CONDITIONS FOR INTEGRABILITY OF A 3-FORM

JIŘÍ VANŽURA

ABSTRACT. We find necessary and sufficient conditions for the integrability of one type of multisymplectic 3-forms on a 6-dimensional manifold.

Let $V$ be a 6-dimensional real vector space. The general linear group $GL(V)$ operates naturally on the space of 3-forms $\Lambda^3 V^*$ by

$$\varphi \alpha(v, v', v'') = \alpha(\varphi^{-1}v, \varphi^{-1}v', \varphi^{-1}v''), \quad \alpha \in \Lambda^3 V^*, \varphi \in GL(V).$$

This action has six orbits, see e.g. [1]. They can be described by their representatives. Let us choose a basis $v_1, \ldots, v_6$ of $V$, and let $\alpha_1, \ldots, \alpha_6$ be the corresponding dual basis. Let us recall that a 3-form $\alpha \in \Lambda^3 V^*$ is called regular or multisymplectic if the linear mapping

$$\iota: V \to \Lambda^2 V^*, \quad \iota(v) = \iota_v \alpha$$

is injective. All the other forms are then called singular. Obviously, all forms belonging to an orbit are either regular or singular. We then speak about regular orbits and singular orbits. We denote $R_+$, $R_-$ and $R_0$ the regular orbits and by $\rho_+, \rho_-, \rho_0$ their representatives. Similarly we denote $S_1$, $S_2$ and $S_3$ the singular orbits and by $\sigma_1$, $\sigma_2$, $\sigma_3$ their representatives.

$$(R_+) \quad \rho_+ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6,$$

$$(R_-) \quad \rho_- = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6,$$

$$(R_0) \quad \rho_0 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6 + \alpha_3 \wedge \alpha_6 \wedge \alpha_4,$$

$$(S_1) \quad \sigma_1 = 0,$$

$$(S_2) \quad \sigma_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3,$$

$$(S_3) \quad \sigma_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5).$$

We recall that a 2-form $\beta$ on a vector space is called decomposable if there exist 1-forms $\gamma$ and $\gamma'$ such that $\beta = \gamma \wedge \gamma'$. It is well known that a 2-form $\beta$ is decomposable if and only if $\beta \wedge \beta = 0$.

With every 3-form $\alpha \in \Lambda^3 V^*$ we can associate a subset $\Delta(\alpha) \subset V$ defined by

$$\Delta(\alpha) = \{v \in V; \iota_v \alpha \wedge \iota_v \alpha = 0\}.$$
In other words $\Delta(\alpha)$ consists of all $v \in V$ such that the 2-form $\iota_v \alpha$ is decomposable.

1. ALGEBRAIC PROPERTIES

We take now an element $\alpha \in R_0$. We find easily that
$$\Delta(\rho_0) = [v_1, v_2, v_3].$$
This shows that the subset $\Delta(\alpha)$ is a 3-dimensional subspace of $V$. For simplicity we denote $V_0 = \Delta(\alpha)$. There is also another possible description of $\Delta(\alpha)$.

1. Lemma. $\Delta(\alpha) = \{ v \in V; (\iota_v \alpha) \wedge \alpha = 0 \}$.

Proof. Obviously it suffices to prove this equality for $\alpha = \rho_0$. We take $v = a_1v_1 + \cdots + a_6v_6$ and we find
$$(\iota_v \rho_0) \wedge \rho_0 = -2a_6\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 + 2a_4\alpha_1 \wedge \alpha_3 \wedge \alpha_5 \wedge \alpha_6 \wedge \alpha_6 - 2a_5\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6.$$
This proves the lemma.

For $\rho_0$, and consequently for every $\alpha \in R_0$ we have the following lemma.

2. Lemma. If $\alpha \in R_0$ and $v, v' \in \Delta(\alpha)$, then $\alpha(v, v', \cdot) = 0$.

Inspired by $\rho_0$ we introduce the following definition.

3. Definition. A basis $w_1, \ldots, w_6$ of $V$ is called canonical basis for $\alpha$ if the following conditions are satisfied
$$\alpha(w_1, w_2, w_3) = 0, \quad \alpha(w_i, w_j, w_k) = 0 \quad \text{for} \quad 1 \leq i < j \leq 3, \quad k = 4, 5, 6,$$
$$\alpha(w_1, w_4, w_5) = 1, \quad \alpha(w_1, w_5, w_6) = 0, \quad \alpha(w_1, w_6, w_4) = 0,$$
$$\alpha(w_2, w_4, w_5) = 0, \quad \alpha(w_2, w_5, w_6) = 1, \quad \alpha(w_2, w_6, w_4) = 0,$$
$$\alpha(w_3, w_4, w_5) = 0, \quad \alpha(w_3, w_5, w_6) = 0, \quad \alpha(w_3, w_6, w_4) = 1,$$
$$\alpha(w_4, w_5, w_6) = 0.$$
A dual basis $\beta_1, \ldots, \beta_6$ to a canonical basis will be called canonical dual basis for $\alpha$.

It is easy to see that $\beta_1, \ldots, \beta_6$ is a canonical dual basis for $\alpha$ if and only if there is
$$\alpha = \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_5 \wedge \beta_6 + \beta_3 \wedge \beta_6 \wedge \beta_4.$$

Because the forms $\alpha$ and $\rho_0$ are equivalent (= belong to the same orbit), it is obvious that

4. Lemma. Every 3-form $\alpha \in R_0$ has a canonical basis.

Nevertheless for the later considerations within the framework of differential geometry we shall present a constructive proof.

Proof. We choose first a complement $V_c$ of $V_0$ in $V$. In this complement we take three linearly independent vectors $z_4, z_5, z_6$. We denote $a = \alpha(z_4, z_5, z_6)$. Because
the form \( \alpha \) is regular, there is \( v_0 \in V_0 \) such that \( \alpha(v_0, z_5, z_6) = b \neq 0 \). Taking \( w_4 = z_4 - (a/b)v_0, w_5 = z_5, \) and \( w_6 = z_6 \) we get
\[
\alpha(w_4, w_5, w_6) = \alpha(z_4 - (a/b)v_0, z_5, z_6)
= \alpha(z_4, z_5, z_6) - (a/b)\alpha(v_0, z_5, z_6) = a - (a/b)b = 0.
\]
Now we have on \( V_0 \) three linear forms, namely the forms \( \alpha(\cdot, w_4, w_5), \alpha(\cdot, w_5, w_6), \) and \( \alpha(\cdot, w_6, w_4) \). The regularity of \( \alpha \) implies again that these three forms are linearly independent. Consequently, there are uniquely determined \( w_1, w_2, w_3 \in V_0 \) such that
\[
\alpha(w_1, w_4, w_5) = 1, \quad \alpha(w_1, w_5, w_6) = 0, \quad \alpha(w_1, w_6, w_4) = 0,
\]
\[
\alpha(w_2, w_4, w_5) = 0, \quad \alpha(w_2, w_5, w_6) = 1, \quad \alpha(w_2, w_6, w_4) = 0,
\]
\[
\alpha(w_3, w_4, w_5) = 0, \quad \alpha(w_3, w_5, w_6) = 0, \quad \alpha(w_3, w_6, w_4) = 1.
\]
The equations \( \alpha(w_1, w_2, w_3) = 0 \) and \( \alpha(w_i, w_j, w_k) = 0 \) for \( 1 \leq i < j \leq 3, k = 4, 5, 6 \) are satisfied automatically by virtue of Lemma [2]

Let us consider two canonical dual bases \( \beta_1, \ldots, \beta_6 \) and \( \beta'_1, \ldots, \beta'_6 \). We can write
\[
\beta'_1 = c_{11}\beta_1 + c_{12}\beta_2 + c_{13}\beta_3 + c_{14}\beta_4 + c_{15}\beta_5 + c_{16}\beta_6
\]
\[
\beta'_2 = c_{21}\beta_1 + c_{22}\beta_2 + c_{23}\beta_3 + c_{24}\beta_4 + c_{25}\beta_5 + c_{26}\beta_6
\]
\[
\beta'_3 = c_{31}\beta_1 + c_{32}\beta_2 + c_{33}\beta_3 + c_{34}\beta_4 + c_{35}\beta_5 + c_{36}\beta_6
\]
\[
\beta'_4 = c_{44}\beta_4 + c_{45}\beta_5 + c_{46}\beta_6
\]
\[
\beta'_5 = c_{54}\beta_4 + c_{55}\beta_5 + c_{56}\beta_6
\]
\[
\beta'_6 = c_{64}\beta_4 + c_{65}\beta_5 + c_{66}\beta_6
\]
We start with the equation
\[
\beta'_1 \wedge \beta'_4 \wedge \beta'_5 + \beta'_2 \wedge \beta'_6 + \beta'_3 \wedge \beta'_6 = \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_5 \wedge \beta_6 + \beta_3 \wedge \beta_6 \wedge \beta_4.
\]
Comparing the coefficients at \( \beta_1 \wedge \beta_4 \wedge \beta_5, \beta_1 \wedge \beta_5 \wedge \beta_6, \) and \( \beta_1 \wedge \beta_6 \wedge \beta_4 \), we obtain
\[
\begin{vmatrix}
c_{21} & c_{44} & c_{45} \\
c_{31} & c_{54} & c_{55} \\
c_{11} & c_{64} & c_{65}
\end{vmatrix}
= 1,
\begin{vmatrix}
c_{21} & c_{45} & c_{46} \\
c_{31} & c_{55} & c_{56} \\
c_{11} & c_{65} & c_{66}
\end{vmatrix}
= 0,
\begin{vmatrix}
c_{21} & c_{46} & c_{44} \\
c_{31} & c_{56} & c_{54} \\
c_{11} & c_{66} & c_{64}
\end{vmatrix}
= 0.
\]
Let us introduce the vectors
\[
z = (c_{21}, c_{31}, c_{11}), z_4 = (c_{44}, c_{54}, c_{64}), z_5 = (c_{45}, c_{55}, c_{65}), z_6 = (c_{46}, c_{56}, c_{66}).
\]
It is obvious that the vectors \( z_4, z_5, z_6 \) are linearly independent. The last two determinant identities show that \( z \) is a linear combination of \( z_5 \) and \( z_6 \) as well as a linear combination of \( z_6 \) and \( z_4 \). This implies that \( z \) is a multiple of \( z_6 \), i.e. \( z = \tau z_6 \).
From the first determinant identity we get then
\[
\begin{vmatrix}
c_{46} & c_{44} & c_{45} \\
c_{56} & c_{54} & c_{55} \\
c_{66} & c_{64} & c_{65}
\end{vmatrix}
= 1.
\]
We denote
\[
\delta = \begin{vmatrix}
  c_{44} & c_{45} & c_{46} \\
  c_{54} & c_{55} & c_{56} \\
  c_{64} & c_{65} & c_{66}
\end{vmatrix}.
\]

From the identity \( z = \tau z_6 \) we get
\[
c_{11} = c_{66} \cdot \delta^{-1}, \quad c_{21} = c_{46} \cdot \delta^{-1}, \quad c_{31} = c_{56} \cdot \delta^{-1}.
\]

Comparing coefficients at the monomials \( \beta_2 \wedge \beta_4 \wedge \beta_5, \beta_2 \wedge \beta_5 \wedge \beta_6, \) and \( \beta_2 \wedge \beta_6 \wedge \beta_4 \) we obtain along the same lines as above
\[
c_{12} = c_{64} \cdot \delta^{-1}, \quad c_{22} = c_{44} \cdot \delta^{-1}, \quad c_{32} = c_{54} \cdot \delta^{-1}.
\]

Further, comparing coefficients at the monomials \( \beta_3 \wedge \beta_4 \wedge \beta_5, \beta_3 \wedge \beta_5 \wedge \beta_6, \) and \( \beta_3 \wedge \beta_6 \wedge \beta_4 \) we have
\[
c_{13} = c_{65} \cdot \delta^{-1}, \quad c_{23} = c_{45} \cdot \delta^{-1}, \quad c_{33} = c_{55} \cdot \delta^{-1}.
\]

It remains to compare coefficients at \( \beta_4 \wedge \beta_5 \wedge \beta_6 \). Here we obtain the identity
\[
(*) \quad \begin{vmatrix}
  c_{14} & c_{15} & c_{16} \\
  c_{44} & c_{45} & c_{46} \\
  c_{54} & c_{55} & c_{56}
\end{vmatrix} + \begin{vmatrix}
  c_{24} & c_{25} & c_{26} \\
  c_{54} & c_{55} & c_{56} \\
  c_{64} & c_{65} & c_{66}
\end{vmatrix} + \begin{vmatrix}
  c_{34} & c_{35} & c_{36} \\
  c_{64} & c_{65} & c_{66} \\
  c_{44} & c_{45} & c_{46}
\end{vmatrix} = 0.
\]

We have thus proved the following

5. Lemma. If \( \beta'_1, \ldots, \beta'_6 \) and \( \beta_1, \ldots, \beta_6 \) are canonical dual bases, then their transition matrix has the form
\[
\begin{pmatrix}
  c_{66} \cdot \delta^{-1} & c_{64} \cdot \delta^{-1} & c_{65} \cdot \delta^{-1} & c_{14} & c_{15} & c_{16} \\
  c_{46} \cdot \delta^{-1} & c_{44} \cdot \delta^{-1} & c_{45} \cdot \delta^{-1} & c_{24} & c_{25} & c_{26} \\
  c_{56} \cdot \delta^{-1} & c_{54} \cdot \delta^{-1} & c_{55} \cdot \delta^{-1} & c_{34} & c_{35} & c_{36} \\
  0 & 0 & 0 & c_{44} & c_{45} & c_{46} \\
  0 & 0 & 0 & c_{54} & c_{55} & c_{56} \\
  0 & 0 & 0 & c_{64} & c_{65} & c_{66}
\end{pmatrix}
\]
satisfying \((*)\). If \( \beta_1, \ldots, \beta_6 \) is a canonical dual basis and \( \beta'_1, \ldots, \beta'_6 \) is a basis of \( V^* \) such that the transition matrix between both bases has the above form and satisfies \((*)\), then \( \beta'_1, \ldots, \beta'_6 \) is also a canonical dual basis.

2. Geometric properties

Now we start to consider a 6-dimensional differentiable manifold \( M \). From now on all structures will be differentiable, i.e. of class \( C^\infty \). A 3-form \( \omega \) on \( M \) will be called a form of class \( R_0 \) if for every \( x \in M \) there is an isomorphism \( h_x : T_x M \rightarrow V \) such that \( h^*_x \rho_0 = \omega_x \). (Quite analogous definitions can be introduced for other types of forms.) We consider now on \( M \) a 3-form of type \( R_0 \). We get easily on \( M \) a 3-dimensional distribution \( D \) defined by \( D_x = \Delta(\omega_x) \). But here we need the following lemma.

6. Lemma. The distribution \( D \) is differentiable.
Proof. Around any point \( x \in M \) we can find a local basis \( X_1, \ldots, X_6 \) of \( TM \). We take a vector field \( X = f_1X_1 + \cdots + f_6X_6 \), where \( f_1, \ldots, f_6 \) are (locally defined) differentiable functions. To find differentiable vector fields \( Y_1, Y_2, Y_3 \) which span the distribution \( D \) it is necessary to solve the equation \((i_X \omega) \wedge \omega = 0\). This leads to a system of six linear homogeneous equations the coefficients of which are differentiable functions. The rest of the proof is then completely standard. \( \square \)

7. Definition. A local basis \( X_1, \ldots, X_6 \) of \( TM \) around a point \( x \in M \) is called local canonical basis for \( \omega \) if the following conditions are satisfied

\[
\begin{align*}
\alpha(X_1, X_2, X_3) &= 0, \quad \alpha(X_i, X_j, X_k) = 0 \quad \text{for} \quad 1 \leq i < j \leq 3, \quad k = 4, 5, 6, \\
\alpha(X_1, X_4, X_5) &= 1, \quad \alpha(X_1, X_5, X_6) = 0, \quad \alpha(X_1, X_6, X_4) = 0, \\
\alpha(X_2, X_4, X_5) &= 0, \quad \alpha(X_2, X_5, X_6) = 1, \quad \alpha(X_2, X_6, X_4) = 0, \\
\alpha(X_3, X_4, X_5) &= 0, \quad \alpha(X_3, X_5, X_6) = 0, \quad \alpha(X_3, X_6, X_4) = 1, \\
\alpha(X_4, X_5, X_6) &= 0.
\end{align*}
\]

8. Proposition. Around every point \( x \in M \) there exists a canonical basis for the 3-form \( \omega \).

Proof. We choose first a complement \( D_c \) of \( D \) in \( TM \). This complement is also a differentiable distribution. In this complement we take locally three linearly independent vector fields \( Y_4, Y_5, Y_6 \). We denote \( f = \omega(Y_4, Y_5, Y_6) \). Because the form \( \omega_x \) is regular, there is \( v_0 \in D_x \) such that \( \omega_x(v_0, Y_{5,x}, Y_{6,x}) = b \neq 0 \). Then we take a vector field \( Y_0 \) around \( x \) lying in \( D \) such that \( X_{0,x} = v_0 \). Obviously, then \( \omega(Y_0, Y_5, Y_6) = g \) is non-zero in a neighborhood of \( x \). Taking \( X_4 = Y_4 - (f/g)Y_0, \quad X_5 = Y_5, \quad \text{and} \quad X_6 = Y_6 \) we get

\[
\begin{align*}
\omega(X_4, X_5, X_6) &= \alpha(Y_4 - (f/g)Y_0, Y_5, Y_6) \\
&= \omega(Y_4, Y_5, Y_6) - (f/g)\omega(Y_0, Y_5, Y_6) = f - (f/g)g = 0.
\end{align*}
\]

Now we have in a neighborhood of \( x \in M \) three 1-forms, namely the forms \( \omega(\cdot, X_4, X_5), \quad \omega(\cdot, X_5, X_6), \quad \text{and} \quad \omega(\cdot, X_6, X_4) \). The regularity of \( \omega_x \) implies again that these three forms are linearly independent. Consequently, there are uniquely determined vector fields \( X_1, X_2, X_3 \) in \( D \) such that

\[
\begin{align*}
\omega(X_1, X_4, X_5) &= 1, \quad \omega(X_1, X_5, X_6) = 0, \quad \omega(X_1, X_6, X_4) = 0, \\
\omega(X_2, X_4, X_5) &= 0, \quad \omega(X_2, X_5, X_6) = 1, \quad \omega(X_2, X_6, X_4) = 0, \\
\omega(X_3, X_4, X_5) &= 0, \quad \omega(X_3, X_5, X_6) = 0, \quad \omega(X_3, X_6, X_4) = 1.
\end{align*}
\]

The equations \( \omega(X_1, X_2, X_3) = 0 \) and \( \omega(X_i, X_j, X_k) = 0 \) for \( 1 \leq i < j \leq 3, \quad k = 4, 5, 6 \) are again satisfied automatically by virtue of Lemma 2. This finishes the proof. \( \square \)

Now it suffices to take dual 1-forms \( \omega_1, \ldots, \omega_6 \) to the vector fields \( X_1, \ldots, X_6 \) and we get the following proposition.
9. **Proposition.** For a 3-form $\omega$ of type $R_0$ on $M$ locally there exist 1-forms $\omega_1, \ldots, \omega_6$ such that

$$\omega = \omega_1 \wedge \omega_4 \wedge \omega_5 + \omega_2 \wedge \omega_5 \wedge \omega_6 + \omega_3 \wedge \omega_6 \wedge \omega_4.$$ 

10. **Example.** On $\mathbb{R}^6$ let us consider the 3-form

$$\omega = dx_1 \wedge (dx_4 + x_1 dx_3) \wedge dx_5 + dx_2 \wedge dx_5 \wedge dx_6 + dx_3 \wedge dx_6 \wedge (dx_4 + x_1 dx_3).$$

We have

$$d\omega = dx_1 \wedge dx_1 \wedge dx_3 \wedge dx_5 + dx_3 \wedge dx_6 \wedge dx_1 \wedge dx_3 = 0.$$ 

On the other hand the distribution $D = \Delta(\omega)$ is spanned by the vector fields

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}$$

and is not integrable. This shows that the closeness of the 3-form $\omega$ does not imply the integrability of the associated distribution $\Delta(\omega)$.

We shall need a version of the Poincaré lemma. On $\mathbb{R}^6$ we take coordinates $(x_1, \ldots, x_6)$ and consider an integrable 3-dimensional distribution $D$ defined by the equations $dx_4 = dx_5 = dx_6 = 0$.

11. **Lemma.** Let $\theta$ be a 2-form on $\mathbb{R}^6$ such that $d\theta = 0$ and $\theta \mid D = 0$. Then there exists a 1-form $\eta$ on $\mathbb{R}^6$ such that $\theta = d\eta$ and $\eta \mid D = 0$.

**Proof.** We denote $\Omega^k$ the vector space of $k$-forms on $\mathbb{R}^6$ and $Z(\Omega^k)$ the subspace consisting of closed forms. It is well known that there exists a linear maping $E : Z(\Omega^2) \rightarrow \Omega^1$ such that for every $\xi \in Z(\Omega^2)$ there is $\xi = dE(\xi)$. The problem is that $E(\theta)$ need not satisfy $E(\theta) \mid D = 0$. But we have

$$dE(\theta) \mid D = \theta \mid D = 0.$$ 

On any leaf $L(c_4, c_5, c_6)$ of the distribution $D$ (i.e. $x_4 = c_4$, $x_5 = c_5$, $x_6 = c_6$) we can again apply the Poincaré lemma and we find that there exists on $L(c_4, c_5, c_6)$ a function $f(c_4, c_5, c_6)$ such that $E(\theta) \mid L(c_4, c_5, c_6) = df(c_4, c_5, c_6)$. Of course, this does not solve our problem. But we can use an obvious parametric version of the Poincaré lemma. We can consider $\mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$. On $\mathbb{R}^3$ we take a family of 1-forms $\zeta_{c_4, c_5, c_6}$ depending on three parameters $c_4, c_5, c_6$. Namely, the 1-form $\zeta_{c_4, c_5, c_6}$ with parameters $c_4, c_5, c_6$ is the form $E(\theta) \mid L(c_4, c_5, c_6)$ transferred to $\mathbb{R}^3$ under the natural identification $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, c_4, c_5, c_6)$. Now the Poincaré lemma with three parameters gives us a three parametric system of functions $f_{c_4, c_5, c_6}$ on $\mathbb{R}^3$ such that $\zeta_{c_4, c_5, c_6} = df_{c_4, c_5, c_6}$. In other words this means that the function $f(x_1, x_2, x_3, x_4, x_5, x_6) = f_{x_4, x_5, x_6}(x_1, x_2, x_3)$ satisfies

$$E(\theta) \mid D = df \mid D.$$ 

Taking now $\eta = E(\theta) - df$ we can see that $d\eta = \theta$ and $\eta \mid D = 0$. 

Let us recall now the following definition.

12. **Definition.** A 3-form $\omega$ of type $R_0$ on a manifold $M$ is called integrable if locally there exist coordinates $x_1, \ldots, x_6$ such that

$$\omega = dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_5 \wedge dx_6 + dx_3 \wedge dx_6 \wedge dx_4.$$
It is obvious that if the 3-form \( \omega \) is integrable then \( \omega \) is closed and the associated distribution \( \Delta(\omega) \) is integrable. Now we are going to prove that these two conditions are also sufficient for the integrability.

13. **Theorem.** A 3-form \( \omega \) of type \( R_3 \) on a manifold \( M \) is integrable if and only if the following two conditions are satisfied

(1) \( d\omega = 0 \),

(2) the distribution \( D = \Delta(\omega) \) is integrable.

**Proof.** We must show that the conditions are sufficient. According to Proposition 9 around every point \( x \in M \) we can find 1-forms \( \omega''_1', \ldots, \omega''_n' \) such that

\[
\omega = \omega''_1' \land \omega''_2' \land \omega''_3' \land \omega''_n' \land \omega''_j' \land \omega''_k' \land \omega''_l' \land \omega''_m' .
\]

Because \( \Delta(\omega) \) is integrable, we can find three functions \( f_1', f_2', f_3' \) such that their differentials \( df_1', df_2', df_3' \) are linearly independent and \( df_1' | D = df_2' | D = df_3' | D = 0 \). Then using Lemma 11 we can find 1-forms \( \omega_1', \omega_2', \omega_3' \) such that

\[
\omega = \omega_1' \land df_1' + \omega_2' \land df_2' + \omega_3' \land df_3' .
\]

We denote \( X_1', \ldots, X_n' \) the canonical basis associated to the canonical dual basis \( \omega_1', \omega_2', \omega_3', df_1', df_2', df_3' \). Obviously, we have

(4) \( 0 = d\omega = d\omega_1' \land df_1' + d\omega_2' \land df_2' + d\omega_3' \land df_3' \).

Applying \( \iota_{X_2'} \iota_{X_1'} \) on both sides we get

\[
0 = (\iota_{X_2'} \iota_{X_1'} d\omega_1') \cdot df_1' \land df_2' + (\iota_{X_2'} \iota_{X_1'} d\omega_2') \cdot df_2' \land df_3' + (\iota_{X_2'} \iota_{X_1'} d\omega_3') \cdot df_3' \land df_4' .
\]

This shows that \( d\omega_1'(X_1', X_2') = d\omega_2'(X_1', X_2') = d\omega_1'(X_1', X_2') = 0 \). Similarly we find that \( d\omega_1'(X_2', X_3') = d\omega_2'(X_2', X_3') = d\omega_3'(X_2', X_3') = 0 \) and \( d\omega_1'(X_3', X_1) = d\omega_2'(X_3', X_1) = d\omega_3'(X_3', X_1) = 0 \). We have thus proved that

\[
d\omega_1' | D = d\omega_2' | D = d\omega_3' | D = 0 .
\]

Consequently \( d\omega_1' \) must have the following form

\[
d\omega_1' = g_{114} \omega_1' \land df_1' + g_{115} \omega_1' \land df_2' + g_{116} \omega_1' \land df_3' + g_{124} \omega_2' \land df_1' + g_{125} \omega_2' \land df_2' + g_{126} \omega_2' \land df_3' + g_{134} \omega_3' \land df_1' + g_{135} \omega_3' \land df_2' + g_{136} \omega_3' \land df_3' + g_{145} \omega_4' \land df_1' + g_{156} \omega_5' \land df_2' + g_{164} \omega_6' \land df_3' .
\]

Similar formulas we can write for \( d\omega_2' \) and \( d\omega_3' \). Now taking into account the equation (4) we find the following identities.

\[
g_{114} + g_{214} + g_{315} = 0 ,
\]
\[
g_{124} + g_{224} + g_{325} = 0 ,
\]
\[
g_{134} + g_{234} + g_{335} = 0 .
\]

Let us consider now the 2-form \( d\omega_1' \). This form is closed because \( d \omega_1' | D = 0 \). According to Lemma 11 there exists a 1-form \( \theta_1 \) such that \( \theta_1 | D = 0 \) and \( d \theta_1 = d\omega_1' \). Again similar considerations are possible with the 2-forms \( d\omega_2' \) and
We can see that the function \( d\omega_3 \). In this way we obtain three 1-forms \( \theta_1, \theta_2, \) and \( \theta_3 \), which can be expressed in the form

\[
\begin{align*}
\theta_1 &= h_{14}d_4' + h_{15}d_5' + h_{16}d_6', \\
\theta_2 &= h_{24}d_4' + h_{25}d_5' + h_{26}d_6', \\
\theta_3 &= h_{34}d_4' + h_{35}d_5' + h_{36}d_6'.
\end{align*}
\]

The 1-forms \( \omega_1' - \theta_1, \omega_2' - \theta_2, \) and \( \omega_3' - \theta_3 \) are closed and consequently we can find functions \( f_1', f_2', f_3' \) such that \( \omega_1' - \theta_1 = df_1', \omega_2' - \theta_2 = df_2', \) and \( \omega_3' - \theta_3 = df_3' \). Now it is obvious that the functions \( f_1', \ldots, f_6' \) represent a local coordinate system. The local dual basis \( df_1', df_2', df_3', df_4', df_5', df_6' \) is a relatively good basis, but unfortunately it need not be a canonical basis. The transition matrix from the canonical basis \( \omega_1', \omega_2', \omega_3', df_4', df_5', df_6' \) to the last basis is

\[
\begin{pmatrix}
1 & 0 & 0 & -h_{14} & -h_{15} & -h_{16} \\
0 & 1 & 0 & -h_{24} & -h_{25} & -h_{26} \\
0 & 0 & 1 & -h_{34} & -h_{35} & -h_{36} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and it may happen that \( h_{16} + h_{24} + h_{35} \neq 0 \).

Considering the equations \( d\omega_1' = d\theta_1, d\omega_2' = d\theta_2, \) and \( d\omega_3' = d\theta_3 \) we get the identities

\[
\begin{align*}
X_1'h_{16} &= g_{116}, & X_2'h_{16} &= g_{126}, & X_3'h_{16} &= g_{136}, \\
X_1'h_{24} &= g_{214}, & X_2'h_{24} &= g_{224}, & X_3'h_{24} &= g_{234}, \\
X_1'h_{35} &= g_{315}, & X_2'h_{35} &= g_{325}, & X_3'h_{35} &= g_{335}.
\end{align*}
\]

Hence we obtain

\[
\begin{align*}
X_1(h_{16} + h_{24} + h_{35}) &= g_{116} + g_{214} + g_{315} = 0, \\
X_2(h_{16} + h_{24} + h_{35}) &= g_{126} + g_{224} + g_{325} = 0, \\
X_3(h_{16} + h_{24} + h_{35}) &= g_{136} + g_{234} + g_{335} = 0.
\end{align*}
\]

We can see that the function \( h = h_{16} + h_{24} + h_{35} \) is constant on the leaves of the foliation associated with the distribution \( D \). In our coordinate system \( f_1', \ldots, f_6' \) this means that \( h \) is a function of variables \( f_4', f_5', f_6' \) only. We can choose a function \( l \) of variables \( f_4', f_5', f_6' \) only such that \( \partial l/\partial f_6' = h \). Now we take a dual basis in the form

\[
df_1' + dl, df_2', df_3', df_4', df_5', df_6'.
\]
The transition matrix of this basis with respect to the basis \( \omega'_1, \omega'_2, \omega'_3, df'_4, df'_5, df'_6 \) is

\[
\begin{pmatrix}
1 & 0 & 0 & -h_{14} + \partial l/\partial f'_4 & -h_{15} + \partial l/\partial f'_5 & -h_{16} + h \\
0 & 1 & 0 & -h_{24} & -h_{25} & -h_{26} \\
0 & 0 & 1 & -h_{34} & -h_{35} & -h_{36} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and obviously satisfies the condition \((*)\). This implies that the dual basis \( df'_1 + dl, df'_2, df'_3, df'_4, df'_5, df'_6 \) is canonical. Now it suffices to set \( f_1 = f'_1 + l, f_2 = f'_2, f_3 = f'_3, f_4 = f'_4, f_5 = f'_5, f_6 = f'_6 \) and we have

\[
\omega = df_1 \wedge df_4 \wedge df_5 + df_2 \wedge df_5 \wedge df_6 + df_3 \wedge df_6 \wedge df_4.
\]

\[ \square \]

Let us assume now that there exists on \( M \) a symmetric connection \( \nabla \) such that \( \nabla \omega = 0 \). Then using \([2, \text{Cor. 8.6}]\), we find that \( d \omega = \text{Alt}(\nabla \omega) = 0 \). Next for arbitrary vector fields \( X, X_1, X_2, Y \) we can calculate

\[
(\nabla_Y (\iota_X \omega)(X_1, X_2)) = Y((\iota_X \omega)(X_1, X_2)) - (\iota_X \omega)(\nabla_Y X_1, X_2) - (\iota_X \omega)(X_1, \nabla_Y X_2)
\]

\[
= Y(\omega(X, X_1, X_2)) - \omega(X, \nabla_Y X_1, X_2) - \omega(X_1, \nabla_Y X_2)
\]

\[
= (\nabla_Y \omega)(X, X_1, X_2) + \omega(\nabla_Y X_1, X_2) + \omega(X, \nabla_Y X_1, X_2)
\]

\[
+ \omega(X, X_1, \nabla_Y X_2) - \omega(X_1, \nabla_Y X_2) - \omega(X, \nabla_Y X_2)
\]

\[
= \omega(\nabla_Y X, X_1, X_2) = (\iota_{\nabla_Y X} \omega)(X_1, X_2).
\]

Now let us assume that a vector field \( X \) lies in the distribution \( D \). We have then \((\iota_X \omega) \wedge \omega = 0\) and consequently

\[
0 = \nabla_Y ((\iota_X \omega) \wedge \omega) = (\nabla_Y (\iota_X \omega)) \wedge \omega = (\iota_{\nabla_Y X} \omega) \wedge \omega,
\]

which show that \( \nabla \) preserves the distribution \( D \). Because the connection \( \nabla \) is symmetric, this implies that the distribution \( D \) is integrable. Together this means that the 3-form \( \omega \) is integrable. We will see that the converse is also true.

14. **Theorem.** A 3-form \( \omega \) of type \( R_0 \) on a paracompact manifold \( M \) is integrable if and only if there exists on \( M \) a symmetric connection \( \nabla \) such that \( \nabla \omega = 0 \).

**Proof.** We must prove that if \( \omega \) is integrable then there exists a symmetric connection \( \nabla \) such that \( \nabla \omega = 0 \). We can cover \( M \) by a locally finite open covering of \( M \) consisting of charts \( \{U^\lambda\}_{\lambda \in I} \) with coordinates \( x_1^\lambda, \ldots, x_6^\lambda \) such that on \( U^\lambda \) we have

\[
\omega = dx_1^\lambda \wedge dx_4^\lambda \wedge dx_5^\lambda + dx_2^\lambda \wedge dx_5^\lambda \wedge dx_6^\lambda + dx_3^\lambda \wedge dx_6^\lambda \wedge dx_4^\lambda.
\]

On each \( U^\lambda \) we take a connection \( \nabla^\lambda \) defined by

\[
\nabla^\lambda_{\partial/\partial x_i^\lambda}(\partial/\partial x_j^\lambda) = 0, \quad i, j = 1, \ldots, 6.
\]
It is obvious that this connection is symmetric and satisfies $\nabla^\lambda \omega = 0$. Now it remains to glue these connections together. We take a partition of unity $\{a^\lambda\}_{\lambda \in I}$ subordinate to the covering $\{U^\lambda\}_{\lambda \in I}$. Then it suffices to define

$$\nabla = \sum_{\lambda \in I} a^\lambda \nabla^\lambda,$$

and we have on $M$ a symmetric connection satisfying $\nabla \omega = 0$. $\square$

References
