ON THE EXISTENCE OF NON-LINEAR FRAMES

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Abstract. A stronger version of the notion of frame in Banach space called Strong Retro Banach frame (SRBF) is defined and studied. It has been proved that if $X$ is a Banach space such that $X^*$ has a SRBF, then $X$ has a Bi-Banach frame with some geometric property. Also, it has been proved that if a Banach space $X$ has an approximative Schauder frame, then $X^*$ has a SRBF. Finally, the existence of a non-linear SRBF in the conjugate of a separable Banach space has been proved.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in the context of nonharmonic Fourier series. Frames now a days are widely used in various branches of mathematics and engineering. Feichtinger and Grochenig [6] generalized the notion of frame to Banach spaces and introduced the concept of atomic decomposition in a Banach space. Also, Grochenig [7] introduced a more general concept namely Banach frame for Banach spaces. For a nice and comprehensive survey of frames and related concepts one may refer to [1, 4].

Various other generalizations of frames for Banach spaces were defined and studied by many authors namely Schauder frames by Han and Larson [8] and also studied by Casazza et al. [2, 3], frames by Terekhin [17]. Banach frames in conjugate Banach spaces, called retro Banach frames, were introduced and studied by Jain et al. [9] and further studied in [13]. Approximative atomic decompositions in Banach spaces were studied in [10]. Schauder frames in conjugate Banach spaces were defined and studied in [12] while approximative Schauder frames were studied in [11]. The notion of Bi-Banach frame in a Banach space was defined and studied in [14] wherein they noted that a Schauder frame for a Banach space is a Bi-Banach frame but the converse is not true.

In the present paper, we shall consider a stronger notion of frame in a Banach space called strong Retro Banach frame (SRBF). It has been proved that if $X$ is a Banach space such that $X^*$ has a SRBF, then $X$ has a Bi-Banach frame with some geometric property. Also, it has been proved that if a Banach space $X$ has an approximative Schauder frame, then $X^*$ has a SRBF. Finally, a result related to

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the existence of a non-linear SRBF in the conjugate of a separable Banach space has been proved.

Throughout this paper $\mathcal{X}$ will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R}, \mathbb{C})$, $\mathcal{X}^*$ denotes the conjugate space of $\mathcal{X}$ and $L(\mathcal{X}, \mathcal{X})$ denote the Banach space of all continuous linear mappings of $\mathcal{X}$ into $\mathcal{X}$. For a sequence $\{x_n\} \subset \mathcal{X}$ and $\{f_n\} \subset \mathcal{X}^*$, $\{x_n\}$ denotes the closed linear span of $\{x_n\}$ in the norm topology of $\mathcal{X}$ and $[\{f_n\}]$ the closed linear span of $\{f_n\}$ in the weak star topology of $\mathcal{X}^*$. A sequence space $S$ is called a BK-space if it is a Banach space and the co-ordinate functionals are continuous on $S$. That is the relations $x_n = \{\alpha_j^{(n)}\}$, $x = \{\alpha_j\} \subset S$, $\lim_{n \to \infty} x_n = x$ imply $\lim_{n \to \infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, 3, \ldots)$. Also, if $V \subset \mathcal{X}^*$, then we define $\gamma(v) = \inf_{x \in \mathcal{X}} \sup_{f \in V, \|f\| \leq 1} |f\left(\frac{x}{\|x\|}\right)|$.

A sequence $\{x_n\} \subset \mathcal{X}$ is said to be a Markusevic basis (M-basis) for $\mathcal{X}$ if $\{x_n\}$ is complete in $\mathcal{X}$ and there exists a sequence $\{f_n\}$ in $\mathcal{X}^*$ biorthogonal to $\{x_n\}$, called an associated sequence of coefficient functional (a.s.c.f.), which is total on $\mathcal{X}$.

**Definition 1.1** ([9]). Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{x_n\} \subset \mathcal{X}$ and $J: \mathcal{X}_d \to \mathcal{X}^*$ be given. The pair $(\{x_n\}, J)$ is called a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{X}_d^*$ if

(a) $\{f(x_n)\} \subset \mathcal{X}_d^*$, for all $f \in \mathcal{X}^*$.

(b) There exist positive constants $A$ and $B$ with $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathcal{X}^*} \leq \|\{f(x_n)\}\|_{\mathcal{X}_d^*} \leq B\|f\|_{\mathcal{X}^*}, \quad \text{for all } f \in \mathcal{X}^*. \tag{1.1}$$

(c) $J$ is a bounded linear operator such that

$$J(\{f(x_n)\}) = f, \quad \text{for all } f \in \mathcal{X}^*.$$
Lemma 1.4 (16). Let $\mathcal{X}$ be a separable normed linear space and let $\{x_n^*\}$ be a sequence in $\mathcal{X}^*$ such that $\frac{x_n^*}{\|x_n^*\|} \rightharpoonup 0$ and that for the linear subspace $[x_n^*]$ of $\mathcal{X}^*$, $\gamma([x_n^*]) \geq 0$. Then there exist a norm $|\cdot|$ on $\mathcal{X}$ equivalent to the initial norm on $\mathcal{X}$ such that $(\mathcal{X}, |\cdot|)$ is strictly convex and satisfies the following property

\[(1.2) \quad \text{If } \lim_{n \to \infty} f_k(x_n) = f_k(x_0)(k = 1, 2, \ldots), \text{ then } \lim_{n \to \infty} |x_n| \geq |x_0|.\]

2. Main result

Approximative Schauder frames in Banach spaces were studied in [11] and the notion of Bi-Banach frame was studied in [14]. In the following definition, we gave a stronger notion called Strong Retro Banach frame (SRBF). The idea of defining this notion is to correlate this notion with the existing notions like approximative Schauder frames and Bi-Banach frames.

Definition 2.1. Let $\{x_n\} \subset \mathcal{X}$ be an exact RBF for $\mathcal{X}^*$ with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Let $X_n = \{x_1, x_2, \ldots, x_n\}$, $n \in \mathbb{N}$. If there exists a sequence $\{v_n\}$, where each $v_n: X_n \to X_n$ is a continuous linear mapping, such that $x = \lim_{n \to \infty} v_n \sum_{i=1}^{n} f_i(x)x_i$, $x \in \mathcal{X}$, then $\{\{x_n\}, \{f_n\}, \{v_n\}\}$ is called a strong RBF (or SRBF) for $\mathcal{X}^*$.

Remark 2.2. If we define $u_n: \mathcal{X} \to \mathcal{X}$, $n \in \mathbb{N}$ by

$$u_n(x) = v_n \sum_{i=1}^{n} f_i(x)x_i, \quad n \in \mathbb{N}.$$  

Then one may observe that if $\{\{x_n\}, \{f_n\}, \{v_n\}\}$ is a SRBF, then $\lim_{n \to \infty} u_n(x) = x$ and $\dim u_n(\mathcal{X}) = \dim (\sum_{i=1}^{n} f_i(x)x_i)(\mathcal{X}) \leq n < \infty$ and so $\{x_n\}$ is an approximative basis of $\mathcal{X}$.

In the following result, we prove that the existence of SRBF in the conjugate of a Banach space guarantees the existence of a Bi-Banach frame in the Banach space along with some geometric property.

Theorem 2.3. Let $X$ be a Banach space and $\{\{x_n\}, \{f_n\}, \{v_n\}\}$ be a SRBF for $\mathcal{X}^*$ with admissible sequence $\{f_n\} \subset \mathcal{X}^*$. Then $\{\{x_n\}, \{f_n\}\}$ is a Bi-Banach frame for $\mathcal{X}$ such that $\gamma([f_n]) > 0$.

Proof. Clearly, by definition of SRBF, $\{x \in \mathcal{X} : f_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$. Therefore, by Lemma 1.3, there exists an associated Banach space $\mathcal{X}_d = \{\{f_n(x)\}\}; x \in \mathcal{X}$ with norm given by $\|\{f_n(x)\}\|_d = \|x\|_\mathcal{X}, x \in \mathcal{X}$. Define $J: \mathcal{X}_d \to \mathcal{X}$ by $J(\{f_n(x)\}) = x$, $x \in \mathcal{X}$. Then $J$ is a bounded linear operator such that $\{\{f_n\}, J\}$ is a Banach frame for $\mathcal{X}$. Hence $\{\{x_n\}, \{f_n\}\}$ is a Bi-Banach frame for $\mathcal{X}$. Let for each $n \in \mathbb{N}$, $v_n: X_n \to X_n$ be a continuous linear mapping given by

$$\lim_{n \to \infty} v_n \sum_{i=1}^{n} f_i(x)x_i = x.$$  

Write $v_n(x_j) = \sum_{i=1}^{n} a_{ji}^{(n)} x_i$, $j = 1, 2, \ldots, n$, $n \in \mathbb{N}$, where
\[ a_{ij}^{(n)} = f_i(v_n(x_j)), \text{ for all } i, j = 1, 2, 3, \ldots, n, \ n \in \mathbb{N}. \]

Thus
\[ v_n \left( \sum_{i=1}^{n} f_i(x)x_i \right) = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ij}^{(n)} f_i(x) \right)x_j, \quad x \in \mathcal{X}, \ n \in \mathbb{N}. \]

Define
\[ h_{n,j} = \sum_{i=1}^{n} a_{ij}^{(n)} f_i, \quad j = 1, 2, 3, \ldots, n, \ n \in \mathbb{N}. \]

Then \[ h_{n,i} \in [f_i]_{i=1}^{n} \quad (j = 1, 2, \ldots, n; \ n \in \mathbb{N}). \] Hence, we conclude that \( \gamma([f_n]) > 0. \)

In the following result, we prove a weak duality type result.

**Theorem 2.4.** Let \( \{x_n\}, \{f_n\}, \{v_n\} \) be a SRBF for \( \mathcal{X}^* \) with admissible sequence \( \{f_n\} \subset \mathcal{X}^* \). Then there exists a sequence of continuous linear mappings \( \{\tau_n\} \) (\( \tau_n : V_n \to V_n \), where \( V_n = [f_1, f_2, \ldots, f_n], \ n \in \mathbb{N} \)) such that
\[
\lim_{n \to \infty} \left( \tau_n \sum_{i=1}^{n} f_i(x)f_i(x) \right) = \lim_{n \to \infty} \left( \tau_n \sum_{i=1}^{n} f_i(x)f_i(x) \right).
\]

**Proof.** For each \( k = 1, 2, 3, \ldots, n, \ n \in \mathbb{N} \), define \( \tau_n : \text{span} \{f_1, f_2, \ldots, f_n\} \to \text{span} \{f_1, f_2, \ldots, f_n\} \) by
\[
\tau_n(f_k) = \sum_{i=1}^{n} a_{ik}^{(n)} f_i = \sum_{i=1}^{n} f_k(v_n(x_i))f_i.
\]

Extend each \( \tau_n \) to \( [f_1, f_2, \ldots, f_n] \). Then
\[
\left( \tau_n \sum_{i=1}^{n} f_i(x)f_i(x) \right) = \sum_{i=1}^{n} f_i(x)\left( \tau_n(f_i) \right)(x)
= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ji}^{(n)} f_j(x) \right)f(x_i)
= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} f_i(v_n(x_j))f_j(x) \right)f(x_i)
= \sum_{i=1}^{n} v_n \left( \sum_{j=1}^{n} f_i(x_j)f_j(x) \right)f(x_i)
= f\left( v_n \left( \sum_{i=1}^{n} f_i(x)x_i \right) \right)
\to f(x) \quad \text{as} \quad n \to \infty.
\]

Approximative Schauder frames were defined and studied in [11]. In the following result, we prove that if a Banach space \( \mathcal{X} \) has an approximative Schauder frame, then its dual space has a SRBF.
Theorem 2.5. If a Banach space $\mathcal{X}$ has an approximative Schauder frame, then $\mathcal{X}^*$ has a SRBF.

Proof. Let $\{u_n\}$ be a sequence of finite rank continuous linear mapping from $\mathcal{X}$ to $\mathcal{X}$ such that $\lim_{n \to \infty} u_n(x) = x$, $x \in \mathcal{X}$. Let $\{x_n\}$ be a Markusevic basis for $\mathcal{X}$ with $a.s.c.f. \{f_n\} \subset \mathcal{X}^*$ such that

$$\bigcup_n u_n^*(\mathcal{X}^*) \subset [f_n]. \quad (2.1)$$

Since each $u_n$ is finite dimensional, we may write

$$u_n(x) = \sum_{i=1}^{p_n} \psi_{ni}(x)\phi_{ni}, \quad x \in \mathcal{X}, \ n \in \mathbb{N},$$

where $\{\phi_{ni}\}_{i=1}^{p_n}$ is a basis for $\{u_n(\mathcal{X})\}$ with associated sequence $\{\psi_{ni}\}_{i=1}^{p_n} \subset \mathcal{X}^*$. Let $\{g_{nj}\}_{j=1}^{p_n}$ be a sequence in $\mathcal{X}^*$ that is biorthogonal to $\{\phi_{ni}\}_{i=1}^{p_n}$. Then

$$u_n^*(g_{nj})(x) = g_{nj}\left(u_n(x)\right)$$

$$= g_{nj}\left(\sum_{i=1}^{p_n} \psi_{ni}(x)\phi_{ni}\right)$$

$$= \sum_{i=1}^{p_n} \psi_{ni}(x)g_{nj}(\phi_{ni})$$

$$= \psi_{nj}(x), \quad j = 1, 2, \ldots, p_n.$$

Hence, $\psi_{nj} \in [f_n], \ j = 1, 2, \ldots, p_n, \ n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be given. Then, for any $\epsilon > 0$, there exists an integer $m_n(\epsilon)$ such that for each $i = 1, 2, \ldots, p_n$ one can find $\overline{\phi}_{ni} \in [x_1, x_2, \ldots, x_{m_n}]$ and $\overline{\psi}_{ni} \in [f_1, f_2, \ldots, f_{m_n}]$ such that

$$\|\phi_{ni} - \overline{\phi}_{ni}\| < \epsilon \quad \text{and} \quad \|\psi_{ni} - \overline{\psi}_{ni}\| < \epsilon, \quad i = 1, 2, 3, \ldots, p_n. \quad (2.2)$$

Write

$$\overline{v}_{m_n}(x) = \sum_{i=1}^{p_n} \overline{\psi}_{ni}(x)\overline{\phi}_{ni}, \quad x \in \mathcal{X}.$$ 

Then

$$\|\overline{v}_{m_n}(x) - u_n(x)\| = \left\|\sum_{i=1}^{p_n} (\overline{\psi}_{ni}(x) - \psi_{ni}(x))\overline{\phi}_{ni} + \sum_{i=1}^{p_n} \psi_{ni}(x)(\overline{\phi}_{ni} - \phi_{ni})\right\|$$

$$\leq \left(\sum_{i=1}^{p_n} \|\overline{\psi}_{ni} - \psi_{ni}\|\|\overline{\phi}_{ni}\| + \sum_{i=1}^{p_n} \|\psi_{ni}\|\|\overline{\phi}_{ni} - \phi_{ni}\|\right) \|x\|, \quad x \in \mathcal{X}.$$ 

Therefore, by (2.2), taking $\epsilon = \frac{1}{n}$ and $\{m_n\}$ to be an increasing sequence, we obtain

$$\|\overline{v}_{m_n} - u_n\| < \frac{1}{n}. \quad (2.3)$$
Now, observe that

\[ f_i \left( x - \sum_{j=1}^{k} f_j(x)x_j \right) = 0, \]

for all \( x \in \mathcal{X}, \ i = 1, 2, \ldots, m_n; \ k = m_n, m_n + 1, \ldots. \)

Also, \( \overline{\psi}_{ni} \in [f_1, f_2, \ldots, f_{m_n}] \). So, for \( x \in \mathcal{X} \), we have

\[
\overline{v}_{m_n} \left( \sum_{j=1}^{k} f_j(x)x_j \right) = \sum_{i=1}^{p_n} \overline{\psi}_{ni} \left( \sum_{j=1}^{k} f_j(x)x_j \right) \overline{\phi}_{ni}
\]

\[
= \sum_{i=1}^{p_n} \overline{\psi}_{ni}(x) \overline{\phi}_{ni}
\]

\[
= \overline{v}_{m_n}(x), \text{ for all } x \in \mathcal{X} \text{ and } k \geq m_n.
\]

Therefore

\[
\lim_{n \to \infty} \overline{v}_{m_n} \left( \sum_{j=1}^{m_n} f_j(x)x_j \right) = \lim_{n \to \infty} \overline{v}_{m_n}(x)
\]

\[
= \lim_{n \to \infty} u_n(x)
\]

\[
= x, \quad x \in \mathcal{X}.
\]

Define a sequence \( \{T_n\} \) by \( T_n(x) = \sum_{i=1}^{n} f_i(x)x_i, \ n \in \mathbb{N} \). Write

\[ v_k = T_k|_{[x_1, x_2, \ldots, x_k]}, \ k = 1, 2, \ldots, m_1 - 1 \]

and

\[ v_k = \overline{v}_{m_n}|_{[x_1, x_2, \ldots, x_k]}, \quad k = m_n, m_n + 1, \ldots, m_n - 1, \quad n = 1, 2, 3 \ldots. \]

Then, each \( v_k \) is a continuous linear mapping defined on \([x_1, x_2, \ldots, x_k]\) with range given by

\[ v_k([x_1, x_2, \ldots, x_k]) = [x_1, x_2, \ldots, x_k]; \quad k = 1, 2, 3, \ldots, m_1 - 1 \]

\[ v_k([x_1, x_2, \ldots, x_k]) \subset [x_1, x_2, \ldots, x_k], \quad k = m_n, m_n + 1, \ldots, m_n + 1 - 1, \quad n \in \mathbb{N}. \]

Using (2.5) and (2.6), we obtain \( \lim_{n \to \infty} \overline{v}_{m_n}(x) = x \). Hence \( \lim_{n \to \infty} v_n(\sum_{i=1}^{n} f_i(x)x_i) = x \).

In view of the proof of Theorem 2.5, one may observe that existence of approximative Schauder frame in a Hilbert space is a sufficient condition for the space having Markusevic basis to have a SRBF. More precisely we have

**Corollary 2.6.** Let \( \mathcal{H} \) be a Hilbert space with an approximative Schauder frame. Then every Markusevic basis of \( \mathcal{H} \) give rise to a SRBF for \( \mathcal{H} \).

In the following example, we show that in general, a SRBF do not have strong duality
Example 2.7. Let $\mathcal{X}$ be a Banach space with a Schauder basis and such that $\mathcal{X}^*$ is separable but fails to have approximative property. Let $\{x_n\}$ be a shrinking Markusevic basis of $\mathcal{X}$ with associated sequence of coefficient functional $\{f_n\} \subset \mathcal{X}^*$. Define

$$u_n(x) = \sum_{i=1}^{n} f_i(x)x_i, \quad x \in \mathbb{N}.$$ 

Then $\{x_n\}$ is an approximative Schauder frame for $\mathcal{X}$ satisfying $\bigcup_{n=1}^{\infty} u_n^*(\mathcal{X}^*) \subset [f_n]$. Therefore $\{(x_n), \{f_n\}, \{v_n\}\}$ is a SRBF for $\mathcal{X}^*$. However, $\mathcal{X}^{**}$ has no SRBF.

One may observe that in Definition 2.1, each $v_n$ is linear. Now, we would like to drop this condition of linearity and in the process define non-linear SRBF.

Definition 2.8. A SRBF $\{(x_n), \{f_n\}, \{v_n\}\}$ is called non-linear SRBF if each $v_n$ is continuous but not necessarily linear.

Finally, we prove the following result related to the existence of a non-linear SRBF.

Theorem 2.9. If $\mathcal{X}$ is a separable Banach Space, then $\mathcal{X}^*$ has a non-linear SRBF.

Proof. Let $\{x_n\}$ be a Markusevic basis with a sequence of coefficient functional $\{f_n\} \subset \mathcal{X}^*$ such that $\gamma([f_n]) > 0$. Then, by Lemma 1.4 there is a norm $\| \cdot \|$ on $\mathcal{X}$ that is equivalent to the original norm $\| \cdot \|_{\mathcal{X}}$ such that $\mathcal{X}$ has strictly convex. Therefore, by Corollary 3.3, page 110, for every finite dimensional subspace $G$ of $\mathcal{X}$ and for every $x \in \mathcal{X} \setminus G$, there is a unique $\pi_G(x) \in G$ such that $|x - \pi_G(x)| = \text{dist}(x, G) = \min_{x \in G} |x - g|$ and such that the mapping $\pi_G : \mathcal{X} \to G$ is continuous (here note that, in general, $\pi_G$ is non-linear). Let $\mathfrak{N}(a, b)$ denote a positive integer depending on $a$ and $b$. For each $n$, choose an increasing sequence of positive integers $\{m_n\}$ with $m_1 = \mathfrak{N}(1, 1)$, $m_2 = \mathfrak{N}(m_1, \frac{1}{2})$, $m_3 = \mathfrak{N}(m_2, \frac{1}{3})$, $\ldots$, $m_n = \mathfrak{N}(m_{n-1}, \frac{1}{n})$, for all $n \geq 2$ and satisfying

$$\text{dist} \left( a, [x_i]_{i=m_n+1}^{m_{n+1}} \right) \leq \left( 1 + \frac{1}{n} \right) \text{dist} \left( a, [x_i]_{i=m_n+1}^{\infty} \right),$$

where $a \in [x_i]_{i=1}^{m_n-1}$. Define $\{v_n\}$ by $v_k = T_k |[x_1, \ldots, x_k]$, $k = 1, 2, \ldots, m_1 - 1$, where $T_k(x) = \sum_{i=1}^{k} f_i(x)x_i$ and for any $b = \sum_{i=1}^{k} a_i x_i \in [x_i]_{i=1}^{k}$, $(k = m_n, m_n + 1, \ldots, m_{n+1} - 1; n \in \mathbb{N})$

$$v_k(b) = \sum_{i=1}^{m_n-1} a_i x_i - \pi_G \left( \sum_{i=1}^{m_{n-1}} a_i x_i \right),$$

where $\mathcal{G} = [x_i]_{i=m_n+1}^{m_{n+1}}$. Then each $v_n$ is continuous (in general, non-linear) with range given by

$$v_k([x_1, \ldots, x_k]) = [x_1, \ldots, x_k], \quad k = 1, 2, 3, \ldots, m_1 - 1$$

$$v_k([x_1, \ldots, x_k]) \subset [x_1, \ldots, x_k], \quad (k = m_n, m_n + 1, \ldots, m_{n+1} - 1; n \in \mathbb{N}).$$
Let \( x \in X \) be any element. Then
\[
 f_i\left(v_k\left(\sum_{j=1}^{k} f_j(x) x_j\right)\right) = f_i(x), \ i = 1, 2, \ldots, m_n; \ k = m_n, m_n+1, \ldots, m_n+1-1; \ n \in \mathbb{N}.
\]
This gives
\[
 (2.7) \quad \lim_{k \to \infty} f_i\left(v_k\left(\sum_{j=1}^{k} f_j(x) x_j\right)\right) = f_i(x), \ i = 1, 2, \ldots.
\]
In view of Lemma 1.4, we have
\[
 (2.8) \quad \lim_{k \to \infty} \left| v_k\left(\sum_{i=1}^{k} f_i(x) x_i\right) \right| \geq |x|.
\]
Also, we have
\[
 v_k\left(\sum_{i=1}^{k} f_i(x) x_i\right) = \left| \sum_{i=1}^{m_n-1} f_i(x) x_i - \pi g \sum_{i=1}^{m_n-1} f_i(x) x_i \right| \\
 = \text{dist} \left( \sum_{i=1}^{m_n-1} f_i(x) x_i, G \right) \\
 \leq \left(1 + \frac{1}{n}\right) \text{dist} \left( \sum_{i=1}^{m_n-1} f_i(x) x_i, [x]_{i=m_n-1+1}^{\infty} \right) \\
 \leq \left(1 + \frac{1}{n}\right) \left| \sum_{i=1}^{m_n-1} f_i(x) x_i + \left( x - \sum_{i=1}^{m_n-1} f_i(x) x_i \right) \right| \\
 = \left(1 + \frac{1}{n}\right) |x|, \ k = m_n, m_n + 1, m_n+1 - 1; \ n \in \mathbb{N}.
\]
Thus, by (2.8), we have
\[
 (2.9) \quad \lim_{k \to \infty} \left| v_k\left(\sum_{i=1}^{k} f_i(x) x_i\right) \right| = |x|.
\]
Hence, we conclude that
\[
 \lim_{k \to \infty} \left| v_k\left(\sum_{i=1}^{k} f_i(x) x_i\right) - x \right| = 0.
\]
Since \( | \cdot | \) is equivalent to the initial norm of \( X \), we obtain
\[
 \lim_{n \to \infty} v_n\left(\sum_{i=1}^{n} f_i(x)x_i\right) = x, \ x \in X.
\]

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