ON THE GENERALIZED CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper a general theorem concerning the \( \psi - |C, \alpha; \delta|_k \) summability factors of infinite series has been proved.

1. Introduction. A sequence \((w_n)\) of positive numbers is said to be \( \delta\)-quasi monotone, if \( w_n \to 0, w_n > 0 \) ultimately and \( \Delta w_n \geq -\delta_n \), where \((\delta_n)\) is a sequence of positive numbers (see [1]). Let \( \sum a_n \) be a given infinite series with partial sums \((s_n)\). We define \( A_n^\alpha \) by identity

\[
\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}.
\]

The sequence-to-sequence transformations given by

\[
u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{-1} s_v,
\]

\[
t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{-1} va_v,
\]

define the \((C, \alpha)\) means of the sequences \((s_n)\) and \((na_n)\), respectively.

The series \( \sum a_n \) is said to be summable \( |C, \alpha|_k \), \( k \geq 1 \) and \( \alpha > -1 \), if (see [3])

\[
\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.
\]

If we take \( \alpha = 1 \), then \( |C, \alpha|_k \) summability is the same as \( |C, 1|_k \) summability. Let \((\psi_n)\) be a sequence of positive real numbers. We say that the series \( \sum a_n \) is said to be summable \( \psi - |C, \alpha; \delta|_k \), \( k \geq 1 \), \( \alpha > -1 \) and \( \delta \geq 0 \), if

\[
\sum_{n=1}^{\infty} \psi^{\delta k+k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.
\]

But since \( t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha) \) (see [4]) condition (5) can also be written as

\[
\sum_{n=1}^{\infty} \psi^{\delta k+k-1} n^{-k} |t_n^\alpha|^k < \infty.
\]

If we take \( \delta = 0 \) and \( \psi_n = n \) (resp. \( \delta = 1 \) and \( \psi_n = n \)), then \( \psi - |C, \alpha; \delta|_k \) summability is the same as \( |C, \alpha|_k \) (resp. \( |C, 1|_k \)) summability.

Remark. Since \((\psi_n)\) is a sequence of positive real numbers the summability
method $\psi - | C, \alpha; \delta |_k$ is a new method and general than the $| C, \alpha; \delta |_k$ summability method. On the other hand $| C, \alpha; \delta |_k$ and $\psi - | C, \alpha; \delta |_k$ summability methods are different from each other. That is they have got different summability fields.

Therefore, we take the sequence $(\psi_n)$ instead of $n$.

2. The following theorem is known.

**Theorem A** ([2]). Let $t^n_\alpha$ be the n-th Cesáro mean of order $\alpha$, with $\alpha \geq 1$, of the sequence $(na_n)$ such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ such that it is $\delta$-quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $| \Delta \lambda_n | \leq | B_n |$ for all $n$.

\[ (7) \quad \sum_{n=1}^{m} | \Delta (n^\alpha) | | B_{n+1} | \log n = O(1), \]

\[ (8) \quad \sum_{n=1}^{m} \frac{1}{n^\alpha} | t^n_\alpha |^k = O(\log m) \text{ as } m \to \infty, \]

then the series $\sum a_n \lambda_n$ is summable $| C, \alpha |_k, k \geq 1$.

3. The aim of this paper is to generalize Theorem A in the following form.

**Theorem.** Let $k \geq 1$ and $\delta \geq 0$. Let $t^n_\alpha$ be the n-th Cesáro mean of order $\alpha$, with $\alpha \geq 1$, of the sequence $(na_n)$ such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ such that it is $\delta$-quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$, $\sum B_n \log n$ is convergent, $| \Delta \lambda_n | \leq | B_n |$ for all $n$ and that condition (7) of Theorem A is satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{-k} \psi_n^{\delta k + k - 1})$ is non-increasing and

\[ (9) \quad \sum_{n=1}^{m} \psi_n^{\delta k + k - 1} n^{-k} | t^n_\alpha |^k = O(\log m) \text{ as } m \to \infty, \]

then the series $\sum a_n \lambda_n$ is summable $\psi - | C, \alpha; \delta |_k$.

If we take $\delta = 0$, $\epsilon = 1$ and $\psi_n = n$ in this theorem, then we get Theorem A.

4. We need the following lemmas for the proof of our theorem.

**Lemma 1** ([5]). If $\sigma > \delta > 0$, then

\[ (10) \quad \sum_{n=v+1}^{m} \frac{A_n^{\delta - \sigma}}{A_n^{\sigma}} = \sum_{n=v+1}^{m} \frac{(n-v)^{\delta - 1}}{n^\sigma} = O(\psi^{\delta - \sigma}) \text{ as } m \to \infty. \]

**Lemma 2** ([2]). Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ which is $\delta$-quasi monotone with $\sum B_n \log n$ is convergent and $| \Delta \lambda_n | \leq | B_n |$ for all $n$, then

\[ (11) \quad | \lambda_n | \log n = O(1) \text{ as } n \to \infty. \]

**Lemma 3** ([2]). Let $\alpha \geq 1$. If $(B_n)$ is $\delta$-quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$ and $\sum B_n \log n$ is convergent, then

\[ (12) \quad m^\alpha B_m \log m = O(1) \text{ as } m \to \infty, \]

\[ (13) \quad \sum_{n=1}^{\infty} n^\alpha | \Delta B_n | \log n < \infty. \]

**Lemma 4** ([2]). Let $t^n_\alpha$ be the n-th Cesáro mean of order $\alpha$, with $\alpha \geq 1$, of the sequence $(na_n)$ such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$. If $n \geq v$, then

\[ (14) \quad | \sum_{p=1}^{v} A_{n-p \alpha}^{-1} t_{n-p \alpha} | \leq A_{n-v \alpha}^{-1} A_v^{\alpha} | t_v |. \]
5. Proof of the Theorem. Let \((T_n^\alpha)\) be the \(n\)-th \((C, \alpha)\), with \(\alpha \geq 1\), means of the sequence \((na_n\lambda_n)\). Then, by (3), we have

\[
T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{15}
\]

Using Abel’s transformation, we get

\[
T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p + \lambda_n \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v
\]

\[
= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p + \lambda_n T_n^\alpha
\]

\[
= T_{n,1}^\alpha + T_{n,2}^\alpha. \tag{16}
\]

Since

\[
|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k (|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k),
\]

to complete the proof of the theorem, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \psi^{\alpha k + k - n} \rho - n^{-k} |T_{n,r}^\alpha|^k < \infty \text{ for } r = 1, 2, \text{ by (6).}
\]

Firstly, when \(k > 1\), using Lemma 4 and after applying Hölder’s inequality with indices \(k\) and \(k'\), where \(\frac{1}{k} + \frac{1}{k'} = 1\), we get that

\[
\sum_{n=2}^{m+1} \psi^{\alpha k + k - n} \rho - n^{-k} |T_{n,1}^\alpha|^k = \sum_{n=2}^{m+1} \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

\[
\leq \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{v=1}^{n-1} \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \sum_{v=1}^{n-1} \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \sum_{v=1}^{n-1} \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

\[
\times \left\{ \sum_{v=1}^{n-1} A_{n-v}^{\alpha-1} \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \psi^{\alpha k} |B_v| |C_v|^k \sum_{n=2}^{m+1} \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

\[
= O(1) \sum_{v=1}^{m} \psi^{\alpha k} |B_v| |C_v|^k \sum_{n=2}^{m+1} \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

\[
= O(1) \sum_{v=1}^{m} \psi^{\alpha k} |B_v| |C_v|^k \sum_{n=2}^{m+1} \psi^{\alpha k + k - n} \rho - n^{-k} \left( \sum_{p=1}^{v} A_{n-v}^{\alpha-1} p a_p \right)^k
\]

by Lemma 1. Thus

\[
\sum_{n=2}^{m+1} \psi^{\alpha k + k - n} \rho - n^{-k} |T_{n,1}^\alpha|^k = O(1) \sum_{v=1}^{m-1} \Delta \psi^{\alpha k} |B_v| |C_v|^k \sum_{p=1}^{v} \psi^{\alpha k + k - p} \rho - p^{-k} |C_p|^k
\]
+ O(1)m^\alpha \left| B_m \left( \sum_{v=1}^{m} \psi_{k+1}^\alpha - v^{\alpha} \right) \right| |k^k

= O(1) \sum_{v=1}^{m-1} \Delta(v^\alpha | B_v) \log v + O(1)m^\alpha \left| B_m \right| \log m

= O(1) \sum_{v=1}^{m-1} v^\alpha | \Delta B_v | \log v + O(1) \sum_{v=1}^{m-1} \Delta(v^\alpha) \left| B_v+1 \right| \log v

+ O(1)m^\alpha \left| B_m \right| \log m = O(1) as m \to \infty,

by virtue of the hypotheses of the Theorem and Lemma 3.

Again, since \mid \lambda_n \mid = O(1), we have that

\sum_{n=1}^{m} \psi_{n+1}^\delta n^{-k} \left| T_{n,2}^\alpha \right| k

= \sum_{n=1}^{m} \psi_{n+1}^\delta n^{-k} \mid \lambda_n \mid \mid \lambda_n \mid \mid t_n^\alpha \mid k

= O(1) \sum_{n=1}^{m} \psi_{n+1}^\delta n^{-k} \mid \lambda_n \mid \mid t_n^\alpha \mid k

= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{p=1}^{n} \psi_{p+1}^\delta p^{-k} \mid t_p^\alpha \mid k

+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \psi_{n+1}^\delta n^{-k} \mid t_n^\alpha \mid k

= O(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid \log n + O(1) \mid \lambda_m \mid \log m

= O(1) \sum_{n=1}^{m-1} \mid B_n \mid \log n + O(1) \mid \lambda_m \mid \log m

= O(1) as m \to \infty,

by virtue of the hypotheses of the Theorem and Lemma 2.

Therefore, we get (16). This completes the proof of the Theorem.

References


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