ON THE CHERN–WEIL HOMOMORPHISM IN FINSLER SPACES

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Dedicated to Professor Árpád Varecza on the occasion of his 60th birthday

Abstract. The aim of this paper is to devise a Chern–Weil-type construction for a Finsler manifold \((M, L)\) which is determined only by the manifold \(M\) and by the Finslerian fundamental function \(L\).

1. Introduction

The focus in this paper is to set up a framework in which the famous Chern–Weil homomorphism can be formulated on a Finsler manifold. Most of the basic notations in this paper are the same as in [GHV73]. Background information on Finsler geometry can be found e.g. in [Mat86] and [AP94].

Let \((M, L)\) be a Finsler space, the horizontal projection determined by the Finslerian fundamental function \(L\) is \(h\). \(h\) can be interpreted as a \(\tau_{TM}\)-valued 1-form on \(TM\): \(h \in \Lambda^1(TM; \tau_{TM}) \cong \text{Hom}(\tau_{TM}; \tau_{TM})\). The horizontal subbundle of \(\tau_{TM}\) will be denoted by \(HM\), \(\text{Sec}HM = X_h(TM)\).

Denote by \((\Lambda(TM), \wedge)\) the graded algebra of differential forms on \(TM\). From \(h\) one can derive a first order graded derivative \(dh: \Lambda(TM) \rightarrow \Lambda(TM)\)

\[
d_h \omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i h^{X_i} \omega(X_0, \ldots, \hat{X}_i, \ldots, X_p) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_h, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)
\]

where \(\omega \in \Lambda^p(TM) (p \geq 1)\) is a \(p\)-form, \(X_i \in \mathfrak{x}(TM) (i = 0 \ldots p)\), \([X, Y]_h = [hX, Y] + [X, hY] - h[X, Y]\), furthermore \(d_h f(X) = (hX)f\) \((f \in \Lambda^0(TM) \equiv C^\infty(TM))\) ([FN56] or [Mic87]). It is easy to see that \(d_h^2 = 0\) iff the Frölicher–Nijenhuis bracket of the operator pair \((h, h)\) is zero: \([h, h] = 0\). In the Finslerian case this condition means that the torsion \(R^1\) of the unique Cartan connection vanishes, i.e. the horizontal distribution is integrable.

In the Finslerian case this special situation was studied in [ACD87] and their main result is the following:

Theorem. If \(R^1 = 0\) then the cohomology groups of \(d_h\) are isomorphic to the de Rham cohomology groups of an integral manifold of the nonlinear connection associated to \(L\).

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In this paper we do not suppose the integrability of the horizontal distribution.

2. Tools

**Forms.** Let $\nabla^C : \mathfrak{X}(T^*M) \times \mathfrak{X}_h T^*M \to \mathfrak{X}_h T^*M$ be the Cartan connection of the Finsler space $(M, \mathcal{L})$. Then $(\nabla, h)$, where $\nabla : \mathfrak{X}(T^*M) \times \mathfrak{X}_h T^*M \to \mathfrak{X}_h T^*M$, $\nabla X = \nabla^C_X Y$, is the so-called $h$-connection of the Finsler space.

By easy calculations, one can show the following statement:

**Proposition 1.** Let $(\nabla, h)$ be the $h$-connection of the Finsler space. The map

$$\nabla : A(TM; H^*) \to A(TM; H^*)$$

where

$$(\nabla \Psi)(X_0, \ldots, X_p) = \sum_{i=0}^p (-1)^i\nabla X_i \Psi(X_0, \ldots, \hat{X}_i, \ldots, X_p) +
+ \sum_{i<j} (-1)^{i+j} \Psi([X_i, X_j]h, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)$$

$(\Psi \in A^p(TM; H^*)) (p > 0)$; for $p = 0$: $(\nabla \sigma)(X) = \nabla X \sigma$ is a first order graded derivation of the graded algebra of $H^*$-valued forms on $TM$ in the sense of [Mic87].

We will use the following construction in the next section. Let $\xi_0, \xi_1, \ldots, \xi_m$ be vector bundles with the same base $B$ and let $\phi \in \text{Hom}(\xi_1, \ldots, \xi_m; \xi_0)$. $\phi$ determines a map $\phi_* \in \text{Hom}(A(B; \xi_1), \ldots, A(B; \xi_m); A(B; \xi_0))$ as follows.

$$\phi_*(\sigma_1, \ldots, \sigma_m) = \phi(\sigma_1, \ldots, \sigma_m)$$

for $\sigma_i \in A^0(B; \xi_i) \equiv \text{Sec} \xi_i$, and for elements in $A^p(B; \xi_i)$ this map is determined by

$$\phi_* (\omega_1 \wedge \sigma_1, \ldots, \omega_m \wedge \sigma_m) = (\omega_1 \wedge \ldots \wedge \omega_m) \wedge \phi_*(\sigma_1, \ldots, \sigma_m)$$

where $\omega_i \in A(B)$, $\sigma_i \in A^0(B; \xi_i)$ $(i = 1 \ldots m)$. $\phi_*$ satisfies the following identity:

$$(1) \quad \phi_*(\psi_1, \ldots, \omega \wedge \psi_1, \ldots, \psi_m) = (-1)^q \omega \wedge \phi_*(\psi_1, \ldots, \psi_m)$$

where $\psi_i \in A^q(B; \xi_i)$, $q_i = p_1 + \ldots + p_{i-1}$ $(i \geq 2)$, $q_1 = 0$, $\omega \in A^q(B)$, and moreover,

$$(2) \quad \phi_*(\psi_1, \ldots, \psi_m)(X_1, \ldots, X_p) =
= \frac{1}{p_1! \ldots p_m!} \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) \phi(\psi_1(X_{\sigma(1)}), \ldots, \psi_m(X_{\sigma(p)}))$$

where $\psi_i \in A^q(B; \xi_i)$, $X_i \in \mathfrak{X}(B)$ $(i = 1 \ldots p)$, $p = p_1 + \ldots + p_m$.

**Invariant polynomials.** Let $F$ be a real vector space. An *invariant polynomial of degree $p$* is a symmetric map

$$f^p \in \text{Hom}(L(F; F), \ldots, L(F; F); \mathbb{R})$$

such that for all $a \in \text{GL}(F)$

$$(3) \quad f^p(\text{Ad}(a)\alpha_1, \ldots, \text{Ad}(a)\alpha_p) = f^p(\alpha_1, \ldots, \alpha_p)$$

where $\alpha_i \in L(F; F)$ $(i = 1 \ldots p)$ is a linear operator and $\text{Ad} : \text{GL}(F) \to \text{GL}(L(F; F))$ is the adjoint representation. By the invariance condition (3) one can extend $f^p$ to the bundle of linear operators over the vector bundle $\xi$ with base manifold $B$ and the typical fiber $F$.

$$f^p \in \text{Hom}(L\xi, \ldots, L\xi; B \times \mathbb{R}) \cong \text{Sec}(L\xi \otimes \cdots \otimes L\xi)^*.$$ 

This $f^p$ is called invariant polynomial in $\xi$ of degree $p$. 


Curvature. Let $R^2 \in \Lambda^2(TM;L_{HM})$ denote the curvature of the Cartan connection.

Proposition 2. The $h$-Finsler connection $(\nabla,h)$ in $HM$ satisfies:

$$[(\nabla R^2)(X,Y,Z)](W) = \mathfrak{S}_{(X,Y,Z)} \{ P^2(R^1(X,Y)(hZ)(W)) \}$$

where $P^2$ is the $h$-curvature, $R^1 = \frac{1}{2}[h,h]$ and $\mathfrak{S}_{(X,Y,Z)}$ is the symbol of the cyclic sum with respect to $X,Y,Z$.

3. Construction of $d h$-closed forms

Theorem. Let $(\nabla,h)$ be the $h$-Finsler connection, $f^p$ an invariant polynomial in $HM$. If $\nabla R^2 = 0$ then $d_h f^p(R^2,\ldots,R^2) = 0$, i.e. $f^p(R^2,\ldots,R^2)$ is a $d_h$-closed 2p-form.

Proof. We have found the adequate ideas, so the proof of the theorem is quite easy. First we prove the following statement:

**Lemma.** Let $f \in \text{Hom}(L_{HM},\ldots,L_{HM}; TM \times \mathbb{R}) \cong \text{Sec}(L_{HM} \otimes \cdots \otimes L_{HM})^*$. If $\nabla_X f = 0$ for any $X \in \mathfrak{X}(TM)$ then

$$d_h f(\Omega_1,\ldots,\Omega_p) = \sum_{i=1}^{p} (-1)^{q_i} f_s(\Omega_1,\ldots,\nabla\Omega_i,\ldots,\Omega_p),$$

where $\Omega_i \in \Lambda^r(TM;L_{HM})$ ($i = 1 \ldots p$), $q_i = r_1 + \cdots + r_{i-1}$ ($i = 2 \ldots p$), $q_1 = 0$. (Concerning $f_s \in \text{Hom}(A(TM;L_{HM}),\ldots,A(TM;L_{HM}); A(TM))$ see (2)!)

Clearly, $\Lambda^r(TM;L_{HM}) \cong \Lambda^r(TM) \otimes \text{Sec}_{L_{HM}}$. If $\alpha_i \in \text{Sec}_{L_{HM}}$ ($i = 1 \ldots p$) then (4) reduces to:

$$d_h f(\alpha_1,\ldots,\alpha_p) = \sum_{i=1}^{p} f_s(\alpha_1,\ldots,\nabla\alpha_i,\ldots,\alpha_p).$$

We have $(d_h f(\alpha_1,\ldots,\alpha_p))(X) = hX f(\alpha_1,\ldots,\alpha_p)$. On the other hand,

$$\sum_{i=1}^{p} f_s(\alpha_1,\ldots,\nabla\alpha_i,\ldots,\alpha_p)(X) \overset{(2)}{=} \sum_{i=1}^{p} f(\alpha_1,\ldots,\nabla(\alpha_i))(X),\ldots,\alpha_p).$$

Together with the previous line this proves the statement.

Let $\Omega_i \in \Lambda^r(TM;L_{HM})$ ($i = 1 \ldots p$), $\Omega_i = \omega_i \wedge \alpha_i$ ($\omega_i \in \Lambda^r(TM)$ $i = 1 \ldots p$), and $q = r_1 + \cdots + r_p$. By induction we infer

$$d_h f_s(\omega_1 \wedge \alpha_1,\ldots,\omega_p \wedge \alpha_p) \overset{(1)}{=}$$

$$= \sum_{i=1}^{p} (-1)^{q_i} \omega_1 \wedge \ldots \wedge d_h \omega_i \wedge \ldots \wedge \omega_p \wedge f_s(\alpha_1,\ldots,\alpha_p) +$$

$$+ (-1)^q \omega_1 \wedge \ldots \wedge \omega_p \wedge d_h f_s(\alpha_1,\ldots,\alpha_p).$$

Similarly,

$$f_s(\omega_1 \wedge \alpha_1,\ldots,\nabla(\omega_1 \wedge \alpha_i),\ldots,\omega_p \wedge \alpha_p) =$$

$$= \omega_1 \wedge \ldots \wedge d_h \omega_1 \wedge \ldots \wedge \omega_p \wedge f_s(\alpha_1,\ldots,\alpha_p) +$$

$$+ (-1)^r (-1)^{r+1} \ldots (-1)^p \omega_1 \wedge \ldots \wedge \omega_p \wedge f_s(\alpha_1,\ldots,\nabla\alpha_i,\ldots,\alpha_p).$$

We proved the lemma.

Now, for an invariant polynomial $f^p$, $\nabla_X f^p = \nabla_{hX} f^p = 0$ and applying the lemma for $\Omega_i = R^2$ we get the statement of the theorem. □
4. Remarks

Pseudocomplexes. For $d_h$ we have a sequence of graded vector spaces

$$(PS) \cdots \longrightarrow \Lambda^{p-1}(TM) \xrightarrow{d_h} \Lambda^{p}(TM) \xrightarrow{d_h} \Lambda^{p+1}(TM) \longrightarrow \cdots$$

where $d_h \circ d_h$ is not necessarily zero. Following I. Vaisman [Vai68], for $(PS)$ we use the name of pseudocomplex. Of course, when $[h, h] = 0$ then $(PS)$ is a usual cochain complex.

In the case of non-vanishing $d^2_h$ the most natural way to define cohomology groups is by

$$H^p(d_h, TM) = \frac{\text{Ker} d_h}{\text{Im} d_h \cap \text{Ker} d_h}.$$  

These $H^p(d_h, TM)$ cohomology groups are usual cohomology groups of several cochain complexes. We put

$$(\tilde{PS}) \cdots \longrightarrow \Lambda^{p-1}(TM) \xrightarrow{\tilde{d}_h} \Lambda^{p}(TM) \xrightarrow{\tilde{d}_h} \Lambda^{p+1}(TM) \longrightarrow \cdots$$

where

$$\Lambda^{p}(TM) = \text{Ker} d_h \circ d_h.$$  

and $\tilde{d}_h$ is the restriction of $d_h$ to $\Lambda^{p}(TM)$. Then it is easy to check that in the case of $(\tilde{PS}) d^2_h = 0$ holds and the cochain complex $(\tilde{PS})$ has the same cocycles and coboundaries as the pseudocomplex $(PS)$ itself ([HL75], [Vai93]).

Finsler spaces with the condition $\nabla R^2 = 0$. There are several examples for Finsler spaces with vanishing curvature $R^2$. This condition implies the required identity $\nabla R^2 = 0$, c.f. Proposition 2. These spaces are the so called Landsberg spaces ([Koz96], [Mat96]).

References


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