TRANSLATION INVARIANT OPERATORS ON HARDY SPACES
OVER VILENKIN GROUPS

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Dedicated to Professor William R. Wade on the occasion of his 60th birthday

Abstract. We show that a number of well known multiplier theorems for Hardy spaces over Vilenkin groups follow immediately from a general condition on the kernel of the multiplier operator. In the compact case, this result shows that the multiplier theorems of Kitada [6], Tateoka [13], Daly-Phillips [2], and Simon [11] are best viewed as providing conditions on the partial sums of the Fourier-Vilenkin series of the kernel rather than explicit conditions on the Fourier-Vilenkin coefficients themselves. The theorem is used to prove an extension of the Marcinkiewicz multiplier theorem for Hardy spaces.

1. Introduction

In this paper the setting will be a locally compact Vilenkin group $G$ of bounded order. Thus $G$ contains a decreasing sequence of compact open subgroups $(G_n)_{n=-\infty}^\infty$ such that

1. $\bigcup_{n=-\infty}^\infty G_n = G$ and $\bigcap_{n=-\infty}^\infty G_n = \{0\}$,
2. $\sup_n \{\text{order}(G_n/G_{n+1})\} < \infty$.

In the case that $G$ is compact, we use the convention that $G_n = G$ if $n \leq 0$. The additive group of a local field is Vilenkin group, as is its ring of integers. In particular, the $p$-adic numbers are a Vilenkin group. In the case that $p = 2$, the ring of integers is also called the dyadic group and the characters the Walsh functions.

Let $\Gamma$ denote the dual group of $G$ and $\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}$. The Haar measures $\mu$ on $G$ and $\lambda$ on $\Gamma$ are chosen so that $\mu(G_n) = \lambda(\Gamma_n) = 1$ and consequently, $\mu(G_n) = (\lambda(\Gamma_n))^{-1} := (M_n)^{-1}$ for each $n \in \mathbb{Z}$. There is a norm on $G$ defined by $|x| = (M_n)^{-1}$ if $x \in G_n \setminus G_{n+1}$. The Fourier transform and inverse Fourier transform respectively are denoted by $\wedge$ and $\vee$, and satisfy

$$\langle \xi_{G_n} \rangle^\wedge = (\lambda(\Gamma_n))^{-1} \xi_{\Gamma_n}$$

where $\xi_A$ denotes the characteristic function of a set $A$. Consequently,

$$\langle \xi_{\Gamma_n} \rangle^\vee = (\lambda(G_n))^{-1} \xi_{G_n}.$$ 

We define distributions according to the theory developed by Taibleson [12] for local fields. Let $S(G)$ be defined as the collection of functions that have compact support and that are constants on the cosets of a $G_n$ ($n \in \mathbb{Z}$). A sequence $(\psi_k)$ in $S(G)$ is said to converge to $\psi \in S(G)$ if there are $n, m \in \mathbb{Z}$ such that every $\psi_k$ is

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constant on the cosets of $G_n$, supp $\psi_k \subset G_n \ (k \in \mathbb{N})$, and $(\psi_k)$ converges uniformly to $\psi$. Continuous linear functionals on $S(G)$ are called distributions. The set of distributions will be denoted by $S^\prime(G)$.

The (atomic) Hardy spaces on $G$ are given as follows. A function $a : G \to C$ is a $p$-atom, $0 < p \leq 1$, if

i) supp $a \subset I_n := x + G_n$ for some $x \in G$, and $n \in \mathbb{N}$,

ii) $||a||_\infty \leq (\mu(I_n))^{-1/p}$,

iii) $\int_G a(x)dx = 0$.

A distribution $f \in S^\prime(G)$ belongs to $H^p(G)$ if $f$ is given by $f = \sum_{i=1}^{\infty} \lambda_i a_i$, where each $a_i$ is a $p$-atom, $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$, and convergence is in $S^\prime(G)$. We set

$$||f||_{H^p} = \inf \left( \sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p}$$

with the infimum taken over all such atomic decompositions of $f$. A function $\varphi \in L^\infty(\Gamma)$ is a (Fourier) multiplier on $H^p(G)$ if there exists a constant $C > 0$ so that for all $f \in H^p(G) \cap L^2(G)$,

$$\left\| (\varphi f^\wedge)^\vee \right\|_{H^p} \leq C \|f\|_{H^p}.$$ 

A multiplier operator $T_\varphi$ is defined for a function $\varphi$ on $\Gamma$ by

$$(T_\varphi f)^\wedge = \varphi \cdot f^\wedge.$$ 

The operator $T_\varphi$ is a convolution operator determined by the distribution $\Phi$ which has kernel $k$ defined by

$$k^\wedge = \varphi.$$ 

The blocks $\Delta_n k$ of the kernel $k$ are defined by $\Delta_n k = (k^\wedge \xi_{1_{n+1}\Gamma_n})^\vee (n \in \mathbb{Z})$. For a multiplier $\varphi$, the blocks are $\Delta_n \varphi = \varphi \xi_{1_{n+1}\Gamma_n}$.

2. RESULTS AND PROOFS

A number of authors have proved multiplier theorems for $H^p(G)$. Among them are Daly, Fridli, Kitada, Oonneweer, Phillips, Quek, Simon, and Tateoka. The results of Kitada [6], Oonneweer-Quek [8], and Tateoka [13] often were phrased in terms of blocks of the kernel belonging to certain Herz spaces along with growth bounds. These were called multiplier theorems; even though, the theorems are most naturally phrased in terms of the corresponding kernel.

First we formulate Theorem 1 which is a general result for a convolution operator with kernel $k$ to be a bounded operator on $H^p(G)$. Then we formulate Theorem 2. From this theorem we will show that all of the previous multiplier results follow in a straightforward manner. Finally, we will use it to prove an $H^p(G)$ version of the classical Marcinkiewicz multiplier theorem.

**Theorem 1.** Let $k$ be locally integrable on $G \setminus \{0\}$ and $0 < p \leq 1$. If either

i) $\sup_N \int_{(G_N)^r} |G_N|^{-1} \left( \int_{G_N} |k(x-y)| \ dy \right)^p dx < \infty$

or

ii) $\sup_N \int_{(G_N)^r} |G_N|^{-1} \left( \int_{G_N} |k(x) - \mathbb{1}_y| \ dy \right)^p dx < \infty$,

then $T_k$ is bounded on $H^p(G)$.

Theorem 1 in the case of $p = 1$ has appeared many places in the literature. For example, Inglis [4] proves a version for totally disconnected groups and a version for local fields appears in the paper of Phillips and Taibleson [9]. In both examples, they were concerned with boundedness questions of operators on $L^r$, $1 < r < \infty$, 


and weak($L^1$) results. As the atomic theory of Hardy spaces was developed, these results were extended to $H^1$. See [5] for an example.

If the kernel $k$ is decomposed into blocks then one can get the following sufficient conditions that turned to be useful in applications.

**Theorem 2.** Let $k$ be locally integrable on $G \setminus \{0\}$ and $0 < p \leq 1$. If either

i) $\sup_N \int_{(G_N)^r} |G_N|^{-1} \left( \int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j k(x-y)| \, dy \right)^p \, dx < \infty$

or

ii) $\sup_N \int_{(G_N)^r} |G_N|^{-1} \left( \int_{G_N} |\sum_{j=N+1}^{\infty} (\Delta_j k(x-y) - \Delta_j k(x))| \, dy \right)^p \, dx < \infty$,

then $T_k$ is bounded on $H^p(G)$.

Condition ii) of Theorem 1 and Theorem 2 is useful in analyzing the boundedness properties of singular integral type operators. For example, in the case of $q$-adic fields $K_q$, Calderon-Zygmund singular integral operators have been studied extensively. See Phillips-Tablelesos [9] for the $L^p(K_q)$, $1 < p < \infty$, case and Daly-Phillips [3] for the $H^p(K_q)$, $0 < p \leq 1$, case. These operators have homogeneity in the kernels $k$: $k(qx) = q^{-j}k(x)$. Thus the kernel can be written as $k = \omega \cdot |\cdot|^{-1}$ with $\omega(qx) = \omega(x)$ for $x \neq 0$. The kernel $k$ is said to be homogeneous of degree $-1$. If the kernel satisfies

$$\int_{|y| \leq 1} \int_{|x| > 1} |k(x-y) - k(x)| \, dx \, dy < \infty$$

then $T_k$ is bounded on $L^p(K_q)$ for $1 < p < \infty$ and $H^1(K_q)$ (see [3]). Using the homogeneity of the kernel, this condition is easily seen to be equivalent to our condition ii) of Theorem 1 for $p = 1$. Also, if one chooses to decompose the kernel into blocks in a manner inconsistent with the subgroup decompositions of $\Gamma$, then one would begin the proof of boundedness using Theorem 1 directly and not use Theorem 2. For example, Wo-Sung Young does so in [15] where she proves a Marcinkiewicz multiplier theorem using dyadic blocks for an arbitrary compact Vilenkin group.

We proceed with listing conditions that are sufficient for the multiplier operator to be bounded on $H^p(G)$, and that have been used by several authors. They all can be considered as consequences of Theorem 1.

**Corollary 3.** If $k$ is locally integrable on $G \setminus \{0\}$ and $0 < p \leq 1$, and

$$\sup_N \sum_{j=N+1}^{\infty} \int_{(G_N)^r} |G_N|^{-1} \left( \int_{G_N} |\Delta_j k(x-y)| \, dy \right)^p \, dx < \infty,$$

then $T_k$ is bounded on $H^p(G)$.

We note that this condition was used by Simon [11] in the special case when $G$ is a compact bounded multiplicative Vilenkin group. He sated the result in terms of $(\Delta_j e)^\omega$ rather than $\Delta_j k$.

In the following corollary we assume that $p = 1$. It was first formalized and used by Kitada [5] and Tateoka [13].

**Corollary 4.** Let $k$ be locally integrable on $G \setminus \{0\}$ and $0 < p \leq 1$. If

$$\sup_N \int_{(G_N)^r} \sum_{j=-\infty}^{\infty} |(\Delta_j k)(x)| \, dx < \infty,$$

then $T_k$ is bounded on $H^1(G)$. 
Namely, they proved that it is enough to start the summation from $N + 1$ instead of $-\infty$.

**Corollary 5.** Let $k$ be locally integrable on $G \setminus \{0\}$ and $0 < p \leq 1$. If

$$\sup_N \int_{(G_N)^c} \sum_{j=N+1}^{\infty} |(\Delta_j k)(x)| \, dx < \infty,$$

then $T_k$ is bounded on $H^1(G)$.

The condition in the following corollary is due to Kitada [5] and Tateoka [13]. We note that it was used for example by Daly and Fridli in [1] for Walsh multipliers.

**Corollary 6.** Let $k$ be locally integrable on $G \setminus \{0\}$ and $0 < p \leq 1$. If

$$\sum_{N=-\infty}^{j} |G_N|^{1-p} \left( \int_{G_N \setminus G_{N+1}} |\Delta_j k(y)| \, dy \right)^p \leq C|G_j|^{1-p},$$

then $T_k$ is bounded on $H^p(G)$.

We will first provide the proofs of the corollaries, assuming Theorem 2, and then provide the proof of Theorem 1 and Theorem 2. For Corollary 3, we use i) from Theorem 2 and the fact $p \leq 1$:

$$\int_{(G_N)^c} |G_N|^{-1} \left( \int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j k(x - y)| \, dy \right)^p \, dx \leq \int_{(G_N)^c} |G_N|^{-1} \left( \int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j k(x - y)| \, dy \right)^p \, dx \leq \int_{N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1} \left( \int_{G_N} |\Delta_j k(x - y)| \, dy \right)^p \, dx.$$

Taking the supremum over $N$, we obtain Corollary 3.

To prove Corollary 5 with the condition of Daly and Phillips [2] for $H^1(G)$, we proceed from Corollary 3 with $p = 1$:

$$\sum_{j=N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1} \int_{G_N} |\Delta_j k(x - y)| \, dy \, dx.$$

As $x \in (G_N)^c$, $y \in G_N$ we have that the value inner integral does not actually depend on $y$. Indeed, $\int_{G_N} |\Delta_j k(x - y)| \, dy \, dx = \int_{x + G_N} |\Delta_j k(t)| \, dt$. The function $|G_N|^{-1} \int_{x + G_N} |\Delta_j k(t)| \, dt$ is nothing but the integral average function of $|\Delta_j k|$ over the cosets of $G_n$. Consequently it is constant on these cosets and its integral over $(G_N)^c$ is equal to the integral of the function, i.e.

$$\sum_{j=N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1} \int_{G_N} |\Delta_j k(x - y)| \, dy \, dx = \sum_{j=N+1}^{\infty} \int_{G_n} |\Delta_j k(t)| \, dt.$$

Thus the condition in Corollary 3 and the Daly-Phillips conditions coincide when $p = 1$. Allowing the above sum to run from $-\infty$ to $\infty$, one obtains the Kitada-Tateoka ([6], [13]) condition, i.e. Corollary 4 for $H^1(G)$.

Applying the same argument to condition from Corollary 3 for $0 < p < 1$, and a Hölder inequality with exponent $1/p$ we obtain the following condition.
Corollary 7. Let \( k \) be locally integrable on \( G \setminus \{0\} \) and \( 0 < p \leq 1 \). Then
\[
\sup_N \sum_{j=N+1}^{\infty} |G_N|^{p-1} \left( \int_{(G_N)^c} |(\Delta_j k(t)| dt \right)^p < \infty
\]
implies that the operator \( T_k \) is bounded on \( H^p(G) \).

The proof of Corollary 6 for \( H^p(G) \) is more involved than the previous. Beginning again with i) from Theorem 2:
\[
U_N := \int_{(G_N)^c} |G_N|^{-1} \left( \int_{G_N} \left| \sum_{j=N+1}^{\infty} \Delta_j k(x-y) \right| dy \right)^p dx
\]
\[
= \sum_{n=-\infty}^{N-1} \int_{G_n \setminus G_{n+1}} |G_N|^{-1} \left( \int_{G_N} \left| \sum_{j=N+1}^{\infty} \Delta_j k(x-y) \right| dy \right)^p dx
\]
\[
\leq |G_N|^{-1} \sum_{n=-\infty}^{N-1} \int_{G_n \setminus G_{n+1}} \left( \int_{G_N} \sum_{j=N+1}^{\infty} |\Delta_j k(x-y)| dy \right)^p dx.
\]
Using the Hölder inequality on the outer integral with \( r = 1/p \) and \( r' = 1/(1 - p) \), we continue with
\[
U_N \leq |G_N|^{-1} \sum_{n=-\infty}^{N-1} \left( \int_{G_n \setminus G_{n+1}} \int_{G_N} \sum_{j=N+1}^{\infty} |\Delta_j k(x-y)| dy \right)^p dx
\]
\[
\times \left( \int_{G} \left| \sum_{n=-\infty}^{N-1} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} \sum_{j=N+1}^{\infty} |\Delta_j k(x-y)| dy \right)^p dx \right)^p.
\]
Making use of the fact that \( x-y \in G_n \setminus G_{n+1} \) when \( N > n, y \in G_N, \) and \( x \in G_n \), we have \( \int_{G_n} |\Delta_j k(x-y)| dy = \int_{x+G_N} |\Delta_j k(t)| dt \). Therefore the inequality becomes
\[
U_N \leq |G_N|^{-p-1} \sum_{n=-\infty}^{N-1} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} \sum_{j=N+1}^{\infty} |\Delta_j k(x)| dx \right)^p
\]
\[
\leq |G_N|^{-p-1} \sum_{j=N+1}^{\infty} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} |\Delta_j k(x)| dx \right)^p.
\]
Since \( j \geq N + 1 \) we have that the inner sum can be estimated above the left side of condition from Corollary 6. It is bounded by \( C|G_j|^{1-p} \). Thus
\[
U_N \leq C|G_N|^{-p-1} \sum_{j=N+1}^{\infty} |G_j|^{1-p} \leq C|G_N|^{-p-1}|G_N|^{1-p} = C.
\]

We now proceed with the proof of Theorem 1 and Theorem 2.

Proofs of Theorems 1 and 2. We note that it is sufficient to show \( T_k(a) \in L^p(G) \). Without the loss of generality we may suppose that \( \text{supp} a \subset G_N, \|a\|_{L^\infty(G)} \leq |G_N|^{-1/p}, \) and \( \int_{G_N} a = 0 \). Set
\[
(1) \quad \|T_k(a)\|_{L^p(G)} = \int_{G_N} |T_k(a)(x)|^p dx + \int_{(G_N)^c} |T_k(a)(x)|^p dx = T_1 + T_2.
\]
For $T_1$ we use the usual $L^2$ argument that exploits the facts that $T_k$ is bounded on $L^2$ and $a \in L^2$:

$$T_1 = \int_G |T_k(a)(x)|^p \xi_G(x) dx$$

$$\leq \left( \int_G |T_k(a)(x)|^2 dx \right)^{\frac{p}{2}} \left( \int_G \xi_G(x) dx \right)^{1-\frac{p}{2}}$$

$$\leq C \|a\|_2^p \|G\|_{1-\frac{p}{2}}$$

$$\leq C \|G\|_{\frac{1}{2},0}^p \|G\|_{1-\frac{p}{2}} = C.$$  

For $T_2$ we will use the boundedness and cancellation properties of the atom $a$. One direction is

$$T_2 = \int_{(G_N)^c} \left| \int_{G_N} k(x-y)a(y) dy \right|^p dx \leq \int_{(G_N)^c} \|G_N\|^{-1} \left( \int_{G_N} \|k(x-y)\| dy \right)^p dx$$

and the other is

$$T_2 = \int_{(G_N)^c} \left| \int_{G_N} (k(x-y) - k(x))a(y) dy \right|^p dx$$

$$\leq \int_{(G_N)^c} \|G_N\|^{-1} \left( \int_{G_N} \|k(x-y) - k(x)\| dy \right)^p dx.$$  

This proves Theorem 1.

Let us take (1) again. To prove Theorem 2 we decompose the kernel $k$ in terms of the blocks of its Fourier-Vilenkin transform $k = \sum_{j=-\infty}^{\infty} \Delta_j k$. Using this decomposition, $T_2$ becomes in the first case

$$\int_{(G_N)^c} \left| \int_{G_N} k(x-y)a(y) dy \right|^p dx \leq \int_{(G_N)^c} \left( \int_{G_N} \sum_{j=-\infty}^{N-1} \Delta_j k(x-y)a(y) dy \right)^p dx$$

$$+ \left| \int_{G_N} \sum_{j=N}^{\infty} \Delta_j k(x-y)a(y) dy \right|^p dx.$$  

Since $\Delta_j k(x-y) = \Delta_j k(x)$ as $j < N$ and $y \in G_N$, and using the fact $\int_{G_N} a = 0$, we have that the first integrand is identically zero. Combining this with our estimates for $T_1$

$$\|T_k(a)\|_{L^p(G)}^p \leq C + \int_{(G_N)^c} \left( \int_{G_N} \sum_{j=-\infty}^{N-1} \Delta_j k(x-y)a(y) dy \right)^p dx = C + U_1.$$  

Using again the fact $\int_{G_N} a = 0$, $U_1$ can be rewritten as

$$U_2 = \int_{(G_N)^c} \left( \int_{G_N} \sum_{j=N}^{\infty} (\Delta_j k(x-y) - \Delta_j k(x))a(y) dy \right)^p dx.$$  

The final estimates for both $U_1$ and $U_2$ follow from $\|a\|_{L^\infty(G)} \leq \|G_N\|^{-1/p}$. Indeed, for $U_1$ we have

$$U_1 \leq \int_{(G_N)^c} \left( \int_{G_N} \sum_{j=N}^{\infty} \Delta_j k(x-y) |a(y)| dy \right)^p dx$$

$$\leq \int_{(G_N)^c} \|G_N\|^{-1} \left( \int_{G_N} \sum_{j=N}^{\infty} \Delta_j k(x-y) |dy| \right)^p dx.$$
This is the required estimate for (i) of Theorem 2. As stated above, the estimate for (ii) of Theorem 2 is obtained in an identical manner from $U_2$.

We will use Theorem 2 in the form of Corollary 5 (Kitada, Tateoka) to prove a version of the Marcinkiewicz multiplier theorem for $H^p(G)$. This will be for the compact multiplicative $G$. Then the dual group $\Gamma = \{\chi_n\}$ can be enumerated in the way that corresponds to the Paley enumeration in the Walsh case. The Dirichlet kernels are defined as $D_n = \sum_{k=0}^{n-1} \chi_k$ ($n \in \mathbb{N}$). For details we refer the reader to [10].

First we will need a lemma that is a type of Sidon inequality. The authors [1] earlier proved a version for the dyadic group and Walsh series.

**Lemma 8.** Let $G$ be compact multiplicative Vilenkin group. If $n, N \in \mathbb{N}$ and $1 < q \leq 2$ then for any numbers $c_k$ ($1 \leq k \leq |\Gamma_n|$), we have

$$\int_{(G_N)^c} \left| \sum_{k=1}^{n} c_k D_k(x) \right| dx \leq C|G_N|^{\frac{1}{2} - \frac{1}{q}} \left( \sum_{k=1}^{n} |c_k|^q \right)^{1/q}. $$

**Proof.** The generalized Rademacher functions, see e.g. [10] for the definition, will be denoted by $r_j$ ($j \in \mathbb{N}$). By means of the Rademacher function the Dirichlet kernels can be decomposed as $D_k = \chi_k \sum_{j=0}^{\infty} \sum_{\ell = m_j - k_j}^{m_j-1} r^{\ell}_{|\Gamma_j|} |G_j|^{-1/2}$. We note that $D_{\Gamma_j} = |G_N|^{-1/2} \zeta_{G_N}$ [10].

Without loss of generality, we may assume $n > N$. Then

$$\int_{(G_N)^c} \left| \sum_{k=1}^{n} c_k D_k(x) \right| dx = \int_{(G_N)^c} \left| \sum_{k=1}^{\infty} c_k \chi_k(x) \sum_{j=0}^{\infty} \sum_{\ell = m_j - k_j}^{m_j-1} r^{\ell}_{|\Gamma_j|} |G_j|^{-1/2} \right| dx$$

$$= \int_{(G_N)^c} \left| \sum_{k=1}^{N-1} c_k \chi_k(x) \sum_{j=0}^{N-1} \sum_{\ell = m_j - k_j}^{m_j-1} r^{\ell}_{|\Gamma_j|} |G_j|^{-1/2} \right| dx$$

$$\leq \sum_{j=0}^{N-1} |G_j|^{-1} \int_{(G_N)^c} \xi_{G_j}(x) \sum_{k=1}^{\infty} \sum_{\ell = m_j - k_j}^{m_j-1} r^{\ell}_{|\Gamma_j|} c_k \chi_k(x) dx.$$

Set

$$k_{j, \ell} = \begin{cases} 1 & \text{if, } m_j - k_j \leq \ell \leq m_j - 1, \\ 0 & \text{if, } 0 \leq \ell < m_j - k_j \end{cases} (j \in \mathbb{N}).$$

Then we have

$$\int_{(G_N)^c} \left| \sum_{k=1}^{n} c_k D_k(x) \right| dx \leq \sum_{j=0}^{N-1} \sum_{\ell = 0}^{m_j-1} |G_j|^{-1} \int_{(G_N)^c} \xi_{G_j}(x) \sum_{k=1}^{\infty} k_{j, \ell} c_k \chi_k(x) dx.$$

Introducing $h_{j, \ell}(x) = \text{sgn} \left( \sum_{k=1}^{\infty} k_{j, \ell} c_k \chi_k(x) \right)$, this becomes

$$\int_{(G_N)^c} \left| \sum_{k=1}^{n} c_k D_k(x) \right| dx \leq \sum_{j=0}^{N-1} \sum_{\ell = 0}^{m_j-1} |G_j|^{-1} \sum_{k=1}^{\infty} k_{j, \ell} c_k \int_{G} \xi_{G_j}(x) h_{j, \ell}(x) \chi_k(x) dx$$

$$= \sum_{j=0}^{N-1} \sum_{\ell = 0}^{m_j-1} |G_j|^{-1} \sum_{k=1}^{\infty} k_{j, \ell} c_k (\xi_{G_j} h_{j, \ell})^\wedge(k).$$
We will apply Hölder’s inequality followed by Hausdorff-Young’s and in the final step the boundedness of the Vilenkin group to obtain

\[
\int_{(G_N)^c} \left| \sum_{k=1}^{[\Gamma_n]} c_k D_k(x) \right| \, dx \leq C \sum_{j=0}^{N-1} \sum_{\ell=0}^{m_j-1} |G_j|^{-1} \left( \sum_{k=1}^{[\Gamma_n]} |c_k|^q \right)^{1/q} \\
\times \left( \sum_{k=1}^{[\Gamma_n]} (|\xi_{G,h,j,x}|^q(k))^{q'} \right)^{1/q'} \\
\leq C \sum_{j=0}^{N-1} \sum_{\ell=0}^{m_j-1} |G_j|^{-1} \left( \sum_{k=1}^{[\Gamma_n]} |c_k|^q \right)^{1/q} \|\xi_{G,h,j}\|_q \\
\leq C \sum_{j=0}^{N-1} \sum_{\ell=0}^{m_j-1} |G_j|^{-1} \left( \sum_{k=1}^{[\Gamma_n]} |c_k|^q \right)^{1/q} \\
\leq C |G_N|^{-1} \left( \sum_{k=1}^{[\Gamma_n]} |c_k|^q \right)^{1/q}.
\]

Our theorem about the generalized Marcinkiewicz condition [7] reads as follows.

**Theorem 9.** Let $G$ be a compact multiplicative Vilenkin group. Suppose that $1 < q \leq 2$ and $p > \frac{q}{2q-1}$. If $\varphi$ is bounded and satisfies

\[
\sum_{j \in \Gamma_{k+1} \setminus \Gamma_k} |\varphi(j + 1) - \varphi(j)|^q \leq C |\Gamma_k|^{-1-q},
\]

then $T_\varphi$ is bounded on $H^p(G)$.

**Remark.** We note that besides the trigonometric and the Vilenkin systems the Marcinkiewicz condition have been studied with respect to some other systems as well. Here we only mention a recent result by Weisz [14] in which the Ciesielski system is considered.

**Proof.** We will show the above Marcinkiewicz condition implies the kernel satisfies the Kitada-Tateoka condition from Corollary 6 to provide boundedness on $H^p(G)$.

Recall that this condition for $G$ compact is

\[
\sum_{n=0}^{k} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} |\Delta_k k(y)| \, dy \right)^p \leq C |G_k|^{1-p}.
\]

We begin with the left-hand side:

\[
I_1 = \sum_{n=0}^{k} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} |\Delta_k k(y)| \, dy \right)^p \\
= \sum_{n=0}^{k} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} \left| \sum_{m=[\Gamma_k]}^{[\Gamma_{k+1}]} \varphi(m) \chi_m(y) \right| \, dy \right)^p.
\]

For the inner sum, we use summation by parts to obtain:

\[
\left| \sum_{m=[\Gamma_k]}^{[\Gamma_{k+1}]} \varphi(m) \chi_m \right| \leq \left| \varphi([\Gamma_k]) D_{[\Gamma_k]} \right| + \left| \varphi([\Gamma_{k+1}]) D_{[\Gamma_{k+1}]} \right| \\
+ \sum_{m=[\Gamma_k]}^{[\Gamma_{k+1}]-1} (\varphi(m + 1) - \varphi(m)) D_m.
\]
Consequently,
\[ I_1 \leq \sum_{n=0}^{k} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} \left| \varphi \left( |\Gamma_k| \right) D_{|\Gamma_k|} (y) \right| + \left| \varphi \left( |\Gamma_{k+1}| \right) D_{|\Gamma_{k+1}|} (y) \right| \, dy \right)^p \\
+ \sum_{n=0}^{k} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} \left| \sum_{m=|\Gamma_k|}^{\sum_{n=|\Gamma_{k+1}|}^{n=1}} (\varphi(m+1) - \varphi(m)) D_m (y) \right| \, dy \right)^p \\
= I_{11} + I_{12}.
\]

For \( I_{11} \), we are integrating over \( G_n \setminus G_{n+1} \) which is contained in the complement of the support of \( D_{|\Gamma_k|} \) and \( D_{|\Gamma_{k+1}|} \) for \( n < k \). So in this case the integral is zero. For \( n = k \), we have
\[ I_{11} = |G_k|^{1-p} \left( \int_{G_k \setminus G_{k+1}} \left| \varphi \left( |\Gamma_k| \right) D_{|\Gamma_k|} (y) \right| + \left| \varphi \left( |\Gamma_{k+1}| \right) D_{|\Gamma_{k+1}|} (y) \right| \, dy \right)^p \]
\[ \leq |G_k|^{1-p} (B |\Gamma_k| |G_k| + 0)^p \]
\[ = B^p |G_k|^{1-p}, \]
where \( B \) is an upper bound for \( |\varphi| \). This is the desired estimate for \( I_{11} \). For \( I_{12} \) we apply the Sidon type inequality in Lemma 8:
\[ I_{12} = \sum_{n=0}^{k} |G_n|^{1-p} \left( \int_{G_n \setminus G_{n+1}} \left| \sum_{m=|\Gamma_k|}^{\sum_{n=|\Gamma_{k+1}|}^{n=1}} (\varphi(m+1) - \varphi(m)) D_m (y) \right| \, dy \right)^p \]
\[ \leq C \sum_{n=0}^{k} |G_n|^{1-p} \left( \left( \sum_{m=|\Gamma_k|}^{\sum_{n=|\Gamma_{k+1}|}^{n=1}} |\varphi(m+1) - \varphi(m)|^q \right)^{1/q} \right)^p \]
\[ \leq C \sum_{n=0}^{k} |G_n|^{1-p} \left( |G_k|^{\frac{1}{q} - 1} \right)^p \]
\[ \leq C \left| G_k \right|^{(1 - \frac{1}{q})p} \sum_{n=0}^{k} |G_n|^{1-2p + \frac{q}{2}} \]
\[ \leq C \left| G_k \right|^{(1 - \frac{1}{q})p} \left| G_k \right|^{1-2p + \frac{q}{2}} \]
\[ \leq C \left| G_k \right|^{(1 - \frac{1}{q})p} \left| G_k \right|^{1-2p + \frac{q}{2}} \]
\[ \leq C \left| G_k \right|^{1-p}, \]
the desired estimate for \( I_{12} \). This completes the proof. \( \square \)

**References**


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