SOME INEQUALITIES FOR RICCI CURVATURE OF CERTAIN SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract. In the present paper, we obtain sharp inequalities between the Ricci curvature and the squared mean curvature for slant, semi-slant and bi-slant submanifolds in Sasakian space forms. Also, estimates of the scalar curvature and the $k$-Ricci curvature respectively, in terms of the squared mean curvature, are proved.

1. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be a Sasakian manifold if it admits an endomorphism $\phi$ of its tangent bundle $T\tilde{M}$, a vector field $\xi$ and a 1-form $\eta$, satisfying:

\[ \phi^2 = -\text{Id} + \eta \otimes \xi, \eta(\xi) = 1, \phi \xi = 0, \eta \circ \phi = 0, \]
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi), \]
\[ (\tilde{\nabla}_X \phi)Y = -(\phi(X)Y)\xi + \eta(Y)X, \tilde{\nabla}_X \xi = \phi X, \]

for any vector fields $X, Y$ on $T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to $g$.

A plane section in $T_p\tilde{M}$ is called a $\phi$-section if it is spanned by $X$ and $\phi X$, where $X$ is a unit tangent vector orthogonal to $\xi$. The sectional curvature of a $\phi$-section is called a $\phi$-sectional curvature. A Sasakian manifold with constant $\phi$-sectional curvature $c$ is said to be a Sasakian space form and is denoted by $\tilde{M}(c)$.

The curvature tensor of $\tilde{M}(c)$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]

\[ \tilde{R}(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y\} + \]
\[ + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \]
\[ + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \}
\]

for any tangent vector fields $X, Y, Z$ on $\tilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and $S^{2m+1}$, with standard Sasakian structures (see [1]).

In [9], A. Lotta has introduced the following notion of slant immersion in almost contact metric manifolds.

Definition. We call a differentiable distribution $\mathcal{D}$ on $M$ a slant distribution if for each $x \in M$ and each nonzero vector $X \in \mathcal{D}_x$, the angle $\theta_\phi(X)$ between $\phi X$ and the vector subspace $\mathcal{D}_x$ is constant, which is independent of the choice of $x \in M$.

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and $X \in D_x$. In this case, the constant angle $\theta_D$ is called the \textit{slant angle} of the distribution $D$.

**Definition.** A submanifold $M$ tangent to $\xi$ is said to be \textit{slant} if for any $x \in M$ and any $X \in T_xM$, linearly independent of $\xi$, the angle between $\phi X$ and $T_xM$ is a constant $\theta \in [0, \frac{\pi}{2}]$, called the \textit{slant angle} of $M$ in $\tilde{M}$.

**Examples of slant submanifolds.** (see [2]).

**Example 1.** For any constant $k$,
\[
x(u, v, t) = 2(e^{ku}\cos u \cos v, e^{ku}\cos u \sin v, e^{ku}\cos u \sin v, e^{ku}\sin u \sin v, t)
\]
defines a slant submanifold of dimension 3 with slant angle $\theta = \arccos \frac{|k|}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-k^2}{3(1+k^2)}$ and non-constant mean curvature given by $\|H\| = \frac{2e^{-ku}}{3\sqrt{1+k^2}}$. Hence, the submanifold is not minimal.

**Example 2.** For any constant $k$,
\[
x(u, v, t) = 2(u, k\cos v, v, k\sin v, t)
\]
defines a slant submanifold $M$ with slant angle $\theta = \arccos \frac{1}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-1}{3(1+k^2)}$, constant mean curvature given by $\|H\| = \frac{|k|}{3(1+k^2)}$. Moreover, the following statements are equivalent:

(a) $k = 0$;
(b) $M$ is invariant;
(c) $M$ is minimal;
(d) $M$ has parallel mean curvature vector.

\textit{Invariant} and \textit{anti-invariant immersions} are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a \textit{proper slant immersion}.

**Definition.** We say that a submanifold $M$ tangent to $\xi$ is a \textit{bi-slant} submanifold of $\tilde{M}$ if there exist two orthogonal distributions $D_1$ and $D_2$ on $M$ such that:

i) $TM$ admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \{\xi\}$.

ii) For any $i = 1, 2$, $D_i$ is slant distribution with slant angle $\theta_i$.

Let $2d_1 = \dim D_1$ and $2d_2 = \dim D_2$.

**Remark.** If either $d_1$ or $d_2$ vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds (and, therefore, invariant and anti-invariant submanifolds) are particular cases of bi-slant submanifolds.

**Examples of bi-slant submanifolds.** (see [2], [3])

**Example 1.** For any $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$,
\[
x(u, v, w, s, t) = 2(u, 0, w, 0, v\cos \theta_1, v\sin \theta_1, s\cos \theta_2, s\sin \theta_2, t)
\]
defines a five-dimensional bi-slant submanifold $M$, with slant angles $\theta_1$ and $\theta_2$, is $\mathbb{R}^9$ with its usual Sasakian structure $(\phi_0, \xi, \eta, g)$. 
Furthermore, it is easy to see that
\[
\begin{align*}
  e_1 &= 2\left(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}\right), \\
  e_2 &= \cos \theta_1(2\frac{\partial}{\partial y^1}) + \sin \theta_1(2\frac{\partial}{\partial y^2}), \\
  e_3 &= 2\left(\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}\right), \\
  e_4 &= \cos \theta_2(2\frac{\partial}{\partial y^3}) + \sin \theta_2(2\frac{\partial}{\partial y^4}), \\
  e_5 &= 2\frac{\partial}{\partial z} = \xi,
\end{align*}
\]
form a local orthonormal frame of \(TM\). We define the distributions \(D_1 = \langle e_1, e_2 \rangle\) and \(D_2 = \langle e_3, e_4 \rangle\).

Then, it is clear that \(TM = D_1 \oplus D_2 \oplus \langle \xi \rangle\) and it can be easily proved that \(D_i\) is a slant distribution with slant angle \(\theta_i\) for any \(i = 1, 2\). In particular, if we consider \(\theta_1 = \theta_2 = \theta\) in the above, it results that \(M\) is a \(\theta\)-slant submanifold.

**Example 2.** For any \(\theta_1 \in [0, \frac{\pi}{2}]\), we chose \(\theta_2 \in (0, \frac{\pi}{2}]\), such that \(\cos \theta_2 = \frac{\cos \theta_1}{\sqrt{2}}\).

Then
\[
x(u, v, w, s, t) = 2(u, 0, w, 0, v \cos \theta_1, v \sin \theta_1, s \cos \theta_2, s \sin \theta_2, t)
\]
defines a five-dimensional bi-slant submanifold \(M\) in \((\mathbb{R}^6, \phi_0, \xi, \eta, g)\), with both slant angles equal to \(\theta_2\), but it is not slant submanifold. In fact we can chose a local orthonormal frame \(\{e_1, \ldots, e_5\}\) of \(TM\) such that
\[
\begin{align*}
  e_1 &= \frac{1}{\sqrt{2}}(2\left(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}\right), \\
  e_2 &= \cos \theta_1(2\left(\frac{\partial}{\partial y^1}) + \sin \theta_1(2\left(\frac{\partial}{\partial y^2}), \\
  e_3 &= 2\left(\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}\right), \\
  e_4 &= \cos \theta_2(2\left(\frac{\partial}{\partial y^3}) + \sin \theta_2(2\left(\frac{\partial}{\partial y^4), \\
  e_5 &= 2\frac{\partial}{\partial z} = \xi.
\end{align*}
\]
Now we define the distributions \(D_1 = \langle e_1, e_2 \rangle\) and \(D_2 = \langle e_3, e_4 \rangle\). It is easy to see that both \(D_1\) and \(D_2\) are slant distribution with the same slant angle \(\theta_2\). Nevertheless, we can obtain that \(M\) is not slant since \(\theta_2 \neq 0\).

**Definition.** We say that \(M\) tangent to \(\xi\) is a **semi-slant** submanifold of \(\tilde{M}\) if there exist two orthogonal distributions \(D_1\) and \(D_2\) on \(M\) such that :

i) \(TM\) admits the orthogonal direct decomposition \(TM = D_1 \oplus D_2 \oplus \langle \xi \rangle\).

ii) The distribution \(D_1\) is an invariant distribution, i.e., \(\phi(D_1) = D_1\).

iii) The distribution \(D_2\) is slant with angle \(\theta \neq 0\).

Let \(2d_1 = \dim D_1\) and \(2d_2 = \dim D_2\).

In [3], the invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds. Moreover, it is clear that, if \(\theta = \frac{\pi}{2}\), then the semi-slant submanifold is a semi-invariant submanifold.

(a) If \(d_2 = 0\), then \(M\) is an invariant submanifold.

(b) If \(d_1 = 0\) and \(\theta = \frac{\pi}{2}\), then \(M\) is an anti-invariant submanifold.

(c) If \(d_1 = 0\) and \(\theta \neq \frac{\pi}{2}\), then \(M\) is a proper slant submanifold, with slant angle \(\theta\).

We say that a semi-slant submanifold is **proper** if \(d_1d_2 \neq 0\) and \(\theta \neq \frac{\pi}{2}\).
Examples of semi-slant submanifolds. (see [3])

Example 1. Let $\mathbb{R}^6$ be the Euclidean space of dimension 6, with the standard metric and the almost complex structure given by $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}$, for any $i = 1, 2, 3$, where $(x^i, y^i)$ denote the Cartesian coordinates.

Let $\mathbb{R}^5 \hookrightarrow \mathbb{R}^6$ be the usual immersion. Then, $C = \frac{\partial}{\partial y^3}$ is the unit normal to $\mathbb{R}^5$ and so, $\xi = -JC = \frac{\partial}{\partial x^3}$.

Now, for any $\theta \neq 0$, we can consider the immersions:

$$\varphi_1 : \mathbb{R}^4 \rightarrow \mathbb{R}^6 : (u, v, t, s) \mapsto (u \cos \theta, u \sin \theta, t, v, 0, s),$$

$$\varphi_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^5 : (u, v, t) \mapsto (u \cos \theta, u \sin \theta, t, v, 0).$$

We can directly prove that $\varphi_1$ is a semi-slant immersion, with complex distribution $D_1 = \left\langle \frac{\partial}{\partial x^3}, \frac{\partial}{\partial y^3} \right\rangle$ and slant distribution, with angle $\theta$,

$$D_2 = \left\langle \cos \theta \frac{\partial}{\partial x^1} + \sin \theta \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1} \right\rangle.$$

On the other hand, $\varphi_2$ is a $\theta$-slant immersion, where $\mathbb{R}^5$ has the almost contact metric structure induced by the described almost Hermitian structure on $\mathbb{R}^6$.

For the other properties and examples of slant, bi-slant and semi-slant submanifolds in Sasakian manifolds, we refer to [2], [3].

Let $M$ be an $n$-dimensional submanifold of a Riemannian manifold $\tilde{M}$. We denote by $K(p)$ the sectional curvature of $M$ associated with a plane section $T_pM, p \in M$, and $\nabla$ the Riemannian connection of $M$. Also, let $h$ be the second fundamental form and $R$ the Riemann curvature tensor of $M$.

Then the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

for any vectors $X, Y, Z, W$ tangent to $M$.

Let $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of the tangent space $T_pM$. We denote by $H$ the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).$$

Also, we set

$$h_{i}^{j} = g(h(e_i, e_j), e_r)$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any tangent vector field $X$ to $M$, we put $\phi X = PX + FX$, where $PX$ and $FX$ are the tangential and normal components of $\phi X$, respectively. We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

Suppose $L$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L$. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of $L$ such that $e_1 = X$. 
Define the *Ricci curvature* $\text{Ric}_L$ of $L$ at $X$ by

$$\text{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k},$$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. We simply called such a curvature a $k$-Ricci curvature.

The *scalar curvature* $\tau$ of the $k$-plane section $L$ is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}.$$

For each integer $k$, $2 \leq k \leq n$, the Riemannian invariant $\Theta_k$ on an $n$-dimensional Riemannian manifold $M$ is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L, X} \text{Ric}_L(X), \quad p \in M,$$

where $L$ runs over all $k$-plane sections in $T_p M$ and $X$ runs over all unit vectors in $L$.

Recall that for a submanifold $M$ in a Riemannian manifold, the *relative null space* of $M$ at a point $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_p M | h(X, Y) = 0, \text{ for all } Y \in T_p M\}.$$

### 2. Ricci curvature and squared mean curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [6]).

We prove similar inequalities for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We consider submanifolds $M$ tangent to the Reeb vector field $\xi$.

**Theorem 2.1.** Let $M$ be an $(n = 2k + 1)$-dimensional $\theta$-slant submanifold tangent to $\xi$ in a $(2m + 1)$-dimensional Sasakian space form $M(c)$. Then:

(i) For each unit vector $X \in T_p M$ orthogonal to $\xi$, we have

$$\text{Ric}(X) \leq \frac{1}{4} \left((n-1)(c+3) + \frac{1}{2}(3 \cos^2 \theta - 2)(c-1) + n^2 \|H\|^2 \right).$$

(ii) If $H(p) = 0$, then a unit tangent vector $X \in T_p M$ orthogonal to $\xi$ satisfies the equality case of (2.1) if and only if $X \in \mathcal{N}_p$.

(iii) The equality case of (2.1) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

**Proof.** Let $X \in T_p M$ be a unit tangent vector $X$ at $p$, orthogonal to $\xi$. We choose an orthonormal basis $e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangent to $M$ at $p$, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1) \frac{c+3}{4} - \frac{3(n-1) \cos^2 \theta - 2n+2}{4} \frac{c-1}{4}.$$
From (2.2), we get

\[ n^2 \|H\|^2 = 2\tau + \sum_{r=1}^{2m+1} [(h_{i1}^r)^2 + (h_{i2}^r + \cdots + h_{nn}^r)^2 + 2 \sum_{i<j} (h_{ij}^r)^2] \]

\[ - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i \leq j \leq n} h_{ii}^r h_{jj}^r - n(n-1) \frac{c+3}{4} - [3(n-1) \cos^2 \theta - 2n + 2] \frac{c-1}{4} \]

\[ = 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m+1} [(h_{i1}^r + \cdots + h_{nn}^r)^2 + (h_{i1}^r - h_{i2}^r - \cdots - h_{nn}^r)^2] + 2 \sum_{r=n+1}^{2m+1} \sum_{i<j} (h_{ij}^r)^2 \]

\[ - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i \leq j \leq n} h_{ii}^r h_{jj}^r - n(n-1) \frac{c+3}{4} - [3(n-1) \cos^2 \theta - 2n + 2] \frac{c-1}{4}. \]

From the equation of Gauss, we find

\[ K_{ij} = \sum_{r=n+1}^{2m+1} [h_{i1}^r h_{22}^r - (h_{12}^r)^2] + 3 \cos^2 \theta \cdot \frac{c-1}{4} + \frac{c+3}{4} \]

and consequently

\[ \sum_{2 \leq i \leq j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i \leq j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} c + 3 \frac{c-1}{4} + [3(n-1) \cos^2 \theta - 3 \cos^2 \theta - 2n + 4] \frac{c-1}{8}. \]

Substituting (2.4) in (2.3), one gets

\[ \frac{1}{2} n^2 \|H\|^2 \geq 2 \text{Ric}(X) - 2(n-1) \frac{c+3}{4} - (3 \cos^2 \theta - 2) \frac{c-1}{4}, \]

which is equivalent to (2.1).

(ii) Assume \( H(p) = 0 \). Equality holds in (2.1) if and only if

\[ \begin{align*}
    h_{12}^r &= \cdots = h_{1n}^r = 0, \\
    h_{i1}^r &= h_{i2}^r + \cdots + h_{nn}^r, r \in \{n+1, \ldots, 2m\}.
\end{align*} \]

Then \( h_{ij}^r = 0 \), for every \( j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\} \), that is \( X \in \mathcal{N}_p \).

(iii) The equality case of (2.1) holds for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if

\[ \begin{align*}
    h_{ij}^r &= 0, i \neq j, r \in \{n+1, \ldots, 2m\}, \\
    h_{i1}^r + \cdots + h_{n}^r - 2h_{ii}^r &= 0, \quad i \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}.
\end{align*} \]

In this case, since \( \xi \) is tangent to \( M \), it follows that a totally umbilical point is totally geodesic.

\[ \square \]

**Theorem 2.2.** Let \( M \) be an \((n = 2d_1 + 2d_2 + 1)\)-dimensional bi-slant submanifold satisfying \( g(X, \phi Y) = 0 \), for any \( X \in T_1 \) and any \( X \in T_2 \), tangent to \( \xi \) in a \((2m + 1)\)-dimensional Sasakian space form \( M(c) \). Then:

(i) For each unit vector \( X \in T_p M \) orthogonal to \( \xi \) and if

a) \( X \) is tangent to \( T_1 \) we have

\[ \text{Ric}(X) \leq \frac{1}{4} ((n-1)(c+3) + \frac{1}{2}(3 \cos^2 \theta_1 - 2)(c-1) + n^2 \|H\|^2) \]

and if
b) $X$ is tangent to $D_2$ we have

$$(2.7') \quad \text{Ric}(X) \leq \frac{1}{4} \left( \frac{1}{4} - 1 \right) \left( n - 1 \right) \left( c + 3 \right) + \frac{1}{2} \left( 3 \cos^2 \theta_2 - 2 \right) \left( c - 1 \right) + n^2 \| H \|^2 \right).$$

(ii) If $H(p) = 0$, then a unit tangent vector $X \in T_p M$ orthogonal to $\xi$ satisfies the equality case of (2.7) and (2.7') if and only if $X \in N_p$.

(iii) The equality case of (2.7) and (2.7') holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

Proof. Let $X \in T_p M$ be a unit tangent vector $X$ at $p$, orthogonal to $\xi$. We choose an orthonormal basis $e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangent to $M$ at $p$, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$(2.8) \quad n^2 \| H \|^2 = 2\tau + \| h \|^2 - \frac{1}{4} \left[ \frac{n(n-1)}{4} + 2 n + 2 \right] \frac{c - 1}{4}.$$ 

From (2.8), we get

$$
2m+1 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} h_{ij}^r - n(n-1) \frac{c + 3}{4} - 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2 \frac{c - 1}{4} = 
$$

$$= 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} \left[ h_{ij}^r - n(n-1) \frac{c + 3}{4} - 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2 \frac{c - 1}{4} \right] + 
$$

$$+ 2 \sum_{r=n+1}^{2m+1} \sum_{1 \leq i < j \leq n} \left[ h_{ij}^r - n(n-1) \frac{c + 3}{4} \right] - [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n + 2 \frac{c - 1}{4}].$$

From the equation of Gauss, we find:

a) if $X$ is tangent to $D_1$

$$K_{ij} = \sum_{r=n+1}^{2m+1} \frac{h_{11}^r h_{22}^r - (h_{12}^r)^2}{2} + 3 \cos^2 \theta_1 \cdot \frac{c - 1}{4} + \frac{c + 3}{4}$$

and consequently

$$(2.10) \quad \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} \frac{h_{ij}^r}{2} + \frac{(n - 1)(n - 2) \frac{c + 3}{4}}{2} + 
$$

$$+ [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_1 - 2n + 4] \frac{c - 1}{8}.$$ 

Substituting (2.10) in (2.9), one gets

$$\frac{1}{2} n^2 \| H \|^2 \geq 2 \text{Ric}(X) - 2(n - 1) \frac{c + 3}{4} - (3 \cos^2 \theta_1 - 2) \frac{c - 1}{4},$$

which is equivalent to (2.7).

b) Similar if $X$ is tangent to $D_2$, we have

$$K_{ij} = \sum_{r=n+1}^{2m+1} \frac{h_{ij}^r}{2} + 3 \cos^2 \theta_2 \cdot \frac{c - 1}{4} + \frac{c + 3}{4}.$$
and consequently

\begin{equation}
\sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r = n+1}^{2m+1} \sum_{2 \leq i < j \leq n} \left[ h_{ij}^r - (h_{ij}^r)^2 \right] + \frac{(n-1)(n-2)(c+3)}{4} + \frac{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_2 - 2n + 4}{8} \cdot \frac{c-1}{4}.
\end{equation}

Substituting (2.11) in (2.9), one gets

\begin{equation}
\frac{1}{2} n^2 \| H \|^2 \geq 2 \text{Ric}(X) - (n-1) \left( \frac{c+3}{4} - (3 \cos^2 \theta_2 - 2) \frac{c-1}{4} \right),
\end{equation}

which is equivalent to (2.7').

(ii) Assume \( H(p) = 0 \). Equality holds in (2.7) and (2.7') if and only if

\begin{equation}
\begin{aligned}
&h_{12}^r = \ldots = h_{1n}^r = 0, \\
&h_{11}^r = h_{22}^r + \ldots + h_{nn}^r, r \in \{n + 1, \ldots, 2m\}.
\end{aligned}
\end{equation}

Then \( h_{ij}^r = 0 \), for every \( j \in \{1, \ldots, n\}, r \in \{n + 1, \ldots, 2m\} \), that is \( X \in \mathcal{N}_p \).

(iii) The equality case of (2.7) and (2.7') holds for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if

\begin{equation}
\begin{aligned}
&h_{ij}^r = 0, i \neq j, r \in \{n + 1, \ldots, 2m\}, \\
&h_{11}^r + \ldots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1, \ldots, n\}, r \in \{n + 1, \ldots, 2m\}.
\end{aligned}
\end{equation}

In this case, since \( \xi \) is tangent to \( M \), it follows that a totally umbilical point is totally geodesic.

\begin{corollary}
Let \( M \) be an \( (n = 2d_1 + 2d_2 + 1) \)-dimensional semi-slant submanifold in a \((2m+1)\)-dimensional Sasakian space form \( \tilde{M}(c) \). Then:

(i) For each unit vector \( X \in T_pM \) orthogonal to \( \xi \) and if a \( X \) is tangent to \( D_1 \) we have

\begin{equation}
\text{Ric}(X) \leq \frac{1}{4} ( (n-1)(c+3) - (c-1) + n^2 \| H \|^2 )
\end{equation}

and if b \( X \) is tangent to \( D_2 \) we have

\begin{equation}
\text{Ric}(X) \leq \frac{1}{4} ( (n-1)(c+3) + \frac{1}{2} (3 \cos^2 \theta - 2)(c-1) + n^2 \| H \|^2 ).
\end{equation}

(ii) If \( H(p) = 0 \), then a unit tangent vector \( X \in T_pM \) orthogonal to \( \xi \) satisfies the equality case of (2.14) and (2.14') if and only if \( X \in \mathcal{N}_p \).

(iii) The equality case of (2.14) and (2.14') holds identically for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if \( p \) is a totally geodesic point.

\end{corollary}

\begin{corollary}
Let \( M \) be an \( (n = 2k+1) \)-dimensional invariant submanifold in a \((2m+1)\)-dimensional Sasakian space form \( \tilde{M}(c) \). Then:

(i) For each unit vector \( X \in T_pM \) orthogonal to \( \xi \), we have

\begin{equation}
\text{Ric}(X) \leq \frac{1}{4} ( (n-1)(c+3) + \frac{1}{2} (c-1) ).
\end{equation}

(ii) A unit tangent vector \( X \in T_pM \) orthogonal to \( \xi \) satisfies the equality case of (2.15) if and only if \( X \in \mathcal{N}_p \).

(iii) The equality case of (2.15) holds identically for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if \( p \) is a totally geodesic point.

\end{corollary}
Corollary 2.5. Let $M$ be an $(n = 2k + 1)$-dimensional anti-invariant submanifold in a $(2m + 1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then:

(i) For each unit vector $X \in T_pM$ orthogonal to $\xi$, we have

$$\text{Ric}(X) \leq \frac{1}{4} \{(n - 1)(c + 3) - (c - 1) + n^2\|H\|^2\}.$$  

(ii) If $H(p) = 0$, then a unit tangent vector $X \in T_pM$ orthogonal to $\xi$ satisfies the equality case of (2.16) if and only if $X \in \mathcal{N}_p$.

(iii) The equality case of (2.16) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

3. $k$-Ricci curvature

In this section, we prove a relationship between the $k$-Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We state an inequality between the scalar curvature and the squared mean curvature for submanifolds tangent to $\xi$.

Theorem 3.1. Let $M$ be an $(n = 2k + 1)$-dimensional $\theta$-slant submanifold in a $(2m + 1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to $\xi$. Then we have

$$\|H\|^2 \geq \frac{2\tau}{n(n - 1)} - \frac{c + 3}{4} - \frac{3(n - 1)\cos^2 \theta - 2n + 2(c - 1)}{4n(n - 1)}.$$  

Proof. We choose an orthonormal basis $\{e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}\}$ at $p$ such that $e_{n+1}$ is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator $A_{n+1}$. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$  

$$A_r = (h_{ij}^r), i, j = 1, \ldots, n, r = n + 2, \ldots, 2m + 1, \text{trace } A_r = \sum_{i=1}^{n} h_{ii}^r = 0.$$  

From (2.2), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - n(n - 1)\frac{c + 3}{4} -$$

$$- [3(n - 1)\cos^2 \theta - 2n + 2]\frac{c - 1}{4}.$$  

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n - 1)\sum_{i} a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2.$$
Since we have that
\[
(3.5) \quad n^2\|H\|^2 \geq 2\tau + n\|H\|^2 - n(n-1) \frac{c+3}{4} - [3(n-1)\cos^2\theta - 2n + 2] \frac{c-1}{4},
\]
which is equivalent to (3.1).

Let \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of \(T_p M\). Denote by \(L_{i_1 \ldots i_k}\) the \(k\)-plane section spanned by \(e_{i_1}, \ldots, e_{i_k}\). It follows from (1.7) and (1.8) that
\[
(3.6) \quad \tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \text{Ric}_{L_{i_1 \ldots i_k}}(e_i),
\]
\[
(3.7) \quad \tau(p) = \frac{1}{c_{n-2}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \tau(L_{i_1 \ldots i_k}).
\]

Combining (1.9), (3.6) and (3.7), we find
\[
(3.8) \quad \tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p).
\]

From (3.6), (3.7) and (3.1), we get the following.

**Theorem 3.2.** Let \(M\) be an \((n = 2d_1 + 2d_2 + 1)\)-dimensional bi-slant submanifold satisfying \(g(X,\phi Y) = 0\), for any \(X \in \mathcal{D}_1\) and any \(X \in \mathcal{D}_2\), in a \((2m + 1)\)-dimensional Sasakian space form \(\tilde{M}(c)\) tangent to \(\xi\). Then we have
\[
(3.9) \quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c + 3}{4} - \frac{[3(d_1 \cos^2\theta_1 + d_2 \cos^2\theta_2) - n + 1](c - 1)}{2n(n-1)}.
\]

**Proof.** The proof is similar with their corresponding statements of Theorem 3.1. \(\square\)

**Theorem 3.3.** Let \(M\) be an \((n = 2d_1 + 2d_2 + 1)\)-dimensional semi-slant submanifold in a \((2m + 1)\)-dimensional Sasakian space form \(\tilde{M}(c)\) tangent to \(\xi\). Then we have
\[
(3.10) \quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c + 3}{4} - \frac{[3(d_1 + d_2 \cos^2\theta) - n + 1](c - 1)}{2n(n-1)}.
\]

**Proof.** The proof is similar with their corresponding statements of Theorem 3.1. \(\square\)

**Theorem 3.4.** Let \(M\) be an \((n = 2k + 1)\)-dimensional \(\theta\)-slant submanifold in a \((2m + 1)\)-dimensional Sasakian space form \(\tilde{M}(c)\) tangent to \(\xi\). Then, for any integer \(k\), \(2 \leq k \leq n\), and any point \(p \in M\), we have
\[
(3.11) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c + 3}{4} - \frac{[3(n-1)\cos^2\theta - 2n + 2](c - 1)}{4n(n-1)}.
\]

**Theorem 3.5.** Let \(M\) be an \((n = 2d_1 + 2d_2 + 1)\)-dimensional bi-slant submanifold in a \((2m + 1)\)-dimensional Sasakian space form \(\tilde{M}(c)\) tangent to \(\xi\). Then, for any integer \(k\), \(2 \leq k \leq n\), and any point \(p \in M\), we have
\[
(3.12) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c + 3}{4} - \frac{[3(d_1 \cos^2\theta_1 + d_2 \cos^2\theta_2) - n + 1](c - 1)}{2n(n-1)}.
\]

**Theorem 3.6.** Let \(M\) be an \((n = 2d_1 + 2d_2 + 1)\)-dimensional semi-slant submanifold in a \((2m + 1)\)-dimensional Sasakian space form \(\tilde{M}(c)\) tangent to \(\xi\). Then, for any integer \(k\), \(2 \leq k \leq n\), and any point \(p \in M\), we have
\[
(3.13) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c + 3}{4} - \frac{[3(d_1 + d_2 \cos^2\theta) - n + 1](c - 1)}{2n(n-1)}.
\]
Corollary 3.7. Let $M$ be an $n$-dimensional invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have
\[ \Theta_k(p) \leq \frac{c + 3}{4} + \frac{c - 1}{4n}. \]

Corollary 3.8. Let $M$ be an $n$-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have
\[ \|H\|^2(p) \geq \Theta_k(p) - \frac{c + 3}{4} + \frac{c - 1}{2n}. \]

Corollary 3.9. Let $M$ be an $n$-dimensional contact CR submanifold ($\theta_1 = 0, \theta_2 = \frac{\pi}{2}$) of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have
\[ \|H\|^2(p) \geq \Theta_k(p) - \frac{c + 3}{4} - \frac{(3d_1 - n + 1)(c - 1)}{2n(n - 1)} \]
where $2d_1 = \dim \mathcal{D}_1$.

References


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