SEMINORM GENERATING RELATIONS AND THEIR MINKOWSKI FUNCTIONALS

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Abstract. We show that instead of the Minkowski functionals of absorbing, balanced, convex subsets of a vector space $X$ it is more convenient to consider first the Minkowski functionals of balanced valued linear relations of $\mathbb{R}_+$ onto $X$.

Introduction

A relation $F$ of the set $\mathbb{R}_+$ of all positive numbers onto a vector space $X$ over $K = \mathbb{R}$ or $\mathbb{C}$ will be called a seminorm generating relation for $X$ if

1. $F(r) + F(s) \subseteq F(r+s)$ for all $r, s \in \mathbb{R}_+$;
2. $\lambda F(r) \subseteq F(tr)$ for all $\lambda \in K$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$.

This definition is mainly motivated by the fact that if $A$ is an absorbing, balanced, convex subset of $X$ and $F_A$ is a relation on $\mathbb{R}_+$ to $X$ such that

$$F_A(r) = rA$$

for all $r \in \mathbb{R}_+$, then $F_A$ is a seminorm generating relation for $X$.

Moreover, if $p$ is a seminorm on $X$ and $F_p$ and $\tilde{F}_p$ are relations on $\mathbb{R}_+$ to $X$ such that

$$F_p(r) = B_{r}^{p}(0) \quad \text{and} \quad \tilde{F}_p(r) = \tilde{B}_{r}^{p}(0)$$

for all $r \in \mathbb{R}_+$, then $F_p$ and $\tilde{F}_p$ are also seminorm generating relations for $X$.

If $F$ is a seminorm generating relation for $X$, then the function $p_F$ defined by

$$p_F(x) = \inf \left( F^{-1}(x) \right)$$

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for all \( x \in X \), will be called the Minkowski functional of \( F \). Namely, if \( A \) is an absorbing, balanced, convex subset of \( X \), then \( p_A = p_{rA} \) is just the usual Minkowski functional of \( A \).

After establishing some easy consequences of the definition of seminorm generating relations, we shall only prove the following basic algebraic properties of the Minkowski functionals.

**Theorem 1.** If \( F \) is a seminorm generating relation for \( X \), then \( p_F \) is a seminorm on \( X \) such that \( F_{p_F} \subset F \subset \overline{F}_{p_F} \).

**Corollary 1.** If \( A \) is an absorbing, balanced, convex subset of \( X \), then \( p_A \) is a seminorm on \( X \) such that \( B^{p_A}_1(0) \subset A \subset \overline{B}^{p_A}_1(0) \).

**Theorem 2.** If \( p \) is a seminorm on \( X \) and \( F \) is a seminorm generating relation for \( X \) such that \( F_p \subset F \subset \overline{F}_p \), then \( p = p_F \).

**Corollary 2.** If \( p \) is a seminorm on \( X \) and \( A \) is an absorbing, balanced, convex subset of \( X \) such that \( B^p_1(0) \subset A \subset \overline{B}^p_1(0) \), then \( p = p_A \).

**Theorem 3.** If \( F \) is a seminorm generating relation for \( X \), then \( p_F \) is a norm if and only if \( \bigcap_{r \in \mathbb{R}_+} F(r) = \{0\} \).

**Corollary 3.** If \( A \) is an absorbing, balanced, convex subset of \( X \), then \( p_A \) is a norm on \( X \) if and only if \( \bigcap_{r \in \mathbb{R}_+} rA = \{0\} \).

**Theorem 4.** If \( F \) is a seminorm generating relation for \( X \), then \( F = F_{p_F} \) if and only if \( F(r) = \bigcup_{s < r} F(s) \) for all \( r \in \mathbb{R}_+ \).

**Corollary 4.** If \( A \) is an absorbing, balanced, convex subset of \( X \), then \( A = B^{p_A}_1(0) \) if and only if \( A = \bigcup_{s < r} sA \).

**Theorem 5.** If \( F \) is a seminorm generating relation for \( X \), then \( F = \overline{F}_{p_F} \) if and only if \( F(r) = \bigcap_{s > r} F(s) \) for all \( r \in \mathbb{R}_+ \).

**Corollary 5.** If \( A \) is an absorbing, balanced, convex subset of \( X \), then \( A = \overline{B}^{p_A}_1(0) \) if and only if \( A = \bigcap_{s > r} sA \).

The topological properties of seminorm generating relations and their Minkowski functionals will be investigated elsewhere.

1. **Prerequisites**

A subset \( F \) of a product set \( X \times Y \) is called a relation on \( X \) to \( Y \). If in particular \( X = Y \), then we simply say that \( F \) is a relation on \( X \). Note that if \( F \) is a relation on \( X \) to \( Y \), then \( F \) is also a relation on \( X \cup Y \).

If \( F \) is a relation on \( X \) to \( Y \), and moreover \( x \in X \) and \( A \subset X \), then the sets \( F(x) = \{ y \in X: (x, y) \in F \} \) and \( F[A] = \bigcup_{x \in A} F(x) \) are called the images of \( x \) and \( A \) under \( F \), respectively.

If \( F \) is a relation on \( X \) to \( Y \), then the sets \( D_F = \{ x \in X : F(x) \neq \emptyset \} \) and \( R_F = F[D_F] \) are called the domain and range of \( F \), respectively. If in
particular $X = D_F$ (and $Y = R_F$), then we say that $F$ is a relation of $X$ into (onto) $Y$.

A relation $F$ on $X$ to $Y$ is said to be a function if for each $x \in D_F$ there exists a unique $y \in Y$ such that $y \in F(x)$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If $F$ is a relation on $X$ to $Y$, then values $F(x)$, where $x \in X$, uniquely determine $F$ since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse relation $F^{-1}$ of $F$ can be defined such that $F^{-1}(x) = \{y \in Y : x \in F(y)\}$ for all $x \in X$.

Throughout in the sequel, $X$ will denote a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. And for any $\lambda \in \mathbb{K}$ and $A, B \subseteq X$ we write $\lambda A = \{\lambda x : x \in A\}$ and $A + B = \{x + y : x \in A, y \in B\}$.

Note that thus two axioms of a vector space may fail to hold for the family $\mathcal{P}(X)$ of all subsets of $X$. Namely, only the one-point subsets of $X$ can have additive inverses. Moreover, in general, we only have $(\lambda + \mu)A \subseteq \lambda A + \mu A$.

If $A$ is a subset of $X$, then we say that:

1. $A$ is absorbing if $X = \bigcup_{r \in \mathbb{R}^+} rA$;
2. $A$ is balanced if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;
3. $A$ is convex if $rA + (1-r)A \subseteq A$ for all $r \in \mathbb{R}^+$ with $r < 1$.

A function $p$ of $X$ into $\mathbb{R}$ is called a seminorm on $X$ if

$$p(\lambda x) \leq |\lambda| p(x) \quad \text{and} \quad p(x + y) \leq p(x) + p(y)$$

for all $\lambda \in \mathbb{K}$ and $x, y \in X$. A seminorm $p$ is called a norm if $p(x) = 0$ implies $x = 0$.

If $p$ is a seminorm on $X$, then for each $r \in \mathbb{R}^+$ the relations $B^p_r$ and $\bar{B}^p_r$, defined by

$$B^p_r(x) = \{y \in X : p(x - y) < r\} \quad \text{and} \quad \bar{B}^p_r(x) = \{y \in X : p(x - y) \leq r\}$$

for all $x \in X$, are called the $r$-sized open and closed $p$-surroundings in $X$, respectively.

Concerning the above basic concepts we shall only need here the following simple theorems.

**Theorem 1.1.** If $A \subseteq X$, then the following assertions hold:

1. if $A$ is convex, then $(r + s)A = rA + sA$ for all $r, s \in \mathbb{R}^+$;
2. if $A$ is balanced, then $\lambda A \subseteq \mu A$ for all $\lambda, \mu \in \mathbb{K}$ with $|\lambda| \leq |\mu|$.

**Remark 1.2.** Therefore, a balanced subset $A$ of $X$ is absorbing if and only $X = \bigcup_{n=1}^\infty nA$. 
Theorem 1.3. If \( p \) is a seminorm on \( X \), then

1. \( p(x) \geq 0 \) for all \( x \in X \);
2. \( p(\lambda x) = |\lambda| p(x) \) for all \( \lambda \in \mathbb{K} \) and \( x \in X \).

Remark 1.4. Therefore, our present definition of a seminorm coincides with the usual one.

Theorem 1.5. If \( p \) is a seminorm on \( X \) and \( r \in \mathbb{R}_+ \), then

1. \( B^p_r(x) = x + B^p_r(0) \) for all \( x \in X \);
2. \( B^p_r(0) \) is an absorbing, balanced and convex subset of \( X \) such that \( B^p_r(0) = rB^p_1(0) \).

Remark 1.6. Moreover, the same statements hold for the closed surroundings \( \bar{B}^p_r \).

2. Seminorm generating relations

Definition 2.1. A relation \( F \) of \( \mathbb{R}_+ \) onto \( X \) will be called a seminorm generating relation for \( X \) if

1. \( F(r) + F(s) \subset F(r + s) \) for all \( r, s \in \mathbb{R}_+ \);
2. \( \lambda F(r) \subset F(tr) \) for all \( \lambda \in \mathbb{K} \) and \( r, t \in \mathbb{R}_+ \) with \( |\lambda| \leq t \).

The above definition is mainly motivated by the following simple

Example 2.2. If \( A \) is an absorbing, balanced, convex subset of \( X \) and \( F_A \) is a relation on \( \mathbb{R}_+ \) to \( X \) such that

\[ F_A(r) = rA \]

for all \( r \in \mathbb{R}_+ \), then \( F_A \) is a seminorm generating relation for \( X \).

Since \( A \) is absorbing, for each \( x \in X \) there exists an \( r \in \mathbb{R}_+ \) such that \( x \in rA \). Hence, it is clear that \( A \neq \emptyset \), and thus \( \mathbb{R}_+ \) is the domain of \( F_A \). Moreover, since \( x \in F_A(r) \), it is clear that \( X \) is the range of \( F_A \).

On the other hand, if \( r, s \in \mathbb{R}_+ \), then by Theorem 1.1 (1) it is clear that

\[ F_A(r + s) = (r + s)A = rA + sA = F_A(r) + F_A(s). \]

Moreover, if \( \lambda \in \mathbb{K} \) and \( r, t \in \mathbb{R}_+ \) such that \( |\lambda| \leq t \), then by Theorem 1.1 (2) it is clear that

\[ \lambda F_A(r) = \lambda(rA) = r(\lambda A) \subset r(tA) = (tr)A = F_A(tr). \]

Now, as an important particular case of Example 2.2, we can also state
Example 2.3. If $p$ is a seminorm on $X$ and $F_p$ and $\tilde F_p$ are relations on $\mathbb{R}_+$ to $X$ such that

$$F_p(r) = B^p_r(0) \quad \text{and} \quad \tilde F_p(r) = \tilde B^p_r(0)$$

for all $r \in \mathbb{R}_+$, then $F_p$ and $\tilde F_p$ are seminorm generating relations for $X$.

From Theorem 1.5(2) we know that $A = B^1_1(0)$ is an absorbing, balanced, convex subset of $X$ such that

$$F_p(r) = B^p_1(0) = r B^p_1(0) = r A = F_A(r)$$

for all $r \in \mathbb{R}_+$. Therefore, $F_p = F_A$, and thus $F_p$ is a seminorm generating relation for $X$ by Example 2.2.

The fact that $\tilde F_p$ is also a seminorm generating relation for $X$ can be proved quite similarly by using Remark 1.6 and Example 2.2.

In the sequel, beside Definition 2.1, we shall only need the following obvious

Theorem 2.4. If $F$ is a seminorm generating relation for $X$, then

1. $0 \in F(r)$ for all $r \in \mathbb{R}_+$;
2. $r F(s) \subseteq F(rs)$ for all $r, s \in \mathbb{R}_+$;
3. $F(r) \subseteq F(s)$ for all $r, s \in \mathbb{R}_+$ with $r \leq s$.

Proof. Since the assertions (1) and (2) are immediate from the homogeneity property 2.1(2) of $F$, we need only note that

$$F(r) = F(r) + \{0\} \subseteq F(r) + F(s-r) \subseteq F(s)$$

for all $r, s \in \mathbb{R}_+$ with $r < s$. Therefore, the assertion (3) also holds.

However, as a converse to Example 2.2, we can also easily prove the following

Theorem 2.5. If $F$ is a seminorm generating relation for $X$, then there exists an absorbing, balanced, convex subset $A$ of $X$ such that $F = F_A$.

Proof. If $r, s \in \mathbb{R}_+$, then by the homogeneity property 2.4(2) of $F$ we have

$$r F(s) \subseteq F(rs).$$

Hence, by writing $r^{-1}$ in place of $r$, and $rs$ in place of $s$, we can see that

$$r^{-1} F(rs) \subseteq F(s).$$

This implies that $F(rs) \subseteq r F(s)$. Therefore, the equality

$$F(rs) = r F(s)$$

is also true. Hence, under the notation $A = F(1)$, it follows that

$$F(r) = r F(1) = r A$$
for all $r \in \mathbb{R}_+$.

Therefore, it remains only to prove that $A$ is an absorbing, balanced and convex subset of $X$. For this, note that if $x \in X$, then since $F$ is onto $X$ there exists an $r \in \mathbb{R}_+$ such that $x \in F(r) = rA$. Therefore, $A$ is absorbing. Moreover, if $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$, then from the homogeneity property 2.1(2) of $F$ we can at once see that

$$\lambda A = \lambda F(1) \subset F(1) = A.$$ 

Therefore, $A$ is balanced. Moreover, if $0 < t < 1$, then by the homogeneity and additivity properties of $F$ it is clear that

$$tA + (1 - t)A = tF(1) + (1 - t)F(1) = F(t) + F(1 - t) \subset F(1) = A.$$ 

Therefore, $A$ is convex.

Now, in addition to Theorem 2.4, we can also easily state the following

**Theorem 2.6.** If $F$ is a seminorm generating relation for $X$, then

1. $F(rs) = rF(s)$ for all $r, s \in \mathbb{R}_+$;
2. $F(r + s) = F(r) + F(s)$ for all $r, s \in \mathbb{R}_+$;
3. $F(r)$ is an absorbing, balanced, convex subset of $X$ for all $r \in \mathbb{R}_+$.

**Remark 2.7.** Note that if $F$ is a balanced valued homogeneous relation of $\mathbb{R}_+$ into $X$, then

$$\lambda F(r) \subset tF(r) = F(tr)$$

for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$. That is, the homogeneity property 2.1(2) also holds.

3. **The Minkowski functionals of seminorm generating relations**

**Definition 3.1.** If $F$ is a seminorm generating relation for $X$, then the function $p_F$ defined by

$$p_F(x) = \inf \{ F^{-1}(x) \}$$

for all $x \in X$, will be called the Minkowski functional or gauge of $F$.

**Example 3.2.** If $A$ is an absorbing, balanced, convex subset of $X$, then we can at once see that

$$p_{rA}(x) = \inf \{ r \in \mathbb{R}_+ : x \in rA \}$$

for all $x \in X$. Therefore, $p_A = p_{rA}$ is just the usual Minkowski functional of $A$. (See, [1, p. 24].)

Therefore, it is not surprising that, as a useful reformulation of a well-known theorem on the Minkowski functionals of sets, we have the following
**Theorem 3.3.** If $F$ is a seminorm generating relation for $X$, then $p_F$ is a seminorm on $X$ such that  

$$F_{p_F} \subset F \subset \tilde{F}_{p_F}.$$ 

*Proof.* If $\lambda \in \mathbb{K}$ and $x \in X$, then by the definition of $p_F$ for each $\varepsilon > 0$  
there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, by noticing that  
$x \in F(r)$ and using the homogeneity property 2.1(2) of $F$, we can infer that  

$$\lambda x \in \lambda F(r) \subset F(t r),$$  

and thus $tr \in F^{-1}(\lambda x)$ for all $t \in \mathbb{R}_+$ with $|\lambda| \leq t$. Hence, since  
$tr < tp_F(x) + t\varepsilon$, it is clear that  

$$p_F(\lambda x) = \inf \left( F^{-1}(\lambda x) \right) < tp_F(x) + t\varepsilon$$  

for all $t \in \mathbb{R}_+$ with $|\lambda| \leq t$. Hence, by letting $t \to |\lambda|$ and $\varepsilon \to 0$, we can  
infer that  

$$p_F(\lambda x) \leq |\lambda|p_F(x).$$  

On the other hand, if $x, y \in X$, then again by the definition of $p_F$ for each $\varepsilon > 0$ there exist  
r \in F^{-1}(x)$ and $s \in F^{-1}(y)$ such that $r < p_F(x) + \varepsilon$ and  
s < p_F(y) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and $y \in F(s)$, and using  
the additivity property 2.1(1) of $F$, we can infer that  

$$x + y \in F(r) + F(s) \subset F(r + s),$$  

and thus $r + s \in F^{-1}(x + y)$. Hence, since $r + s < p_F(x) + p_F(y) + 2\varepsilon$, it is clear that  

$$p_F(x + y) = \inf \left( F^{-1}(x + y) \right) < p_F(x) + p_F(y) + 2\varepsilon,$$  

and thus  

$$p_F(x + y) \leq p_F(x) + p_F(y).$$  

Therefore, $p_F$ is a seminorm on $X$.

Finally, if $r \in \mathbb{R}_+$ and $x \in F_{p_F}(r) = B_{p_F}^r(0)$, i.e., $p_F(x) < r$, then again  
by the definition of $p_F$ there exists $s \in F^{-1}(x)$ such that $s < r$. Hence, by  
the monotonicity property 2.4(3) of $F$, it is clear that $x \in F(s) \subset F(r)$. Therefore  
$F_{p_F}(r) \subset F(r)$.

On the other hand, if $r \in \mathbb{R}_+$ and $x \in F(r)$, then $r \in F^{-1}(x)$. Therefore,  
by the definition of $p_F$, we have $p_F(x) \leq r$, and hence $x \in B_{p_F}^r(0) = \tilde{F}_{p_F}(r)$.  
Therefore, $F(r) \subset \tilde{F}_{p_F}(r)$ is also true.

Now, as an immediate consequence of Example 2.2 and Theorem 3.3, we can also state the following more familiar
Corollary 3.4. If $A$ is an absorbing, balanced, convex subset of $X$, then $p_A$ is a seminorm on $X$ such that
\[ B_1^{p_A}(0) \subset A \subset \bar{B}_1^{p_A}(0). \]

In addition, to Theorem 3.3, it is also worth proving the following

Theorem 3.5. If $p$ is a seminorm on $X$ and $F$ is a seminorm generating relation for $X$ such that
\[ F_p \subset F \subset \tilde{F}_p, \]
then $p = p_F$.

Proof. If $x \in X$, then for each $r \in \mathbb{R}_+$, with $p(x) < r$, we have
\[ x \in B_r^p(0) = F_p(r) \subset F(r). \]
Therefore, $r \in F^{-1}(x)$, and thus
\[ p_F(x) = \inf \{ F^{-1}(x) \} \leq r. \]
Hence, by letting $r \to p(x)$, we can infer that $p_F(x) \leq p(x)$.

On the other hand, by the definition of $p_F(x)$, for each $\varepsilon > 0$ there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, we can see that
\[ x \in F(r) \subset \tilde{F}_p(r) = \bar{B}_r^p(0). \]
Therefore, $p(x) \leq r < p_F(x) + \varepsilon$, and thus $p(x) \leq p_F(x)$ is also true.

Remark 3.6. In particular, by Theorem 3.5, we have $p = p_{\bar{F}_r} = p_{\sigma_r}$ for every seminorm $p$ on $X$.

Moreover, as an immediate consequence of Example 2.1 and Theorem 3.5, we can also state

Corollary 3.7. If $p$ is a seminorm on $X$ and $A$ is an absorbing, balanced, convex subset of $X$ such that
\[ B_1^p(0) \subset A \subset \bar{B}_1^p(0), \]
then $p = p_A$.

4. Some further properties of the Minkowski functionals

Theorem 4.1. If $F$ is a seminorm generating relation for $X$, then the following assertions are equivalent:

1. $p_F$ is a norm;
2. $\bigcap_{r \in \mathbb{R}_+} F(r) = \{ 0 \}$.

Proof. If $x \in F(r)$, and hence $r \in F^{-1}(x)$ for all $r \in \mathbb{R}_+$, then by the definition of $p_F$ we have $p_F(x) = 0$. Hence, if the assertion (1) holds, it follows that $x = 0$. Therefore, since $0 \in F(r)$ for all $r \in \mathbb{R}_+$, the assertion (2) also holds.

While, if $x \in X$ such that $p_F(x) = 0$, then by the definition of $p_F$ for each $r \in \mathbb{R}_+$ there exists $s \in F^{-1}(x)$ such that $s < r$. Hence, by the monotonicity property of $F$, it is clear that $x \in F(s) \subset F(r)$. Therefore, if the assertion (2) holds, then $x = 0$, and thus the assertion (1) also holds.
Corollary 4.2. If $A$ is an absorbing, balanced, convex subset of $X$, then the following assertions are equivalent:

1. $p_A$ is a norm;
2. $\bigcap_{r \in \mathbb{R}_+} rA = \{0\}$.

Remark 4.3. Note that, since $A$ is balanced, we may write $\mathbb{K} \setminus \{0\}$ in place of $\mathbb{R}_+$ in the assertion (2).

Theorem 4.4. If $F$ is a seminorm generating relation for $X$, then the following assertions are equivalent:

1. $F = F_{p_F}$;
2. $F(r) = \bigcup_{s \leq r} F(s)$ for all $r \in \mathbb{R}_+$.

Proof. If $r \in \mathbb{R}_+$ and $x \in F(r)$, and the assertion (1) holds, then we have $x \in F_{p_F}(r) = B^{p_F}_r(0)$. Hence, it follows that

$$\inf (F^{-1}(x)) = p_F(x) < r.$$ 

Therefore, there exists an $s \in F^{-1}(x)$ such that $s < r$. Hence, it follows that $x \in F(s)$. Therefore,

$$F(r) \subset \bigcup_{s \leq r} F(s).$$

Now, since the converse inclusion is immediate from the monotonicity property of $F$, it is clear that the assertion (2) also holds.

While, if $r \in \mathbb{R}_+$ and $x \in F(x)$, and the assertion (2) holds, then there exists an $s < r$ such that $x \in F(s)$, and hence $s \in F^{-1}(x)$. Therefore,

$$p_F(x) = \inf (F^{-1}(x)) \leq s < r,$$

and hence $x \in B^{p_F}_r(0) = F_{p_F}(r)$. Consequently, we have $F(r) \subset F_{p_F}(r)$. Now, since the converse inclusion is always true by Theorem 3.3, it is clear that the assertion (1) also holds.

Corollary 4.5. If $A$ is an absorbing, balanced, convex subset of $X$, then the following assertions are equivalent:

1. $A = B^{p_A}_1(0)$;
2. $A = \bigcup_{s \leq 1} sA$.

Theorem 4.6. If $F$ is a seminorm generating relation for $X$, then the following assertions are equivalent:

1. $F = \tilde{F}_{p_F}$;
2. $F(r) = \bigcap_{s \geq r} F(s)$ for all $r \in \mathbb{R}_+$.

Proof. If $r \in \mathbb{R}_+$, and $x \in X$ such that $x \in F(s)$, i.e., $s \in F^{-1}(x)$ for all $s > r$, then

$$p_F(x) = \inf (F^{-1}(x)) \leq r,$$
and hence \( x \in \tilde{B}_r^p (0) = \tilde{F}_p (r) \). Therefore, if the assertion (1) holds, then we also have \( x \in F (r) \). Consequently,
\[
\bigcap_{s>r} F(s) \subset F(r).
\]

Hence, since the converse inclusion is immediate from the monotonicity property of \( F \), it is clear that the assertion (2) also holds.

While, if \( r \in \mathbb{R}_+ \), and \( x \in \tilde{F}_p (r) \), i.e., \( x \in \tilde{B}_r^p (0) \), then
\[
\inf \left( F^{-1}(x) \right) = p_r(x) \leq r.
\]
Therefore, for each \( s > r \) there exists a \( t \in F^{-1}(x) \) such that \( t < s \). Hence, by the monotonicity property of \( F \), it is clear that \( x \in F(t) \subset F(s) \). Therefore,
\[
\tilde{F}_p (r) \subset \bigcap_{s>r} F(s).
\]

Hence, if the assertion (2) holds, it follows that \( \tilde{F}_p (r) \subset F(r) \). Now, since the converse inclusion is always true by Theorem 3.3, it is clear that the assertion (1) also holds.

**Corollary 4.7.** If \( A \) is an absorbing, balanced, convex subset of \( X \), then the following assertions are equivalent:

1. \( A = \tilde{B}_1^p (0) \); 
2. \( A = \bigcap_{s>1} sA \).

**Remark 4.8.** The above corollaries are not established in the standard books on functional analysis.

**References**


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