ON PARTIALLY PSEUDO SYMMETRIC $K$-CONTACT RIEMANNIAN MANIFOLDS

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Abstract. A Riemannian manifold $(M,g)$ is semi-symmetric if $(R(X,Y) \circ R)(U,V,W) = 0$. It is called pseudo-symmetric if $R \circ R = \mathcal{F}$, $\mathcal{F}$ being a given function of $X,...,W$ and $g$. It is called partially pseudo-symmetric if this last relation is fulfilled by not all values of $X,...,W$. Such manifolds were investigated by several mathematicians: I.Z. Szabó, S. Tanno, K. Nomizu, R. Deszcz and others.

In this paper we investigate $K$-contact Riemannian manifolds. In these manifolds the structure vector field $\xi$ plays a special role, and this motivates our interest in the partial pseudo-symmetry of these manifolds. We also investigate the case when $R \circ R$ is replaced by $R \circ S$ ($S$ being the Ricci tensor). We obtain conditions in order that our manifold be: (1) Sasakian or Sasakian of constant curvature 1 (in case of $R \circ R$); (2) an Einstein manifold (in case of $R \circ S$). – Our investigation is closely related to the results of S. Tanno.

1. Introduction

A Riemannian manifold $(M, g)$ is called locally symmetric if its curvature tensor $R$ is parallel: $\nabla R = 0$ ($\nabla$ denotes the Levi-Civita connection). As a proper generalization of locally symmetric manifolds the notion of semi symmetric manifolds was defined by

$$ (R(X,Y) \circ R)(U,V,W) = 0, \quad X,...,W \in \mathcal{X}(M) $$

(or shortly by $R \circ R = 0$) and studied by many authors, e.g. [6], [5], [7], [11], [13]). A complete intrinsic classification of these spaces was given by Z.I. Szabó [9].

It is interesting to investigate the semi-symmetry of special Riemannian manifolds. K. Nomizu proved [6] that if $M$ is a complete, connected hypersurface in a Euclidean space $\mathbb{R}^{n+1}$ ($n > 3$) and it satisfies $R \circ R = 0$, then it is locally symmetric. For the case of a compact Kähler manifold M. Ogawa [7] proved that if it is semi-symmetric, then it must be locally-symmetric, that is there exists no proper semi-symmetric compact Kähler manifold. In the case of contact manifolds S. Tanno [11]–[13] showed among others that there exists no proper semi-symmetric (or semi-Ricci-symmetric) $K$-contact manifold. Namely in $K$-contact manifolds $R \circ R = 0 \Rightarrow \nabla R = 0$, and $R \circ S = 0 \Rightarrow \nabla S = 0$, where $S$ denotes the Ricci curvature tensor.

R. Deszcz and others [2]–[4] weakened the notion of semi-symmetry (respectively semi-Ricci-symmetry) and introduced the notion of pseudo-symmetric spaces by

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requiring in place of \( R \circ R = 0 \) (respectively \( R \circ S = 0 \)) the relation

\[(P) \quad (R(X, Y) \circ R)(U, V, W) = f([\langle X \wedge Y \rangle \circ R](U, V, W))\]

only, where \( f \in C^\infty(M) \). – S. Tanno [11], [13], M. Okumura [8], T. Takahashi [10] and others investigated cases, where (1) is satisfied only by certain special vector fields, e.g. (I) \( Y = U = W = \xi \), (II) \( Y = U = \xi \) or (III) \( Y = \xi \). These spaces need not be semi-symmetric. We call them partially semi symmetric or partially pseudo symmetric, as the restrictions refer to (1), respectively (P).

In this paper we consider partially pseudo-symmetric \( K \)-contact manifolds, where (P) is satisfied with the restrictions (I), (II) or (III) only. Since here \( f \) need not be zero, these spaces are more general, than the corresponding partially semi-symmetric spaces. We want to show that even for these spaces – i.e. for spaces with the weaker condition of partial pseudo-symmetry – similar conclusions can be drawn as in the case of partial semi-symmetry. If

\[(N) \quad f + 1 \text{ never vanishes,}\]

then in case of (I) a partially pseudo-symmetric \( K \)-contact manifold turns out to be Sasakian (Theorem 1); in case of (II) it is moreover of constant curvature 1, and hence it is locally symmetric (Theorem 3). We also obtain that among the \( K \)-contact Riemannian manifolds exactly the partially pseudo-symmetric ones with restriction (I) and (N) are Sasakian (Theorem 2). In the last Section 4 we prove that a partially pseudo-Ricci-symmetric \( K \)-contact manifold with the restriction \( Y = V = \xi \) and with never vanishing function \( f - 1 \) is an Einstein manifold (Theorem 4), and we show that the relation \( R(X, \xi) \circ S = (X \wedge \xi) \circ S \) holds in every Sasakian manifold (Theorem 5).

2. Preliminaries

Let \((M, \varphi, \xi, \eta, g)\) be an \( n \) dimensional contact Riemannian manifold. It satisfies the relations

\[(2) \quad \eta(X) = \langle X, \xi \rangle, \quad \eta(\xi) = 1, \text{ where } \langle X, \xi \rangle \equiv g(X, Y)\]

(We list only those basic relations which will be used in the sequel.) A contact manifold is said to be \( K \)-contact, if \( \xi \) is a Killing vector field [11]. In this case we have [11], [1], [5]:

\[(3) \quad \nabla_X \xi = -\varphi_X\]
\[(4) \quad R(X, \xi)\xi = -X + \eta(X)\xi\]
\[(5) \quad \langle R(X, \xi)Y, \xi \rangle = -\langle X, Y \rangle + \eta(X)\eta(Y)\]
\[(6) \quad S(X, \xi) = 2n\eta(X).\]

If a \( K \)-contact manifold satisfies the relation

\[(7) \quad R(X, \xi)Y = -\langle X, Y \rangle \xi + \eta(Y)X,\]

then \( M \) is called a Sasakian manifold ([5], p. 273, Theorem 5.2).

The endomorphism \( X \wedge Y \) and the homeomorphisms \( R \circ R \) and \( (X \wedge Y) \circ R \) are defined by

\[(8) \quad (X \wedge Y)Z \equiv \langle Y, Z \rangle X - \langle X, Z \rangle Y\]
\[(9) \quad (R(X, Y) \circ R)(U, V, W) \equiv R(X, Y)(R(U, V)W - R(R(X, Y)U, V)W)
\[(10) \quad ((X \wedge Y) \circ R)(U, V, W) \equiv (X \wedge Y)(R(U, V)W) - R((X \wedge Y)U, V)W
- R(U, (X \wedge Y)V)W - R(U, V)((X \wedge Y)W).\]

(9) and (10) are kind of derivations.
3. Partially pseudo-symmetric $K$-contact manifolds

We want to investigate partially pseudo-symmetric $K$-contact manifolds $(M, \varphi, \xi, \eta, g)$ which satisfy (P) with certain restrictions.

A) Let at first the restriction be (I) $Y = U = W = \xi$. Then we get

$$R(X, \xi) \circ R(\xi, V, \xi) = f[(X \wedge \xi) \circ R(\xi, V, \xi)].$$

Such $K$-contact manifolds with vanishing $f$ were considered by S. Tanno [11]. He proved that such a partially semi-symmetric space is Sasakian. – Our condition is weaker than that of Tanno, for with us $f$ in (11) need not be zero (our space is partially pseudo-symmetric, and not partially semi-symmetric). We want to show that from this weaker condition we can draw similar conclusions.

**Theorem 1.** A partially pseudo-symmetric $K$-contact manifold, where (P) is restricted by (I) $Y = U = W = \xi$ and $f + 1$ never vanishes (i.e. condition (N) is satisfied) is a Sasakian manifold.

**Proof.** We denote the expression in the bracket on the right-hand side of (11) by $A$, and we calculate it. By (10)

$$A = (X \wedge \xi)(R(\xi, V)\xi) - R((X \wedge \xi)\xi, V)\xi - R(\xi, (X \wedge \xi)\xi - R(\xi, V)((X \wedge \xi)\xi).$$

Using for the individual terms of $A$ the relations of Section 2 we obtain

$$(X \wedge \xi)(R(\xi, V)\xi) \stackrel{(4)}{=} (X \wedge \xi)(V - \eta(\xi)\eta) \stackrel{(8)}{=} (\xi, V)X - (X, V)\xi - \eta(\xi)\xi - \eta(\xi)\eta(X)\xi;$$

$$R((X \wedge \xi)\xi, V)\xi \stackrel{(2),(8)}{=} R(X - \eta(X)\xi, V)\xi \stackrel{(4)}{=} R(X, V)\xi - \eta(X)\eta(\xi)\eta(\xi),$$

$$R(\xi, (X \wedge \xi)\xi \stackrel{(4)}{=} (X \wedge \xi)\xi - \eta((X \wedge \xi)\xi \stackrel{(8)}{=} \eta(\xi)X - (X, V)\xi - \eta(\xi)\xi \eta(\xi)\eta(\xi),$$

$$R(\xi, V)((X \wedge \xi)\xi \stackrel{(8)}{=} R(\xi, V)(X - \eta(X)\xi) \stackrel{(4)}{=} R(\xi, V)X - \eta(X)(V - \eta(V)\xi).$$

From these

$$A = -(R(X, V)\xi + R(\xi, V)X + \eta(V)X - 2\eta(X)V + (X, V)\xi).$$

However, after similar considerations, the left-hand side of (11) leads to the same expression with opposite sign (see also Tanno [11] (2.8)). Thus (11) reduces to

$$-A = fA.$$

With respect to our condition (N) this can hold only if $A = 0$. However $A = 0$ yields

$$(X \wedge \xi)\xi = \eta(X)V - (X, V)\xi,$$

and this means (see Tanno [11] p. 451 (2.11)) that $M$ is a Sasakian manifold. □

In consequence of (12) every $K$-contact Riemannian manifold with restriction (I) and with $f = -1$ is partially pseudo-symmetric. So we have proved the

**Proposition 1.** In every $K$-contact Riemannian manifold

$$R(X, \xi) \circ R = (\xi \wedge X) \circ R \quad \text{over} \quad (\xi, V, \xi).$$

Now we easily obtain
Theorem 2. Among the K-contact Riemannian manifolds exactly the partially pseudo-symmetric ones with restriction (I) $Y = U = W = \xi$ are Sasakian if in (P) $f + 1$ never vanishes.

Proof. If a K-contact Riemannian manifold $M$ is partially pseudo-symmetric with restriction (I) and (N), then according to our Theorem 1 $M$ is Sasakian.

Conversely, let $M$ be a Sasakian manifold. Tanno deduces (13) from $A = 0$ by making use of (5) ([11], p. 451). Now (13) holds, since $M$ is Sasakian. Also (5) is true, since any Sasakian manifold is K-contact. Then one can invert this consideration, and from (13) obtain $A = 0$. Thus (12) is true also with (N), and this, by repeated application of (2), (4) and (8), leads to (11), i.e. a Sasakian manifold is partially pseudo-symmetric with restriction (I) and (N). □

B) Let now the restriction on (P) be (II) $Y = U = \xi$. Then we get

$$
R(X, \xi) \circ R(\xi, V, W) = f[(X \wedge \xi) \circ R(\xi, V, W)].
$$

Such K-contact manifolds with vanishing $f$ were considered by S. Tanno. He proved ([11], Theorem 2.3 or [13]) that for a K-contact manifold $M$ the following four conditions are equivalent (i) $M$ is of constant curvature 1; (ii) $\nabla R = 0$; (iii) $R(X, Y) \circ R = 0 \forall X, Y$; (iv) $R(X, \xi) \circ R = 0$. His proof runs as follows (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is clear. Then he assumes (iv): $(R(X, \xi) \circ R)(U, V, W) = 0$ and proves (i). At this step, however he uses $(R(X, \xi) \circ R)(\xi, V, W) = 0$ only, which is (14) with $f = 0$. We want to prove that also from our somewhat weaker condition (14), where $f$ is not necessarily zero, one can draw a similar conclusion.

Theorem 3. A partially pseudo-symmetric K-contact manifold, where (P) is restricted by (II): $Y = U = \xi$, and $f + 1$ never vanishes, is a locally symmetric Sasakian manifold with constant curvature 1.

Proof. Of course the broader is the restriction on the vector fields in (P), the wider is the family of spaces satisfying the restricted (P). Thus the family of the K-contact manifolds satisfying (14) is a part of those satisfying (12). Thus (14) $\Rightarrow$ (11).

Suppose (N). Then, according to our Theorem 1, from (11) follows (13), which means that $M$ is Sasakian.

Let us denote the expression in the bracket on the right-hand side of (14) by $C$ and calculate it. By (10)

$$
C = (X \wedge \xi)(R(\xi, V)W) - R((X \wedge \xi)\xi, V)W - R(\xi, (X \wedge \xi)V)W

- R(\xi, V)((X \wedge \xi)W).
$$

Using for the individual terms of $C$ the relations of Section 2 and (13) we obtain

$$
(X \wedge \xi)(R(\xi, V)W) \overset{(13)}{=} (X \wedge \xi)(\eta(W)V - \langle V, W \rangle \xi) \overset{(8)}{=} \\
= \eta(W)\eta(V)X - \eta(W)\langle X, V \rangle \xi - \langle V, W \rangle X + \langle V, W \rangle \eta(X)\xi,
$$

$$
R((X \wedge \xi)\xi, V)W \overset{(8)}{=} R(X, \xi)W - \eta(X)R(\xi, V)W \overset{(13)}{=} \\
= R(X, V)W - \eta(X)\eta(W)V - \langle V, W \rangle \xi, 
$$

$$
R(\xi, (X \wedge \xi)V)W \overset{(13)}{=} \eta(W)((X \wedge \xi)V) - \langle (X \wedge \xi)V, W \rangle \xi \overset{(8)}{=} \\
= \eta(W)\eta(V)X - \eta(W)\langle X, V \rangle \xi - \eta(V)X - \langle X, V \rangle \xi, W \xi,
$$

$$
R(\xi, V)((X \wedge \xi)W) \overset{(13)}{=} R(\xi, V)(\eta(W)V - \langle X, V \rangle \xi) \overset{(13)}{=} \\
= \eta(W)(\eta(X)V - \langle X, V \rangle \xi) - \langle X, W \rangle (V - \eta(V)\xi).
$$

From these

$$
C = -R(X, V)W - \langle V, W \rangle X + \langle X, W \rangle V.
$$
However the left-hand side of (14) gives, after similar calculations, the same expression with an opposite sign. Thus (14) reduces to
\[-C = f(p)C, \quad \forall p \in M.\]
Since \( f(p) = -1 \) is now excluded, \( C = 0 \), i.e.
\[R(X,V)W = (X,W)V - (V,W)X,\]
which means that \( M \) is of constant curvature 1.

Finally the mentioned theorem of Tanno gives that \( \nabla R = 0 \). i.e. \( M \) is locally symmetric. \( \square \)

Concerning the Sasakian manifolds of constant curvature 1, with the help of the previous results and considerations we easily obtain the following

**Proposition 2.** In every Sasakian manifold of constant curvature 1

(15) \[\((\xi \wedge X) \circ R)(\xi, V, \xi) \equiv 0\]

*Proof.* Every Sasakian manifold \( M \) is \( K \)-contact. Hence, by our Proposition 1

\[(R(X,\xi) \circ R)(\xi, V, \xi) = ((\xi \wedge X) \circ R)(\xi, V, \xi).\]

However if \( M \) is of constant curvature 1, then \( R(X,\xi) \circ R = 0 \) according to (iv) of Tanno. So we obtain (15). \( \square \)

4. **Partially pseudo-Ricci-symmetric \( K \)-contact manifolds**

A \( K \)-contact manifold \( (M, \varphi, \eta, \xi, g) \) is said to be *pseudo-Ricci-symmetric* if it satisfies

(PR) \[(R(X,Y) \circ S)(U,V) = f((X \wedge Y) \circ S)(U,V),\]

where \((R(X,Y) \circ S)(U,V) \equiv - S(R(X,Y)U,V) - S(U,R(X,Y)V)\]

and \((X \wedge Y) \circ S)(U,V) \equiv - S((X \wedge Y)U,V) - S(U,(X \wedge Y)V).\]

Thus (PR) has the following more developed form

\begin{equation}
S(R(X,Y)U,V) + S(U,R(X,Y)V)
\end{equation}

\[= f[S((X \wedge Y)U,V) + S(U,(X \wedge Y)V)].\]

We want to investigate partially pseudo-Ricci-symmetric \( K \)-contact manifolds which satisfy (PR) with the restriction \( Y = V = \xi \). So we have

\[S(R(X,\xi)U,\xi) + S(U,R(X,\xi)\xi) = f[S((X \wedge \xi)U,\xi) + S(U,(X \wedge \xi)\xi)].\]

Applying (4), (6) and (8) we obtain

\[2n\eta(R(X,\xi)U) + S(U,X) - \eta(X)S(U,\xi)\]

\[= f[\eta(U)S(X,\xi) - (X, U)S(\xi, \xi) + S(U, X) - \eta(X)S(U, \xi)].\]

The first term of (17) becomes by (2)

\[2n(R(X,\xi)U,\xi) \xrightarrow{\eta} 2n(-(X,U) + \eta(X)\eta(U)).\]

Applying again (6) in four terms of (7), after a reduction we obtain

\[S(U,X) - 2n(X,U) = f[S(U,X) - 2n(X,U)].\]

This can hold only if either: (a) \( f = 1 \) or (b) \( S(U,X) = 2ng(X,U) \) (i.e \( S = 2ng \)).

However (b) means that \( M \) is an Einstein manifold. Thus we have proved
Theorem 4. A partially pseudo-Ricci-symmetric $K$-contact manifold with the restriction $Y = V = \xi$ and with never vanishing function $f - 1$ is an Einstein manifold.

We still want to prove the following

Theorem 5. The relation

(18) $\left( R(X, \xi) \circ S \right)(U, V) = \left( (X \wedge \xi) \circ S \right)(U, V)$

is satisfied on every Sasakian manifold $M$.

This means that every Sasakian manifold is partially pseudo-Ricci-symmetric with the restriction $Y = \xi$ and $f = 1$.

Proof. Putting $Y = \xi$ and $f = 1$ in (16), we obtain a more developed form of (18). Since $M$ is Sasakian we have (13). Applying this, (6) and (8), a direct computation, similar to the previous ones shows that the sides of (18) are identical. □

This also shows that there exist $K$-contact manifolds satisfying $R(X, \xi) \circ S = (X \wedge \xi) \circ S$.

References


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