweer introduced modified fractional differentiation and integration on compact Vilenkin groups $G_p$ of order $p \geq 2$ and proved fundamental theorem of dyadic calculus.

In this paper we give a brief outline of known results concerning dyadic derivatives and integrals.

We also define modified dyadic strong and pointwise integrals and derivatives of fractional order on $R_+$ and formulate some results concerning their properties.

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1. Notations and definitions

For a number \( x \in R_+ \equiv [0, +\infty) \) we consider dyadic expansion

\[
x = \sum_{n=-\infty}^{+\infty} 2^{-n-1} x_n,
\]

where \( x_n \) equals to 0 or 1. Note that \( x_n = 0 \) for \( n \leq n(x) \), where \( n(x) \in Z = \{0, \pm 1, \pm 2, \ldots \} \). If \( x \) is dyadic rational, then we take its finite expansion, i.e. \( x_n = 0 \) for \( n \geq n_0(x) > -\infty \). We define dyadic sum of two numbers \( x, y \in R_+ \) by the operation \( \oplus \) as follows: \( x \oplus y = z \), where \( z_n = x_n + y_n \mod 2 \) for all \( n \in Z \).

Let us put \( \{x, y \} = \sum_{n=-\infty}^{+\infty} x_n y_{-n-1} \) and define the generalized Walsh functions

\[
\psi(x, y) \equiv \psi_{t}(x, y) = (-1)^{t(x, y)} \quad \text{for} \quad (x, y) \in R_+ \times R_+.
\]

They were introduced by N.J. Fine \[8\]. It is evident that \( \psi(x, y) = \psi(y, x) \), \( \psi(x, y) = \pm 1 \) for \( x, y \in R_+ \). The functions \( w_n(x) \equiv \psi(x, n) \), \( n \in Z_+ \), are called the Walsh-Paley functions. They are 1-periodic on \( R_+ \). It is evident that \( w_0(x) \equiv 1 \) on \( R_+ \). The system \( \{w_n(x)\}_{n=0}^{+\infty} \) is orthonormal on \([0, 1]\), i.e.

\[
\int_0^1 w_m(x) w_m(x) \, dx = \delta_{m,n},
\]

where \( \delta_{m,n} \) is Kronecker symbol, i.e. \( \delta_{m,n} = 0 \) for \( m \neq n \) and \( \delta_{n,n} = 1 \).

Let be given a function \( f \in L([0, 1]) \). We denote by \( \sum_{n=0}^{+\infty} \hat{f}(n) w_n(x) \) its Fourier series with respect to the Walsh-Paley system, where \( \hat{f}(n) = \int_0^1 f(x) w_n(x) \, dx \), \( n \in Z_+ \), are Walsh-Fourier coefficients of the function \( f \).

For the function \( f \in L(R_+) \) N.J. Fine \[8\] introduced its Walsh transform by the equality

\[
F[f] (x) \equiv \hat{f}(x) = \int_{R_+} \psi(x, y) \, f(y) \, dy.
\]

If \( f \in L^p(R_+) \), \( 1 < p \leq 2 \), then its Walsh transform is defined as the limit as \( n \to +\infty \) of the sequence \( \int_0^{2^n} f(y) \psi(x, y) \, dy \) in the norm of the space \( L^p(R_+) \), where \( 1/p + 1/q = 1 \).

For \( f \in L(R_+) \), \( g \in L^p(R_+) \), \( 1 \leq p \leq +\infty \), we set

\[
(f \ast g)(x) = \int_{R_+} f(x \oplus y) g(y) \, dy, \quad x \in R_+,
\]

i.e. \( f \ast g \) is dyadic convolution of \( f \) and \( g \). Let us note that \( f \ast g \in L^p(R_+) \), \( (f \ast g) = \hat{f}\hat{g} \).

The function \( f \in L(R_+) \) is called \( W \)-continuous at the point \( x \in R_+ \), if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x \oplus y) - f(x)| < \varepsilon \) for \( 0 < y < \delta \) (see \[9\], Chapter 1).

Let us note that the Wash-Fourier transform \( \hat{f} \) of every function \( f \in L(R_+) \) is \( W \)-continuous on \( R_+ \) (see \[9\], Theorem 6.1.5).

We call the point \( x \in R_+ \) dyadic Lebesgue point of local integrable function \( f \), if \( f \) is defined at the point \( x \) and

\[
\lim_{n \to +\infty} 2^n \int_0^{2^{-n}} |f(x \oplus t) - f(x)| \, dt = 0.
\]
Almost all points of local integrable function are its dyadic Lebesgue points. If a function \( f \) is \( W \)-continuous at the point \( x \in R_+ \), then \( x \) is its dyadic Lebesgue point. (There is also a concept of Walsh-Lebesgue point of an integrable on \([0, 1]\) function, see [26], [27]).

Let us define the generalized Walsh-Dirichlet integral of the function \( f \in L(R_+) \) by the equality

\[
S_0(f)(x) = \int_0^y \tilde{f}(t) \psi(x, t) \, dt.
\]

**Theorem.** If \( x \in R_+ \) is dyadic Lebesgue point of the function \( f \in L(R_+) \), then

\[
\lim_{n \to +\infty} S_{2^n}(f)(x) = f(x).
\]

The statement of this theorem was proved at the page 430 in [10] for the points of \( W \)-continuity of the function \( f \). But the proof is valid also for dyadic Lebesgue points.

It follows from this theorem that if \( f, \tilde{f} \in L(R_+) \), then \( f(x) = \int_{R_+} \psi(x, y) \tilde{f}(y) \, dy \)

almost everywhere (a.e.) on \( R_+ \).

Let \( \Delta = \{ \Delta_n \} \) denote the set of all dyadic intervals \( \Delta_n \equiv [k2^{-n}, (k + 1)2^{-n}) \), \( k \in Z_+, n \in Z \). Let us introduce dyadic maximal function

\[
M_d(f)(x) = \sup_{x \in I \in \Delta} \left\{ \frac{1}{|I|} \int_I f(t) \, dt \right\}, \quad x \in R_+,
\]

and dyadic Hardy space

\[
H(R_+) = \{ f \in L(R_+) : M_d(f) \in L(R_+) \}.
\]

The norm on \( H(R_+) \) is \( \| f \|_{H(R_+)} = \| M_d(f) \|_{L(R_+)} \).

By similar way dyadic Hardy space \( H([0, 1]) \) is defined.

Below \( C_W(R_+) \) is the space of uniformly \( W \)-continuous functions on \( R_+ \). The norm on the \( C_W(R_+) \) is \( \| f \|_{C_W(R_+)} = \sup_{x \in R_+} |f(x)| \). The symbol \( C_W[0, 1) \) will denote the space of uniformly \( W \)-continuous functions on \([0, 1)\) with the norm \( \| f \|_{C_W[0, 1)} = \sup_{x \in [0, 1)} |f(x)| \). For the sake of convenience we shall consider the spaces \( C_W[0, 1) \) and \( C_W(R_+) \) as the spaces \( L^p[0, 1) \) or \( L^p(R_+) \) respectively for \( p = +\infty \).

2. **The known concepts of dyadic derivatives and integrals**


   **Definition 2.1.** Let be given the function \( f \in L[0, 1) \) and a point \( x \in [0, 1) \). If there exists finite limit

   \[
   d^{(1)} f(x) = \lim_{n \to +\infty} \sum_{m=0}^n 2^{m-1} \left[ f(x) - f(x \oplus 2^{-m-1}) \right],
   \]

   then \( d^{(1)} f(x) \) is called dyadic derivative of the function \( f \) at the point \( x \). The dyadic derivatives of higher order are defined by recurrence formulae

   \[
   d^{(m)} f(x) = d^{(1)}(d^{(m-1)} f)(x), \quad m = 2, 3, \ldots
   \]

   P.L. Butzer and H.J. Wagner proved that each Walsh-Paley function has dyadic derivative at each point \( x \in [0, 1) \) and \( d^{(1)} w_n(x) = n w_n(x) \) for \( n \in Z_+ \).

   The notion of strong dyadic \( L^p \)-derivative was introduced by P.L. Butzer and H.J. Wagner [2] by the following way.
Definition 2.2. If for the function \( f \in L^p[0,1) \), \( 1 \leq p \leq +\infty \), the limit
\[
D^{(1)}(f)_{L_p} \equiv \langle L^p \rangle - \lim_{n \to +\infty} \sum_{m=0}^{n} 2^{m-1}[f(\cdot) - f(\cdot + 2^{-m-1})]
\]
eexists in the norm of the space \( L^p[0,1) \), then it is called \( L^p[0,1) \)-derivative of the function \( f \). The strong dyadic \( L^p \)-derivatives of higher order are defined by recurrence formula \( D^{(m)}(f)_{L_p} = D^{(1)}((D^{(m-1)}f)_{L_p}) \), \( m = 2, 3, \ldots \)

It is proved in [2] that every Walsh function has strong dyadic \( L^p[0,1) \)-derivative of arbitrary order \( r \in N \) for each \( 1 \leq p \leq +\infty \) and \( D^{(r)}(w_n)_{L_p} = n^r w_n \), \( n \in Z_+ \).

P.L. Butzer and H.J. Wagner [2] proved the following

Theorem 2.1. If a function \( f \in L^p[0,1) \), \( 1 \leq p \leq +\infty \), has strong dyadic \( L^p[0,1) \)-derivative \( D^{(r)}(f)_{L_p} = g \), then \( g(n) = n^r f(n) \), \( n \in Z_+ \), where \( f(n) \) are Walsh-Fourier coefficients of the function \( f \).

C.W. Onneweer [11] generalized the concepts of pointwise dyadic derivative and strong dyadic \( L^p[0,1) \)-derivative to functions defined on Vilenkin groups.

For functions \( f \) defined on \( R_+ \) the natural analogue of pointwise dyadic derivative \( d^{(1)}f(x) \) is
\[
df(x) = \lim_{n \to +\infty} \sum_{m=-n}^{n} 2^{m-1}(f(x) - f(x \oplus 2^{-m-1})).
\]
(see [3]). P.L. Butzer and H.J. Wagner [3] proved that the generalized Walsh functions have dyadic derivative at each point. More precisely \( d\psi_y(x) = y\psi_y(x) \), \( x \in R_+ \).

For the functions \( f \in L^p[0,1) \), \( 1 \leq p \leq +\infty \) the strong dyadic \( L^p(R_+) \)-derivative is defined as follows:
\[
D(f)_{L^p(R_+)} = \lim_{n \to +\infty} \sum_{m=-n}^{n} 2^{m-1}[f(\cdot) - f(\cdot \oplus 2^{-m-1})],
\]
where the limit is taken in the norm of the space \( L^p(R_+) \) (see [5]). The notion of \( L^p(R_+) \)-derivative \( D^{(r)}(f)_{L^p(R_+)} \) of higher order \( r = 2, 3, \ldots \) is defined by recurrence formula.

It is known that if \( f \in L^p(R_+) \), \( p = 1 \) or 2, and \( D(f)_{L^p(R_+)} \) exists, then
\[
D(f)_{L^p(R_+)}(x) = x f(x). \quad (\text{For } p = 1 \text{ it was proved by P.L. Butzer and H.J. Wagner [3]; for } p = 2 \text{ see J. Pál [12]).})
\]
C.W. Onneweer [6] introduced modified pointwise and strong dyadic derivatives for functions defined on dyadic group \( G \) or dyadic field \( K \). (The characters of dyadic field \( K \) are generalized Walsh functions \( \psi_y(\cdot), y \in R_+ \), and the characters of the group \( G \) are Walsh-Paley functions \( w_n, n \in Z_+ \).)

He proved that the characters of dyadic group \( G \) or dyadic field \( K \) are differentiable in his sense at each point and they are eigenfunctions of modified differential operator \( \delta \). For example, he proved the equalities
\[
\delta(w_0)(y) = 0, \quad \delta(w_n)(y) = 2^k w_n(y), \quad 2^k \leq n < 2^{k+1}, \quad k \in Z_+, \quad y \in G.
\]

In another article [7] C.W. Onneweer introduced modified fractional differentiation and integration on compact groups of order \( p \geq 2 \) and proved fundamental theorem of dyadic calculus.

Dyadic integrals. The dyadic integral for functions defined on the interval \([0,1)\) was introduced by P.L. Butzer and H.J. Wagner [2] as follows. Let us set
\[
W_r(x) = 1 + \sum_{n=1}^{+\infty} \frac{w_n(x)}{n^r}, \quad r \in N.
\]
It is evident that $W_r \in L[0, 1]$, $r \in N$. If $f \in L^p[0, 1]$, $1 \leq p \leq +\infty$, then there exists dyadic convolution

$$I_r(f) = (f \ast W_r)(x) = \int_0^1 f(y) W_r(x \oplus y) \, dy, \quad r \in N$$

and $I_r(f) \in L^p[0, 1]$. The function $I_r(f)$ is called dyadic strong integral of order $r$ of the function $f$ in the space $L^p[0, 1]$.

It follows from $(\ast)$ that for $f \in L^p[0, 1]$, $1 \leq p \leq +\infty$ its dyadic integral $I_r(f)$ has Walsh-Paley series of the form

$$\hat{f}(0) + \sum_{n=0}^{+\infty} \frac{\hat{f}(n)}{n^r} w_n.$$


**Theorem 2.2.** Let $f \in L^p[0, 1]$, $1 \leq p \leq +\infty$ and $\hat{f}(0) = 0$.

a) If there exists $L^p[0, 1]$-derivative $D^{(r)}(f)_{L^p}$ of some order $r \in N$, then $I_r(D^{(r)}(f))_{L^p} = f$.

b) One has $D^{(r)}(I_r(f))_{L^p} = f$ for all $r \in N$.

J. Pál and P. Simon [13] generalized the concept of strong dyadic $L^p[0, 1]$-integral to functions defined on Vilenkin groups. They proved a generalization of the Theorems 2.1 (for $p = 1$) and 2.2, using the concept of strong dyadic $L^p[0, 1]$-derivative for functions defined on Vilenkin groups due to C.W. Oonneweer.

F. Schipp [14] proved that dyadic strong integral has pointwise dyadic derivative a.e. More precisely the following theorem is valid.

**Theorem 2.3.** If $f \in L[0, 1]$, then $d^{(r)}(I_1(f))(x) = f(x)$ a.e. on $[0, 1]$, where $I_1(f)$ is dyadic strong integral of first order of the function $f$ in the space $L[0, 1]$.

For the functions $f \in L^p(R_+)$, $1 \leq p \leq +\infty$, the strong dyadic integral was defined by H. J. Wagner [5] as follows. For $n \in Z_+$ we set

$$W_n(x) = \lim_{k \to +\infty} \int_{2^{-k} - n}^{2^{-k}} \frac{1}{t} \psi_x(t) \, dt, \quad x \in R_+.$$

It has been proved in [5] that this limit exists a.e. on $R_+$ and also in $L(R_+)$-metric. Therefore there exists dyadic convolution

$$(f \ast W_n)(x) = \int_{R_+} f(t) W_n(x \oplus t) \, dt, \quad n \in Z_+, \quad (***)$$

and $f \ast W_n \in L^p(R_+)$, if $f \in L^p(R_+)$, $1 \leq p \leq +\infty$.

**Definition 2.3.** If for a function $f \in L^p(R_+)$, $1 \leq p \leq +\infty$, the sequence $(***)$ converges in $L^p(R_+)$-metric to a function $g \in L^p(R_+)$ as $n \to +\infty$, then $g \equiv I(f)$ is called strong dyadic integral of the function $f$ in the space $L^p(R_+)$ or shortly $L^p(R_+)$-integral of the function $f$.

The notion of $L^p(R_+)$-integral $I_r(f)$ of higher order $r = 2, 3, \ldots$ is defined by recurrence formula.

The following results were proved by H.J. Wagner [5]:

**Theorem 2.4.** For two functions $f, g \in L(R_+)$ the equality $g = I(f)$ holds if and only if $\hat{g}(0) = 0$ and $\hat{g}(x) = \hat{f}(x)/x$, $x > 0$, where $I(f)$ is $L(R_+)$-integral of the function $f$. 

Theorem 2.5. Let be given a function \( f \in L(R_+) \).

a) If \( L(R_+) \)-integral \( I(f) \) exists, then \( D(I(f))_{L(R_+)} = f \).

b) If \( L(R_+) \)-derivative \( D(f)_{L(R_+)} \) exists and \( f(0) = 0 \), then \( I(D(f))_{L(R_+)} = f \).

J. Pál and F. Schipp [15] proved the following theorem.

Theorem 2.6. If a function \( L(R_+) \) has strong dyadic integral \( g = I(f) \) in the space \( L(R_+) \), then \( I(f) \) has pointwise dyadic derivative a.e. on \( R_+ \) and \( d(I(f))(x) = f(x) \) a.e. on \( R_+ \).

3. Modified dyadic integral

and derivative of fractional order on \( R_+ \)

Strong and pointwise derivatives and integrals of fractional order on \( R_+ \). In this subsection we formulate our results most of which are analogues of the results of C.W. Onneweer [7] concerning the functions defined on compact groups \( G_p \) of order \( p = 2, 3, \ldots \)

For \( x > 0 \) we set \( h(x) = 2^{-n}, 2^n \leq x < 2^{n+1}, n \in Z \). It is evident that \( x^{-1} \leq h(x) < 2x^{-1} \).

Lemma 3.1. If \( \alpha > 0 \) and \( n \in Z \), then for each \( x > 0 \) there exists finite limit

\[
W_\alpha^n(x) = \lim_{n \to +\infty} \int_{2^{-n}}^{2^n} (h(y))^\alpha \psi_x(y) dy.
\]

More precisely, \( W_\alpha^n(x) = -2^{\alpha-1}n \) for \( 2^{n-1} \leq x < 2^n \),

\[
W_\alpha^n(x) = -2^{\alpha-1}n + 2(1-2^{-\alpha})\sum_{i=0}^{k} 2^{(n-i)(\alpha-1)}
\]

for \( 2^{n-k-1} \leq x < 2^{n-k-1}, k = 0, 1, \ldots \) and \( W_\alpha^n(x) = 0 \) for \( x \geq 2^n \).

We shall write \( f(x) \approx g(x), x \to a \), if \( f(x) = O(g(x)), x \to a \), and \( g(x) = O(f(x)) \). Then we have the following corollary from the lemma 3.1.

Corollary 3.1. 1) If \( 0 < \alpha < 1 \), \( n \in Z \), then \( W_\alpha^n(x) \approx x^{\alpha-1}, x \to +0 \);

2) \( W_\alpha^n(x) \approx \log_2 x, x \to +0 \); 3) if \( \alpha > 1 \), then \( W_\alpha^n(x) \) is bounded on \( R_+ \);

4) \( W_\alpha^n(x) \) is MSDI of order \( \alpha \) of the function \( f \) in the space \( L^p(R_+) \).

Definition 3.1. If \( \alpha > 0 \), \( f, g \in L^p(R_+) \), and \( \lim_{n \to +\infty} \| f \ast W_\alpha^n - g \|_{L^p(R_+)} = 0 \), then the function \( g = J_\alpha(f) \) is called modified strong dyadic integral (MSDI) of order \( \alpha \) of the function \( f \) in the space \( L^p(R_+) \).

Theorem 3.1. Let \( f, g \in L(R_+) \) and \( \alpha > 0 \). Then the function \( g \) is MSDI of order \( \alpha \) of the function \( f \) in the space \( L(R_+) \), if and only if \( \tilde{g}(x) = \tilde{f}(x)(h(x))^\alpha \) for \( x > 0 \).

Let us set for \( \alpha > 0 \), \( n \in Z \):

\[
\Lambda_\alpha^n(x) = \int_0^{2^n} (h(t))^{-\alpha} \psi(x, t) dt, \quad x \in R_+.
\]

Lemma 3.2. For \( \alpha > 0 \), \( n \in Z \) we have \( \Lambda_\alpha^n \in L(R_+) \cap L^\infty(R_+) \).

Definition 3.2. If \( \alpha > 0 \), \( f, \varphi \in L^p(R_+) \), \( 1 \leq p \leq +\infty \), and

\[
\lim_{n \to +\infty} \| f \ast \Lambda_\alpha^n - \varphi \|_{L^p(R_+)} = 0,
\]

then the function \( \varphi = D^\alpha(f) \) is called modified strong dyadic derivative (MSDD) of order \( \alpha \) of the function \( f \) in the space \( L^p(R_+) \).
Theorem 3.2. Let $\alpha > 0$ and $f, \varphi \in L^p(R_+)$, $1 \leq p \leq 2$. Then the function $\varphi$ is MSDD of order $\alpha$ of the function $f$ in the space $L^p(R_+)$ if and only if
\[ \varphi(x) = \tilde{f}(x) (h(x))^{-\alpha} \]
a.e. on $R_+$.

This theorem is a corollary from the $R_+$-version of a theorem of C.W. Onneweer (see [22], Theorem 3).

Theorem 3.3. Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDD $D^\alpha(f)$ of order $\alpha$ in the space $L(R_+)$. If $\tilde{f}(0) = 0$, then the equality $J_\alpha(D^\alpha(f)) = f$ holds.

Theorem 3.4. Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDI $J_\alpha(f)$ of order $\alpha$ in the space $L(R_+)$. Then the equality $D^\alpha(J_\alpha(f)) = f$ is valid.

The Theorems 3.3 and 3.4 are $R_+$-version of fundamental theorem of dyadic calculus (see Theorem 2.2 above).

Theorem 3.5. Let $\alpha > 0$, $\beta > 0$ and $f \in L(R_+)$. Then $D^\alpha(D^\beta(f)) = D^{\alpha+\beta}(f)$ (respectively $J^\alpha(J^\beta(f)) = J^{\alpha+\beta}(f)$), if the left side of this equality exists.

Theorem 3.6. The functions $a_{m,n}(x) = \psi(x, m2^{-n})X_{[0,2^n)}(x)$, $m \in N$, $n \in Z$, for each $\alpha > 0$ are eigenfunctions of the operators $J_\alpha$ and $D^\alpha$ with eigenvalues $2^{-\alpha}$ and $2^{\alpha}$ respectively. Here $X_E$ is indicator function of the set $E$ and $r = r(m, n) \in Z$ is uniquely determined by the embedding $[m2^{-n}, (m+1)2^{-n}] \subset [2^r, 2^{r+1})$.

Let us denote by $L_{J_\alpha}(R_+)$ or $L_{D^\alpha}(R_+)$ the natural domain of the operator $J_\alpha$ or $D^\alpha$ respectively, i.e. the set of all functions $f \in L(R_+)$ for which $J_\alpha(f)$ or $D^\alpha(f)$ respectively exists. It is evident that $L_{J_\alpha}(R_+)$ and $L_{D^\alpha}(R_+)$ are linear subspaces in $L(R_+)$. It follows from the Theorem 3.6 that
\[ J_\alpha(a_{1,n}) = 2^{n\alpha}a_{1,n}, \quad D^\alpha(a_{1,n}) = 2^{-n\alpha}a_{1,n}, \quad n \in Z, \quad \alpha > 0. \]
Therefore we have

Corollary 3.2. The linear operators $J_\alpha: L_{J_\alpha}(R_+) \rightarrow L(R_+)$ and $D^\alpha: L_{D^\alpha}(R_+) \rightarrow L(R_+)$ are unbounded for each $\alpha > 0$.

Let us define the pointwise dyadic derivative of fractional order. According to the lemma 3.2 we have $\Lambda_n^\alpha \in L^\infty(R_+) \cap L(R_+)$ for $\alpha > 0$, $n \in Z$. Therefore the dyadic convolution $(\Lambda_n^\alpha * f)(x)$ exists at each point $x \in R_+$ for all $\alpha > 0$, $n \in Z$, if $f \in L(R_+)$ or $f \in L^\infty(R_+)$. Taking into account this fact we may to introduce the following definition.

Definition 3.3. Let $\alpha > 0$, $x \in R_+$ and $f \in L(R_+)$ or $f \in L^\infty(R_+)$. If there exists finite limit $d^\alpha(f)(x) \equiv \lim_{n \to +\infty} (\Lambda_n^\alpha * f)(x)$, then we shall say that the function $f$ has the modified dyadic derivative (MDD) $d^\alpha(f)(x)$ of order $\alpha$ at the point $x$.

Theorem 3.7. For each $\alpha > 0$ and fixed $y \in R_+$ the Walsh generalized function $\psi_y(\cdot)$ has MDD of order $\alpha$ at each point $x \in R_+$. More precisely, $d^\alpha(\psi_y)(x) \equiv 0$ on $R_+$ and $d^\alpha(\psi_y)(x) = (h(y))^{-\alpha}\psi_y(x)$ for $x \in R_+$, $y > 0$.

For the case $\alpha = 1$ these results were published in [21].

Theorem 3.8. If $\alpha > 0$ and the function $f \in L(R_+)$ is such that $(h(x))^{-\alpha}\tilde{f}(x) \in L(R_+)$, then at each point $x \in R_+$ it has MDD of order $\alpha$ equal to $\int_0^{+\infty} (h(y))^{-\alpha}\tilde{f}(y)\psi(x, y) \, dy$.

Theorem 8 is an analogue of the following theorem of P.L. Butzer and H.J. Wagner [4].
Theorem 3.9. Under the assumption \( \sum_{n=0}^{\infty} n|\alpha_n| < +\infty \) the series \( \sum_{n=0}^{\infty} a_n w_n(x) \) is absolutely and uniformly convergent on \([0, 1]\) to a function \( f \), which has dyadic derivative \( d^{(1)} f(x) \) for all \( x \in [0, 1] \) and \( d^{(1)} f(x) = \sum_{n=0}^{\infty} n a_n w_n(x) \).

Pointwise and strong dyadic term by term differentiation of Walsh series was investigated by W.R. Wade [23], V.A. Skvorčov and W.R. Wade [24], C.H. Powell and W.R. Wade [25].

Let us define the pointwise dyadic integral of fractional order. According to the corollary 3.1 we have \( W_n^\alpha \in L(R_+) \). Therefore the dyadic convolution \( (W_n^\alpha * f)(x) \) exists at each point \( x \in R_+ \) for all \( \alpha > 0 \), \( n \in \mathbb{Z} \), if \( f \in L^\infty(R_+) \). Taking into account we may to introduce the following definition.

Definition 3.4. If \( \alpha > 0 \), \( x \in R_+ \) and for a function \( f \in L^\infty(R_+) \) there exists finite limit \( j_\alpha(f)(x) \equiv \lim_{n \to +\infty} (f * W_n^\alpha)(x) \), then we say that the function \( f \) has modified dyadic integral (MDI) of order \( \alpha \) at the point \( x \) equal to \( j_\alpha(f)(x) \).

Theorem 3.10. For each \( \alpha > 0 \) and fixed \( y \in R_+ \) the generalized Walsh function \( \psi_y(\cdot) \) has MDI of order \( \alpha \) at each point \( x \in R_+ \). More precisely, \( j_\alpha(\psi_y)(x) \equiv 0 \) on \( R_+ \) and \( j_\alpha(\psi_y)(x) = (h(y))^{\alpha} \psi_y(x) \) for \( x \in R_+, y > 0 \).

Definition 3.5. Let \( \alpha > 0 \) and \( x \in R_+ \). If for the function \( f \in L(R_+) \) there exists finite limit
\[
d^{(\alpha)}(f)(x) \equiv \lim_{n \to +\infty} \int_0^{2^n} (h(y))^{-\alpha} \tilde{f}(y)\psi(x, y) \, dy,
\]
then we shall call it dyadic \( \alpha \)-derivative of the function \( f \) at the point \( x \).

Theorem 3.11. Let \( \alpha > 0 \) and the function \( f \in L(R_+) \) has MSDI \( J_\alpha(f) \) of order \( \alpha \) in the space \( L(R_+) \). Then at each Lebesgue point \( x \in R_+ \) of the function \( f \), hence a.e. on \( R_+ \), the equality \( d^{(\alpha)}(J_\alpha(f))(x) = f(x) \) holds.

Theorem 3.12. Let \( \alpha > 0 \) and the function \( f \in L(R_+) \) has MSDD \( D^{(\alpha)}(f) \) of order \( \alpha \) in the space \( L(R_+) \). Then at each Lebesgue point \( x \in R_+ \) of the function \( f \), hence a.e. on \( R_+ \), the equality \( j_\alpha(D^{(\alpha)}(f))(x) \equiv f(x) \) holds.

Dyadic integral in dyadic Hardy space. The following theorem of Hardy [16] is well known.

Theorem 3.13. If the function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to the Hardy space \( H(|z| < 1) \) on the unit disc \( |z| < 1 \) of the complex plane \( C \) and \( f(e^{it}) \) is its
boundary function on the unit circle $|z| = 1$, then
\[
\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \frac{1}{2} \int_{0}^{2\pi} |f(e^{it})| \, dt.
\]

An analogue of this theorem has been proved by E. Hille and J.D. Tamarkin [17].

**Theorem 3.14.** If the function $f(z)$ belongs to the Hardy space $H(R^2_+)$ on the upper half-plane $R^2_+ = \{ z \in C : \text{Im} \, z > 0 \}$ and $\hat{f}(x)$ is Fourier transform of its boundary function $f(x)$ on real axis, then the following inequality holds
\[
\int_{R^2_+} \left| \frac{\hat{f}(x)}{x} \right| \, dx \leq \frac{1}{2} \int_{-\infty}^{+\infty} |f(x)| \, dx.
\]

An extension of the Theorem 3.14 on Hardy space $H^p(R)$, $0 < p \leq 1$, is also known.

**Theorem 3.15.** If $f \in H^p(R)$, then
\[
\int_{R^2_+} |\hat{f}(x)|^p x^{p-2} \, dx \leq C_p \| f \|^p_{H^p(R)}.
\]
(See [18], p.342).

**Problem.** What are the least constants in right-hand sides of the inequalities of the Theorems 3.13–3.15?

N.R. Ladhawala [19] proved a dyadic analogue of the Theorem 3.13 in the following form.

**Theorem 3.16.** If the function $f$ belongs to dyadic Hardy space $H([0,1))$, then
\[
\sum_{n=1}^{+\infty} \frac{|\hat{f}(n)|}{n} \leq 12\sqrt{2} \| f \|_{H^p(R)}.
\]
where $\hat{f}(n)$ are Walsh–Fourier coefficients of the function $f$.

A dyadic analogue of the Theorem 3.14 was proved in [20]:

**Theorem 3.17.** If $f \in H(R_+)$, then the following inequality holds
\[
\int_{R^2_+} \left| \frac{\hat{f}(x)}{x} \right| \, dx \leq 50\sqrt{2} \| f \|_{H(R_+)}.
\]

**Problem.** 1) What is the least constant in right-hand side of the former inequality?
2) To extend this inequality on dyadic Hardy space $H^p(R_+)$, $0 < p < 1$, i.e. to prove dyadic analogue of the Theorem 3.15.

**Definition 3.7.** Let us define the functions $(f \ast W^\alpha_n)^*(x)$, $n \in Z_+$, by the equality
\[
(f \ast W^\alpha_n)^*(x) \equiv \int_{2^{-n} \leq t \leq 2^n} (f \ast W^\alpha_n)(t) \psi_x(t) \, dt
\]
\[
= \int_{2^{-n} \leq t \leq 2^n} \hat{f}(t)(h(t))^\alpha \psi_x(t) \, dt, \quad x \in R_+,
\]
where $f \in L(R_+)$. If there exists the limit
\[
J^*_n(f)(x) \equiv \lim_{n \to +\infty} (f \ast W^\alpha_n)^*(x),
\]
which is uniform on $R_+$, then we say that the function $f$ has uniform modified dyadic integral (UMDI) of order $\alpha$ on $R_+$. 

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As a corollary from the Theorem 3.17 we obtain:

**Theorem 3.18.** Each function $f \in H(R_+)$ has UMDI of first order on $R_+$. More precisely, the operator $J_1^*: H(R_+) \to C_w(R_+)$ is bounded and

$$\|J_1^*(f)\|_{C_w(R_+)} \leq 100\sqrt{2}\|f\|_{H(R_+)}.$$ 

In [21] the following theorem is proved.

**Theorem 3.19.** The functions $\psi(x, m2^{-n})X_{[0, 2^n)}(x)$, $m \in N$, $n \in Z$, belong to the space $H(R_+)$ and their linear hull $L$ is dense in this space.

If a function $f \in H(R_+)$ has modified dyadic strong integral $J_1(f)$ in the space $L(R_+)$, then $J_1(f)(x) = J_1^*(f)(x)$ a.e. on $R_+$. But the functions $\psi(x, m2^{-n})X_{[0, 2^n)}(x)$, $m \in N$, $n \in Z$, have modified strong dyadic integral of first order in the space $L(R_+)$ (see Theorem 3.6 above). Therefore by setting $\tilde{J}_1(f) \equiv J_1(f)$ we can deduce from the Theorems 3.1, 3.17 and 3.19 the following result.

**Theorem 3.20.** The operator $\tilde{J}_1 : L \to L(R_+)$ is bounded and its operator norm does not exceed $100\sqrt{2}$. Therefore it can be extended continuously on the space $H(R_+)$ without changing its operator norm.

This theorem may be considered as a dyadic analogue of the Theorem 3.14.

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**References**


Department of Higher Mathematics, Moscow Engineering Physics Institute, 115409, Moscow, Kashirskoe shosse, 31, Russia

E-mail address: golubov@mail.mipt.ru