FEKETE-SZEGÖ FUNCTIONAL FOR NON-BAZILEVIČ FUNCTIONS

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Abstract. Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) be an analytic function in the unit disk \( U \) and let the class of non-Bazilevič functions, for \( 0 < \lambda < 1 \), be described with \( \text{Re} \left\{ f'(z) \left( z/f(z) \right)^{1+\lambda} \right\} > 0, z \in U \). In this paper we obtain sharp upper bound of \( |a_2| \) and of the Fekete-Szegö functional \( |a_3 - \mu a_2^2| \) for the class of non-Bazilevič functions and for some of its subclasses.

1. Introduction and preliminaries

Let \( A \) denote the class of analytic functions in the unit disk \( U = \{ z : |z| < 1 \} \) normalized such that \( f(0) = f'(0) - 1 = 0 \), i.e., of type \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \).

A function \( f \in A \) is said to be of Bazilevič type if for a starlike function \( g \in A \) (\( g \in A \) is starlike if and only if \( \text{Re} \left\{ zg'(z)/g(z) \right\} > 0, z \in U \)) we have

\[
\text{Re} \left\{ f'(z) \left( z/f(z) \right)^{\alpha+i\gamma-1} \left( g(z)/z \right)^{-\alpha} \right\} > 0,
\]

\( z \in U \) (see more in [1]). This class and its subclasses were widely studied in the past decades. Specially, in [4] sharp upper bound of the Fekete-Szegö functional \( |a_3 - \mu a_2^2| \) is obtained for all real \( \mu \) when \( \gamma = 0 \). That result was partially extended in [2] to a wider subclass satisfying

\[
\text{Re} \left\{ \frac{zf'(z)}{f'(z)g(z)} \right\} > \beta, \quad z \in U,
\]

where \( \alpha > 0 \) and \( 0 \leq \beta < 1 \).

In [5], Obradović introduced a class of functions \( f \in A \) that for \( 0 < \lambda < 1 \) is defined by

\[
\text{Re} \left\{ f'(z) \left( z/f(z) \right)^{1+\lambda} \right\} > 0, \quad z \in U.
\]

Recently, in his talk at the Conference ‘Computational Methods and Function Theory 2001’, he called this functions to be of non-Bazilevič type. By now, this class was studied in a direction of finding necessary conditions over \( \lambda \) that embeds this class into the class of univalent function or its subclasses, which is still an open problem. Here we will find sharp upper bound of \( |a_2| \) and of the Fekete-Szegö functional \( |a_3 - \mu a_2^2| \) for the class of non-Bazilevič functions and for some its subclasses. In that purpose we will need the following lemma.

Lemma 1. ([6], p.166, formula (10)) ([3], p.41) Let \( p \in P \), that is, \( p \) be analytic in \( U \), be given by \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) and \( \text{Re} p(z) > 0 \) for \( z \in U \). Then

\[
|p_2 - p_1^2/2| \leq 2 - |p_1|^2/2
\]

and \( |p_n| \leq 2 \) for all \( n \in \mathbb{N} \).

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2. Main results

**Theorem 1.** Let \( f \in \mathcal{A}, \) \( 0 < \lambda < 1 \) and \( 0 \leq \alpha < 1. \) If
\[
(1) \quad \text{Re}\left\{ f'(z)\frac{(z/f(z))}{z} \right\} > \alpha, \quad z \in \mathcal{U},
\]
then \( |a_2| \leq 2(1 - \alpha)/(1 - \lambda) \) and for all \( \mu \in \mathbb{C} \) the following bound is sharp
\[
|a_3 - \mu a_2^2| \leq \frac{2(1 - \alpha)^2(1 + \lambda)^2}{2(1 - \lambda)^2 - \mu^2}.
\]

**Proof.** Condition (1) is equivalent to
\[
f'(z) = \frac{(f(z)/z)^{1+\lambda}[(1 - \alpha)p(z) + \alpha]}{z}, \quad z \in \mathcal{U},
\]
for some \( p \in \mathcal{P}. \) Equating coefficients we obtain \( a_2 = p_1(1 - \alpha)/(1 - \lambda), \)
\[
a_3 = \frac{1 - \alpha}{2 - \lambda}p_2 + \frac{(1 - \alpha)^2(1 + \lambda)}{2(1 - \lambda)^2 - \mu^2}
\]
and further
\[
a_3 - \mu a_2^2 = \frac{1 - \alpha}{2 - \lambda} \left( p_2 - \frac{1}{2}p_1^2 \right) + \frac{(1 - \alpha)(1 - \lambda)^2 + (1 - \alpha)^2(1 + \lambda - 2\mu)(2 - \lambda)}{2(2 - \lambda)(1 - \lambda)^2} p_2^2.
\]

Now, using Lemma 1 we receive \( |a_3 - \mu a_2^2| \leq H(x) = A + ABx^2/4 \) where \( x = |p_1| \leq 2, \) \( A = 2(1 - \alpha)/(2 - \lambda) > 0, \) \( B = (|C| - (1 - \lambda)^2)/(1 - \lambda)^2 \) and \( C = (1 - \lambda)^2 + (1 - \alpha)(1 + \lambda - 2\mu)(2 - \lambda). \) So, we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} H(0) = A, & |C| \leq (1 - \lambda)^2 \\ H(2) = A|C|/(1 - \lambda)^2, & |C| \geq (1 - \lambda)^2 \end{cases}
\]

Equality is attained for functions given by
\[
f'(z) \left( \frac{z}{f(z)} \right)^{1+\lambda} = \frac{1 + z^2(1 - 2\alpha)}{1 - z^2}
\]
and
\[
f'(z) \left( \frac{z}{f(z)} \right)^{1+\lambda} = \frac{1 + z(1 - 2\alpha)}{1 - z}
\]
respectively. \( \square \)

For \( \alpha = 0 \) we have the following corollary.

**Corollary 1.** Let \( f \in \mathcal{A} \) and \( 0 < \lambda < 1. \) If
\[
\text{Re}\left\{ f'(z)\frac{(z/f(z))^{1+\lambda}}{z} \right\} > 0, \quad z \in \mathcal{U},
\]
then \( |a_2| \leq 2/(1 - \lambda) \) and for all \( \mu \in \mathbb{C} \) the following bound is sharp
\[
|a_3 - \mu a_2^2| \leq \frac{2}{2 - \lambda} \max\left\{ 1, \left| \frac{1 + (1 + \lambda - 2\mu)(2 - \lambda)}{2(1 - \lambda)^2} \right| \right\}.
\]

Now we will consider one subclass of the class of non-Bazilević function.

**Theorem 2.** Let \( f \in \mathcal{A}, \) \( 0 < \lambda < 1 \) and \( 0 < k \leq 1. \) If
\[
(2) \quad \left| f'(z)\frac{(z/f(z))^{1+\lambda} - 1}{z} \right| < k, \quad z \in \mathcal{U},
\]
then \( |a_2| \leq k/(1 - \lambda) \) and for all \( \mu \in \mathbb{C} \) the following bound is sharp
\[
|a_3 - \mu a_2^2| \leq \frac{k}{2 - \lambda} \max\left\{ 1, \left| \frac{k(2 - \lambda)}{(1 - \lambda)^2} \right| \left| \frac{1 + \lambda}{2 - \mu} \right| \right\}.
\]
Proof. Similarly as in the proof of Theorem 1, condition (2) implies that there exists a function \( p \in \mathcal{P} \) such that for all \( z \in \mathcal{U} \)
\[
f'(z) = (f(z)/z)^{1+\lambda}(2k/(1 + p(z)) + 1 - k).
\]
Equating the coefficients we obtain
\[
a_2 = -kp_1/(2(1 - \lambda)),
\]
and
\[
a_3 - \mu a_2^2 = -\frac{k}{2(2 - \lambda)} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{k^2 p_1^2}{4(1 - \lambda)^2} \left( \frac{1 + \lambda}{2} - \mu \right).
\]
So, \(|a_3 - \mu a_2^2| \leq H(x) = A + Bx^2/4\) where \( x = |p_1| \leq 2, A = k/(2 - \lambda) > 0, B = k^2|C|/(1 - \lambda)^2 - k/(2 - \lambda) \) and \( C = (1 + \lambda)/2 - \mu \). Therefore
\[
|a_3 - \mu a_2^2| \leq \begin{cases} H(0) = A, & \text{if } |C| \leq (1 - \lambda)^2/(k(2 - \lambda)) \\ H(2) = Ak(2 - \lambda)|C|/(1 - \lambda)^2, & \text{if } |C| \geq (1 - \lambda)^2/(k(2 - \lambda)) \end{cases}
\]
Here equality is attained for the functions given by \( f'(z)(z/f(z))^{1+\lambda} = 1 - kz^2 \) and \( f'(z)(z/f(z))^{1+\lambda} = 1 - kz \), respectively.

For \( k = 1 \) we receive the following corollary.

**Corollary 2.** Let \( f \in \mathcal{A} \) and \( 0 < \lambda < 1 \). If
\[
|f'(z)(z/f(z))^{1+\lambda} - 1| < 1, \quad z \in \mathcal{U},
\]
then \(|a_2| \leq 1/(1 - \lambda)\) and for all \( \mu \in \mathbb{C} \) the following bound is sharp
\[
|a_3 - \mu a_2^2| \leq \frac{1}{(2 - \lambda)^2} \max \left\{ 1, \frac{2 - \lambda}{(1 - \lambda)^2} \left| \frac{1 + \lambda}{2} - \mu \right| \right\}.
\]

**References**


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