ENERGY GAPS FOR EXPONENTIAL YANG-MILLS FIELDS

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Abstract. In this paper, some inequalities of Simons type for exponential Yang-Mills fields over compact Riemannian manifolds are established, and the energy gaps are obtained.

1. Introduction

Let $M$ be an $m$-dimensional Riemannian manifold, $G$ an $r_0$-dimensional Lie group, $E$ a Riemannian vector bundle over $M$ with structure group $G$, $g_E \subseteq \text{End}(E)$ the adjoint vector bundle, whose fiber type is $g$, the Lie algebra of $G$. We denote the space of $g_E$-valued $p$-forms by $\Omega^p(g_E)$. Let $\nabla$ be a connection on $E$, then, the curvature $\nabla^2 \in \Omega^2(g_E)$ is defined by $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ for tangent vector fields $X$, $Y$ on $M$.

Extend the connection $\nabla$ into an exterior differential operator $d\nabla: \Omega^p(g_E) \to \Omega^{p+1}(g_E)$ as follows: for each $\omega \in \Omega^p(M)$ and $\sigma \in \Omega^0(g_E)$, let

$$d\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge \nabla \sigma,$$

and extend to all members of $\Omega^p(g_E)$ by linearity.

When $G$ is a subgroup of $O(r_0)$, the Killing form in $g$ is negatively defined, and hence induces an inner product in $g_E$. This inner product and the Riemannian metric of $M$ define an inner product $\langle \cdot, \cdot \rangle$ in $\Omega^p(g_E)$. The exterior differential operator $d\nabla: \Omega^p(g_E) \to \Omega^{p+1}(g_E)$ has a formal adjoint operator $\delta\nabla: \Omega^{p+1}(g_E) \to \Omega^p(g_E)$ with respect to the $L^2$-inner product $\langle \varphi, \psi \rangle = \int_M \langle \varphi, \psi \rangle$. Take a local orthonormal frame field $\{e_1, \ldots, e_m\}$ on $M$. Then, for any $\varphi \in \Omega^p(E)$ and any local tangent vector fields $X_0, X_1, \ldots, X_p$ to $M$, we have

$$(d\nabla \varphi)_{X_0, X_1, \ldots, X_p} = \sum_{k=0}^p (-1)^k \langle \nabla_{X_k} \varphi \rangle_{X_0, \ldots, X_k, \ldots, X_p},$$

$$(\delta\nabla \varphi)_{X_0, \ldots, X_{p-1}} = \sum_{k=1}^m \langle \nabla_{e_k} \varphi \rangle_{e_k, X_1, \ldots, X_{p-1}}.$$
The Laplacian acting on $\Omega^p(g_E)$ is defined by $\Delta = d \circ \delta + \delta \circ d : \Omega^p(g_E) \to \Omega^p(g_E)$. If $\varphi \in \Omega^p(g_E)$ satisfies $\Delta \varphi = 0$, we call it a harmonic $p$-form with values in $g_E$.

Let $C_E$ be the collection of all metric connections on $E$, and fix a connection $\nabla_0 \in C_E$. Then, any connection $\nabla \in C_E$ can be expressed as $\nabla = \nabla_0 + A$, where $A \in \Omega^1(g_E)$. The Yang-Mills functional is defined as: For $\nabla \in C_E$,

$$S(\nabla) = \frac{1}{2} \int_M |R^\nabla|^2.$$

A connection $\nabla \in C_E$ is called a Yang-Mills connection, if it is a critical point of the Yang-Mills functional, and the associated curvature tensor is called a Yang-Mills field.

The Euler-Lagrange equation of the Yang-Mills functional $S(\cdot)$ can be written as

$$\delta R^\nabla = 0.$$

Hence, by Bianchi identity $d R^\nabla = 0$, a Yang-Mills field is a harmonic 2-form with values in $g_E$.

The following gap property for Yang-Mills fields is obtained in [2]:

**Theorem 1.** Let $R^\nabla$ be a Yang-Mills field on $S^m (m \geq 5)$ satisfying that

$$\| R^\nabla \|_{L^\infty}^2 \leq \frac{1}{2} \left( \frac{m}{2} \right),$$

then $R^\nabla \equiv 0$.

Denote the Riemannian curvature operator of $M$ by $R$, the Ricci operator by $\text{Ric}$. Let $C = \text{Ric} \wedge I + 2R$, where $I$ is the identity transformation on $TM$, and define the Ricci-Riemannian curvature operator $C : \Omega^2(g_E) \to \Omega^2(g_E)$ as follows: for $\varphi \in \Omega^2(g_E)$ and $X, Y, Z \in \Gamma(M)$,

$$(C(\varphi))_{X,Y} = \frac{1}{2} \sum \varphi_{e_j,C_{X,Y}(e_j)}.$$  

Here,

$$(\text{Ric} \wedge I)_{X,Y} = \text{Ric}(X) \wedge Y + X \wedge \text{Ric}(Y),$$

and $X \wedge Y$ is identified as a skew-symmetric linear transformation by

$$(X \wedge Y)(Z) = \langle X, Z \rangle Y - \langle Y, Z \rangle X.$$

In the following, that $C \geq \lambda$ means that $\langle C(\varphi), \varphi \rangle \geq \lambda |\varphi|^2$ for each $\varphi \in \Omega^2(g_E)$.

In [13], an inequality of Simons type for Yang-Mills fields is obtained:

**Theorem 2.** Let $M^m (m \geq 3)$ be a compact Riemannian manifold with $C \geq \lambda$. Then, for each Yang-Mills field $R^\nabla$, we have

$$\int_M |\nabla R^\nabla|^2 \leq \int_M \left( -\frac{4(m-2)}{\sqrt{m(m-1)}} |R^\nabla| - \lambda \right) |R^\nabla|^2.$$

If $m \geq 5$, the equality holds if and only if $R^\nabla \equiv 0$.

This inequality implies a gap property (see [13]):
Corollary 3. Let $M^m$ and $\lambda$ be as in Theorem 2. Let $R^\nabla \in \Omega^2(g_E)$ be a Yang-Mills field over $M$. If $m \geq 3$ and $\|R^\nabla\|^2_{L^\infty} < \frac{\lambda^2m(m-1)}{16(m-2)^2}$, then we have $R^\nabla = 0$. If $m \geq 5$ and $\|R^\nabla\|^2_{L^\infty} \leq \frac{\lambda^2m(m-1)}{16(m-2)^2}$, then we also have $R^\nabla = 0$.

When $M = S^m$, we have $\lambda = 2(m-2)$. Therefore Corollary 3 implies Theorem 1.

A $p$-Yang-Mills functional is defined by $S_p(\nabla) = \frac{1}{p} \int_M |R^\nabla|^p$, the critical points of which are called $p$-Yang-Mills connections, and the associated curvature tensors are called $p$-Yang-Mills fields. The article [4] investigated the gaps of $p$-Yang-Mills fields of Euclidean and sphere submanifolds, and generalized the related results of [2].

Theorem 4 (See [4, 13]). Let $M^m$ be a submanifold of $\mathbb{R}^{m+k}$ or $S^{m+k}$. If $C \geq 2(m-2)$, and if $R^\nabla$ is a $p$-Yang-Mills field ($p \geq 2$) with $\|R^\nabla\|^2_{L^\infty} \leq \frac{1}{2} \frac{(m)}{2}$ $(m \geq 5)$, then we have $R^\nabla \equiv 0$.

Theorem 4 is also a generalization of Theorem 1.

An exponential Yang-Mills functional is defined by $S_e(\nabla) = \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right)$, an exponential Yang-Mills connection is a critical point of $S_e$, and an exponential Yang-Mills field is the curvature $R^\nabla$ of an exponential Yang-Mills connection $\nabla \in C_E$. The Euler-Lagrange equation of $S_e(\cdot)$ is

\[
\delta^\nabla \left[ \exp \left( \frac{|R^\nabla|^2}{2} \right) R^\nabla \right] = 0.
\]

Some $L^2$-energy gaps are obtained for four dimensional Yang-Mills fields, see for example [5, 6, 7, 11, 12] etc. The existence of $L^{m/2}$-energy gaps for Yang-Mills fields over $m$-dimensional compact or non-compact but complete Riemannian manifolds are verified independently under some non-positive curvature conditions in [15] and [9]. P.M.N. Feehan prove an existence of $L^{m/2}$-energy gaps over compact manifolds without any curvature assumptions in [8]. Recently, we estimate the $L^p$-energy gaps for $p \geq m/2$ over the unit sphere $S^m$ and the $m/2$-energy gaps over the hyperbolic space $\mathbb{H}^m$ in [14].

In this paper, we establish some inequalities of Simons type for exponential Yang-Mills fields over compact Riemannian manifolds. Then, we use these inequalities to obtain some energy gaps.

2. Inequalities of Simons type for exponential Yang-Mills fields

Take a local orthonormal frame field $\{e_i\}_{i=1,\ldots,m}$ on $M$. We adopt the convention of summation, and let indices $i, j, k, l, u$ run in $\{1, \ldots, m\}$.

For each $\varphi \in \Omega^2(g_E)$, let

\[
\Re(\varphi)_{X,Y} = \sum \left\{ [R_{e_j,X}, \varphi_{e_j,Y}] - [R_{e_j,Y}, \varphi_{e_j,X}] \right\}.
\]

Then, we have (see [2])

\[
\Delta^\nabla \varphi = \nabla^* \nabla \varphi + C(\varphi) + \Re(\varphi),
\]
where, $\nabla^* \nabla = - \nabla_{e_i} \nabla_{e_i} + \nabla_{D_{e_i} e_i}$ is the rough Laplacian ($D$ is the Levi-Civita connection of $M$). Hence we have

$$\frac{1}{2} \Delta |\varphi|^2 = \langle \Delta \nabla \varphi, \varphi \rangle - |\nabla \varphi|^2 - \langle C(\varphi), \varphi \rangle - \langle \Re \nabla \varphi, \varphi \rangle.$$  

(10)

By a straightforward calculation, we get

$$\Delta \exp \left( \frac{|\varphi|^2}{2} \right) = - \exp \left( \frac{|\varphi|^2}{2} \right) |\varphi|^2 |\nabla| |\varphi|^2$$

$$+ \exp \left( \frac{|\varphi|^2}{2} \right) \langle \Delta \nabla \varphi, \varphi \rangle - \exp \left( \frac{|\varphi|^2}{2} \right) |\nabla|^2$$

$$- \exp \left( \frac{|\varphi|^2}{2} \right) \langle C(\varphi), \varphi \rangle - \exp \left( \frac{|\varphi|^2}{2} \right) \langle \Re \nabla \varphi, \varphi \rangle.$$  

(11)

Integrating both sides of (11), we have

**Lemma 5.** For each $\varphi \in \Omega^2(\mathfrak{g}_E)$, we have

$$\int_M \exp \left( \frac{|\varphi|^2}{2} \right) |\varphi|^2 |\nabla||\varphi|^2 = \int_M \exp \left( \frac{|\varphi|^2}{2} \right) \langle \Delta \nabla \varphi, \varphi \rangle$$

$$- \int_M \exp \left( \frac{|\varphi|^2}{2} \right) \langle C(\varphi), \varphi \rangle - \int_M \exp \left( \frac{|\varphi|^2}{2} \right) \langle \Re \nabla \varphi, \varphi \rangle.$$  

(12)

In [13], we establish the following inequality:

**Lemma 6.** For $\varphi \in \Omega^2(\mathfrak{g}_E)$, let

$$\rho(\varphi) = \sum \langle [\varphi_{e_i,e_j}, \varphi_{e_j,e_k}], \varphi_{e_k,e_i} \rangle.$$  

(13)

Then, we have

$$|\rho(\varphi)| \leq \frac{4(m-2)}{\sqrt{m(m-1)}} |\varphi|^3.$$  

(14)

If $m \geq 5$, the inequality is strict unless $\varphi = 0$.

Applying Lemma 6 to Lemma 5, we can obtain the following inequality of Simons type for exponential Yang-Mills fields:

**Theorem 7.** Let $M^m$ ($m \geq 3$) be a Riemannian $m$-manifold, and $R^\nabla$ be an exponential Yang-Mills field over $M^m$. If $C \geq \lambda$, then we have

$$\int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^2 |\nabla|R^\nabla|^2 + \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |\nabla R^\nabla|^2$$

$$\leq \int_M \left( \frac{4(m-2)}{\sqrt{m(m-1)}} |R^\nabla| - \lambda \right) \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^2.$$  

(15)
Proof. By Bianchi identity, we have
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) \langle \Delta \nabla R^\nabla, R^\nabla \rangle = \int_M \left\langle \delta^\nabla R^\nabla, \delta^\nabla \left( \exp \left( \frac{|R^\nabla|^2}{2} \right) R^\nabla \right) \right\rangle. \]
Because $R^\nabla$ is an exponential Yang-Mills field, we have
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) \langle \Delta \nabla R^\nabla, R^\nabla \rangle = 0. \]
Hence by (12) we have
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |\nabla R^\nabla| \leq \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |\nabla R^\nabla|^2. \]
If $\mathcal{C} \geq \lambda$, then we get
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) \langle \mathcal{C}(R^\nabla), R^\nabla \rangle \leq -\lambda \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^2. \]
For $m \geq 3$, from Lemma 6 we have
\[ -\frac{4(m-2)}{\sqrt{m(m-1)}} |R^\nabla|^3 \leq \pm \rho(R^\nabla) \leq \frac{4(m-2)}{\sqrt{m(m-1)}} |R^\nabla|^3. \]
So we have
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) \rho(R^\nabla) \leq \frac{4(m-2)}{\sqrt{m(m-1)}} \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^3. \]
Hence from (17) and (18) we have (15). □

Corollary 8. Let $M^m$ ($m \geq 3$) be a Riemannian $n$-manifold, and $R^\nabla$ be an exponential Yang-Mills field over $M^m$. If $\mathcal{C} \geq \lambda$, then we have
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |\nabla R^\nabla|^2 + 4 \int_M |\nabla \exp \left( \frac{|R^\nabla|^2}{4} \right)|^2 \leq \int_M \left( \frac{4(m-2)}{\sqrt{m(m-1)}} |R^\nabla| - \lambda \right) \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^2. \]
Proof. Because
\[ \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^2 |\nabla |R^\nabla||^2 \]
\[ = \int_M \exp \left( \frac{|R^\nabla|^2}{2} \right) |\nabla \frac{|R^\nabla|^2}{2}|^2 = 4 \int_M |\nabla \exp \left( \frac{|R^\nabla|^2}{4} \right)|^2, \]
then, from \([15]\) we have
\[
4 \int_M |\nabla \exp \left( \frac{|R|^2}{4} \right)|^2 + \int_M \exp \left( \frac{|R|^2}{2} \right) |\nabla R|^2 \leq \int_M \left( \frac{4(m-2)}{m(m-1)} |R\nabla| - \lambda \right) \exp \left( \frac{|R|^2}{2} \right) |\nabla R|^2.
\]
(20)

By \(|\nabla R|^2 \geq |\nabla |R\nabla|^2|\) and (20) we get (19). \(\square\)

By Theorem [7] we have

**Corollary 9.** Let \(M^m (m \geq 3)\) be a Riemannian \(m\)-manifold, and \(R\nabla\) be an exponential Yang-Mills field over \(M^m\). Suppose that \(C \geq \lambda\). Then, if \(\|R\nabla\|_{L^\infty}^2 \leq \frac{m(m-1)\lambda^2}{10(m-2)^2}\), we have \(\nabla R\nabla = 0\). Especially, on \(S^m\), if \(\|R\nabla\|_{L^\infty}^2 < \frac{1}{2} \left( \frac{m}{2} \right)\), we have \(R\nabla = 0\).

3. Energy gaps for exponential Yang-Mills fields

Let \(M^m\) be an \(m\)-dimensional compact Riemannian manifold. We say that the \(q\)-Sobolev inequality holds on \(M^m\) with \(k_1, k_2\) if for all \(u \in C^\infty(M^m)\) we have
\[
\|\nabla u\|^2 \geq k_1 \|u\|^2 - k_2 \|\nabla u\|^2.
\]
(21)

On the unit sphere \(S^m\), the following Sobolev inequality holds (see [11][10]): for \(2 \leq q \leq 2m/(m-2)\),
\[
\|u\|^2 \leq \frac{q-2}{m}\omega_m^{1-2/q} \|\nabla u\|^2 + \frac{1}{\omega_m} \|\nabla u\|^2,
\]
where \(\omega_m\) is the volume of the unit sphere \(S^m\). Hence we have

**Lemma 10.** On \(S^m\), for \(2 < q \leq 2m/(m-2)\), the \(q\)-Sobolev inequality holds with \(k_1 = \frac{m\omega_m^{1-2/q}}{q-2}\), \(k_2 = \frac{m}{q-2}\).

Denote
\[
d_{a,m,r} = \min \left\{ k_1, \frac{k_1 a}{k_2} \right\},
\]
where \(\frac{1}{r} + \frac{1}{q} = 1\).

In [13], we prove the following

**Lemma 11.** Let \(T\) be a tensor over a compact Riemannian manifold \(M^m\) where the \(2q\)-Sobolev inequality holds with \(k_1, k_2\) for \(2 < 2q \leq \frac{2m}{m-2}\). Assume that there exist a positive constant \(a\) and a function \(f\) on \(M\), such that
\[
\|\nabla |T|\|^2 \leq -a \|T\|^2 + \|f |T|\|^2\|_1.
\]
(23)

If \(\|f \|_r < d_{a,m,r}\), then we have \(T = 0\), where \(r = \frac{q}{q-1} \geq \frac{m}{2}\).

**Theorem 12.** Let \(M^m(m \geq 3)\) be a compact Riemannian manifold with \(C \geq \lambda > 0\), where \(2q\)-Sobolev inequality holds with \(k_1\) and \(k_2\) for \(2 < 2q \leq \frac{2m}{m-2}\). Suppose that \(R\nabla\) is an exponential Yang-Mills field over \(M\). If \(\|R\nabla \exp \left( \frac{|R\nabla|^2}{2} \right)\|_r < \frac{\sqrt{m(m-1)}}{4(m-2)} d_{\lambda,m,r}\),
then we have \(R\nabla = 0\), where \(r = \frac{q}{q-1} \geq \frac{m}{2}\).
Proof. By (19) we have
\[
\int_M |\nabla |R^\nabla||^2 \leq \int_M \left( \frac{4(m-2)}{\sqrt{m(m-1)}} |R^\nabla| - \lambda \right) \exp \left( \frac{|R^\nabla|^2}{2} \right) |R^\nabla|^2.
\]
Let \( u = |R^\nabla| \), then
\[
\int_M |\nabla u|^2 \leq \int_M \left( \frac{4(m-2)}{\sqrt{m(m-1)}} u \exp \left( \frac{u^2}{2} \right) - \lambda \right) \exp \left( \frac{u^2}{2} \right) u^2.
\]
So, we have
\[
\int_M |\nabla u|^2 \leq \int_M \left( \frac{4(m-2)}{\sqrt{m(m-1)}} u \exp \left( \frac{u^2}{2} \right) - \lambda u^2 \right)
= \int_M \left( \frac{4(m-2)}{\sqrt{m(m-1)}} u \exp \left( \frac{u^2}{2} \right) - \lambda \right) u^2
\]
i.e. \( \int_M |\nabla u|^2 \leq \int_M f u^2 - \lambda \int_M u^2 \), where \( f = \frac{4(m-2)}{\sqrt{m(m-1)}} u \exp \left( \frac{u^2}{2} \right) \). Then by Lemma 11 we can get the theorem. 

Corollary 13. Suppose that \( R^\nabla \) is an exponential Yang-Mills field over \( S^m \) \((m \geq 3)\). If
\[
\left\| R^\nabla \exp \left( \frac{|R^\nabla|^2}{2} \right) \right\|_r < \sqrt{\frac{m(m-1)}{4(m-2)}} \omega_m^\frac{1}{2} \min \left\{ \frac{m(r-1)}{2}, 2(m-2) \right\}
\]
then, we have \( R^\nabla = 0 \), where \( r \geq \frac{m}{2} \).

Proof. On \( S^m \), \( \lambda = 2(m-2) \), and the \( 2q \)-Sobolev inequality holds for \( 2 < 2q \leq \frac{2m}{m-2} \) with \( k_1 = \frac{n \omega_m^{1-2/2q}}{2q-2} = \frac{m(r-1)}{2} \omega_m^{1/r} \), \( k_2 = \frac{m}{2q-2} = \frac{m(r-1)}{2} \). By a straightforward calculation, we get
(24) \[
d_{2(m-2),m,r} = \omega_m^{1/r} \min \left\{ \frac{m(r-1)}{2}, 2(m-2) \right\}
\]
and \( d_{2(m-2),m,\infty} = 2(m-2) \).

Then by Theorem 12 if
\[
\left\| R^\nabla \exp \left( \frac{|R^\nabla|^2}{2} \right) \right\|_r < \sqrt{\frac{m(m-1)}{4(m-2)}} d_{2(m-2),m,r},
\]
then we have \( R^\nabla = 0 \).

Especially, if
\[
\left\| R^\nabla \exp \left( \frac{|R^\nabla|^2}{2} \right) \right\|_\infty < \sqrt{\frac{m(m-1)}{4(m-2)}} d_{2(m-2),m,\infty} = \sqrt{\frac{m(m-1)}{2}},
\]
we have \( R^\nabla = 0 \). 

\[\square\]
References


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