GRADIENT ESTIMATES OF LI YAU TYPE FOR A GENERAL HEAT EQUATION ON RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we consider gradient estimates on complete non-compact Riemannian manifolds $(M, g)$ for the following general heat equation

$$u_t = \Delta_V u + au \log u + bu$$

where $a$ is a constant and $b$ is a differentiable function defined on $M \times [0, \infty)$. We suppose that the Bakry-Émery curvature and the $N$-dimensional Bakry-Émery curvature are bounded from below, respectively. Then we obtain the gradient estimate of Li-Yau type for the above general heat equation. Our results generalize the work of Huang-Ma (4) and Y. Li (6), recently.

1. Introduction

Recently, the weighted Laplacian on smooth metric measure spaces has been attracted by many researchers. Recall that a triple $(M, g, e^{-f} dv)$ is called a smooth metric measure space if $(M, g)$ is a Riemannian manifold, $f$ is a smooth function on $M$ and $dv$ is the volume form with respect to $g$. On smooth metric measure spaces, the weighted Laplace operator is defined by

$$\Delta_f \cdot := \Delta - \langle \nabla f, \nabla \cdot \rangle$$

where $\Delta$ is the Laplace operator on $M$. On $(M, g, e^{-f} dv)$, the Bakry-Émery curvature $\text{Ric}_f$ and the $N$-dimensional Bakry-Émery curvature $\text{Ric}_f^N$ are defined by

$$\text{Ric}_f := \text{Ric} + \text{Hess} f, \quad \text{Ric}_f^N := \text{Ric}_f - \frac{1}{N} \nabla f \otimes \nabla f$$

where $\text{Ric}$, $\text{Hess} f$ are the Ricci curvature and the Hessian of $f$ on $M$, respectively.

An important generalization of the weighted Laplace operator on Riemannian manifolds is the following operator

$$\Delta_V \cdot := \Delta + \langle V, \nabla \cdot \rangle$$

where $\nabla$ and $\Delta$ are respectively the Levi-Civita connection and the Laplace-Beltrami operator with respect to $g$, $V$ is a smooth vector field on $M$. In [1] and [6], the

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authors introduced two curvatures
\[ \text{Ric}_V := \text{Ric} - \frac{1}{2} \mathcal{L}_V g, \text{Ric}_V^N := \text{Ric} - \frac{1}{N} V \otimes V \]
where \( N \in \mathbb{N} \) is a positive constant and \( \mathcal{L}_V \) is the Lie derivative associated to the vector field \( V \). When \( V = -\nabla f \) then two curvatures \( \text{Ric}_V, \text{Ric}_V^N \) become the Bakry-Émery curvature and the \( N \)-dimensional Bakry-Émery curvature, respectively.

In this paper, let \( (M, g) \) be a Riemannian manifold and \( V \) be a smooth vector field on \( M \). We consider the following general heat equation
\[
(1.1) \quad u_t = \Delta_V u + au \log u + bu
\]
where \( a \) is a constant and \( b \) is a function defined on \( M \times [0, +\infty) \) which is differentiable on \( M \times (0, +\infty) \). When \( M \) is a compact manifold and \( b = 0 \), Li (6) studied gradient estimates of Li-Yau type for equation (1.1). His results can be considered as a generalization of the famous work of Li and Yau (5). Moreover, Li also studied gradient estimates of Hamilton type for the equation (1.1) when \( a = b = 0 \) on complete noncompact manifolds. In the general case, when \( a, b \) are constants and \( M \) is a complete noncompact manifold, Huang and Ma introduced a gradient estimate of Li-Yau type which is independent of \( K \). Here \( K > 0 \) such that \(-K\) is the lower bound of the \( N \)-dimensional Bakry-Émery curvature. Then, they derived the Gaussian lower bound of the heat kernel for the equation \( u_t = \Delta_V u \). Recently, Dung and the author investigated gradient estimates of Hamilton-Souplet-Zhang type. Our work is a generalization of the results of Huang-Ma, Y. Li and other mathematicians, see [3, 5, 6] for further discussion and the references there in.

Motivated by the above result, it is very natural for us to look for gradient estimates of Li-Yau type for the general heat equation (1.1). In this paper, under some natural conditions on the curvatures, we are able to extend the work of Huang-Ma and Li to complete noncompact manifolds. Our main theorem is as follows.

**Theorem 1.1.** Let \( (M, g) \) be a complete noncompact \( n \)-dimensional Riemannian manifold with \( \text{Ric}_V \) bounded from below by the constant \(-K := -K(2R)\), where \( R > 0 \), \( K(2R) > 0 \) in the geodesic ball \( B(p, 2R) \) centered at some fixed point \( p \in M \) and \( V \) be a smooth vector field on \( M \) such that \( |V| \leq L \) for some positive constant \( L \in \mathbb{R} \). Suppose that \( a \) is a real constant, \( b \) is a differentiable function defined on \( M \times [0, +\infty) \) and the general heat equation
\[
\frac{\partial u}{\partial t} = \Delta_V u + au \log u + bu
\]
has a positive solution \( u \) on \( M \times [0, +\infty) \). Then, for all \( x \in B(p, R) \), \( t \in (0, +\infty) \), we have
If $a \leq 0$, then
\[
\beta \frac{\|\nabla u\|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{n}{2(1-\delta)\beta} \left\{ \frac{n c_1^2}{16 \delta \beta (1 - \beta) R^2} + A + \frac{1}{t} + \frac{6 \beta \theta}{n} \right\} + \frac{\beta L^2}{(1 - \beta)N} - \frac{a}{2} + \frac{\theta \beta}{2(1 - \beta)} + \sqrt{\frac{\theta \beta (1 + \beta - a)}{n}}
\]

if $a \geq 0$, then
\[
\beta \frac{\|\nabla u\|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{n}{2(1-\delta)\beta} \left\{ \frac{n c_1^2}{16 \delta \beta (1 - \beta) R^2} + A + \frac{6 \beta \theta}{n} \right\} + \frac{\beta L^2}{(1 - \beta)N} + a + \frac{\theta \beta}{2(1 - \beta)} + \sqrt{\frac{\theta \beta (1 + \beta + a)}{n}}
\]

where $c_1$ and $c_2$ are positive constants, $\beta = e^{-2Kt}$, $0 < \delta < 1$, $\theta := \max\{|b|, |b_t|, |\nabla b|\} \in \mathbb{R}$ and $A$ is defined by
\[
A = \left( n - 1 + \sqrt{(n - 1)KR + LR} \right) c_1 + c_2 + 2c_1^2 \frac{n}{R^2}.
\]

The paper is organized as follows. In the section 2, we give a proof of Theorem 1.1. In section 3, we point out that we can recover the main theorem in [4] by using Theorem 1.1. Moreover, we also show some applications to give gradient estimates of solution of some general heat equations and prove a Harnack inequality for such a solution. This is an extension of the work of Huang-Ma and Li.

2. Gradient estimate of Li Yau type

To begin with, let us recall the following Laplacian comparison theorem in [1].

**Theorem 2.1** ([1]). Let $(M, g)$ be a complete noncompact Riemannian manifold with $\text{Ric}_V$ bounded from below by the constant $-K := -K(2R)$, where $R > 0$, $K(2R) > 0$ in the geodesic ball $B(p, 2R)$ with radius $2R$ around $p \in M$. Suppose that $V$ is a smooth vector field on $M$ satisfying $\langle V, \nabla \rho \rangle \leq v(\rho)$ for some nondecreasing function $v(\cdot)$, where $\rho(x)$ is the distance from a fixed point $p$ to the considered point $x$. Then
\[
\Delta_V \rho \leq \sqrt{(n - 1)K} + \frac{n - 1}{\rho} + v(\rho).
\]

Noting that if $v(\cdot)$ is bounded by a positive constant $L$ then we have
\[
\Delta_V \rho \leq \sqrt{(n - 1)K} + \frac{n - 1}{\rho} + L.
\]

To prove the Theorem 1.1, we first derive the following important lemma.

**Lemma 2.2.** Let $(M, g)$ be a complete noncompact Riemannian manifold with $\text{Ric}_V$ bounded from below by the constant $-K := -K(2R)$, where $R > 0$, $K(2R) > 0$ in the geodesic ball $B(p, 2R)$ with radius $2R$ around $p \in M$ and $V$ is a smooth
vector field on \(M\) such that \(|V|\) is bounded by a positive constant \(L\). For the smooth function \(w = \log u\), where \(u\) be a positive solution to (1.1) then
\[
\Delta V F - F_t \geq t \left\{ \frac{2\beta}{n} (\Delta V w)^2 + \left( \frac{-2\beta L^2}{N} + a(\beta - 1) \right) |\nabla w|^2 - 2\beta \langle \nabla w, \nabla b \rangle + b_t - ab \right\}
- 2 \langle \nabla w, \nabla F \rangle - aF - \frac{F}{t},
\]
where \(F = t(\beta |\nabla w|^2 + aw - w_t)\).

**Proof.** Let \(w = \log u\) with \(u\) be the positive solution to (1.1) then
\[
w_t = |\nabla w|^2 + \Delta V w + aw + b.
\]
Hence,
\[
\Delta V w_t = -2 \langle \nabla w, \nabla w_t \rangle - aw_t + w_{tt} - b_t.
\]
and
\[
\Delta V w = (\beta - 1)|\nabla w|^2 - \frac{F}{t} - b
\]
\[
= \left(1 - \frac{1}{\beta}\right)(-aw + w_t) - \frac{F}{t\beta} - b.
\]

Since \(\text{Ric}_V \geq -K\), \(|V| \leq L\) and \(V\text{-Bochner-Weitzenböck}\) formula (see [6]) implies
\[
\Delta_V |\nabla w|^2 \geq \frac{2}{n}(\Delta_V w)^2 - 2\left(K + \frac{L^2}{N}\right)|\nabla w|^2 + 2 \langle \nabla w, \nabla \Delta_V w \rangle.
\]

By the definition \(F\), it is easy to show that
\[
F_t = \frac{F}{t} + t(-2\beta |\nabla w|^2 + 2\beta \langle \nabla w, \nabla w_t \rangle + aw_t - w_{tt})
\]
\[
\Delta V F = t(\beta \Delta V (|\nabla w|^2) + a\Delta V w - \Delta_V w_t).
\]

Therefore,
\[
\Delta V F - F_t = t(\beta \Delta V (|\nabla w|^2) + a\Delta V w - \Delta_V w_t) - \frac{F}{t}
\]
\[
- t\left(-2\beta |\nabla w|^2 + 2\beta \langle \nabla w, \nabla w_t \rangle + aw_t - w_{tt}\right).
\]

Combining (2.3), (2.5) and (2.6), we obtain
\[
\Delta V F - F_t \geq t \left\{ \frac{2\beta}{n} (\Delta V w)^2 + \left( \frac{-2\beta L^2}{N} - 2\beta a \left(1 - \frac{1}{\beta}\right) \right) |\nabla w|^2 - 2\beta \langle \nabla w, \nabla b \rangle + a^2 (1 - \frac{1}{\beta}) w + a \left(1 - \frac{1}{\beta}\right) w_t - ab + b_t \right\}
- 2 \langle \nabla w, \nabla F \rangle + \left( -\frac{a}{\beta} - \frac{1}{t}\right) F.
\]

On the other hand, by direct computation, we have
\[
- a^2 (1 - \frac{1}{\beta}) w + a \left(1 - \frac{1}{\beta}\right) w_t = -\frac{aF}{t} + \frac{aF}{t\beta} + a(\beta - 1)|\nabla w|^2.
\]
Substituting (2.9) into (2.8), we get

\[
\Delta_v F - F_t \geq t \left\{ \frac{2\beta}{n} (\Delta_v w)^2 + \left( \frac{-2\beta L^2}{N} - a(\beta - 1) \right) |\nabla w|^2 - 2\beta \langle \nabla w, \nabla b \rangle + b_t - ab \right\} \\
- 2 \langle \nabla w, \nabla F \rangle - aF - \frac{F}{t}.
\]

The proof is complete. \(\square\)

Now, we prove the Theorem 1.1.

**Proof of Theorem 1.1.** Let \(\xi(r)\) be a cut-off function such that \(\xi(r) = 1\) for \(r \leq 1\), \(\xi(r) = 0\) for \(r \geq 2\), \(0 \leq \xi(r) \leq 1\), and

\[
0 \geq \xi^{\frac{1}{2}}(r) \frac{\xi'(r)}{\xi} \geq -c_1,
\]

\[
\xi''(r) \geq -c_2
\]

for positive constants \(c_1\) and \(c_2\).

Put \(\varphi(x) = \xi(\varphi(x)/R)\), it is easy to see that

\[
(2.10) \quad \frac{|\nabla \varphi|^2}{\varphi} = \frac{1}{\xi} \frac{\left( \xi(r) \right)^2}{R^2} |\nabla \rho(x)|^2 \leq \frac{(-c_1)^2}{R^2} = \frac{c_1^2}{R^2}.
\]

Hence, by the inequality (2.2), we have

\[
\Delta_v \varphi = \frac{\xi(r)'' |\nabla \rho|^2}{R^2} + \frac{\xi(r)' \Delta_v \rho}{R} \geq -c_2 + \left( \frac{-c_1}{R} \right) \left( \sqrt{(n-1)K + \frac{n-1}{\rho}} + L \right) \\
= - \frac{R \left[ \sqrt{(n-1)K + \frac{n-1}{\rho} + L} \right] c_1 + c_2}{R^2} \geq - \frac{(n-1 + \sqrt{(n-1)KR + LR}) c_1 + c_2}{R^2}.
\]

(2.11)

For \(T \geq 0\), let \((x,t)\) be a point in \(B_{2R}(p) \times [0,T]\) at which \(\varphi F\) attains its maximum. At the point \((x,t)\), we have

\[
\begin{cases}
\nabla (\varphi F) = 0 \\
\Delta_v (\varphi F) \leq 0 \\
F_t \geq 0
\end{cases}
\]

Since \(\nabla (\varphi F) = \varphi \nabla F + F \nabla \varphi = 0\), this implies \(\nabla F = -F \varphi^{-1} \nabla \varphi\). It follows that

\[
\Delta_v (\varphi F) = \varphi \Delta_v F + F \Delta_v \varphi - 2F \varphi^{-1} |\nabla \varphi|^2 \leq 0.
\]
Substituting (2.10) and (2.11) into the above inequality, we obtain

$$\varphi \Delta_V F \leq F \left( \frac{2|\nabla \varphi|^2}{\varphi} - \Delta_V \varphi \right)$$

(2.12)

$$\leq F \left( \frac{(n-1 + \sqrt{(n-1)KR + LR})c_1 + c_2 + 2c_1^2}{R^2} \right) = FA$$

where $A = \frac{(n-1 + \sqrt{(n-1)KR + LR})c_1 + c_2 + 2c_1^2}{R^2}$.

Combining Lemma 2.2 and (2.12), we infer

$$FA \geq \varphi \Delta_V F \geq \varphi \Delta_V F - F_t$$

$$\geq t\varphi \left\{ \frac{2\beta}{n} (\Delta_V w)^2 + \left( \frac{-2\beta L^2}{N} - a(\beta - 1) \right) |\nabla w|^2 - 2\beta \left( \nabla w, \nabla b \right) + b_t - ab \right\}$$

(2.13)

$$+ \varphi \left\{ -2 \left( \nabla w, \nabla F \right) - aF - \frac{F}{t} \right\}.$$  

Here we used $F_t \leq 0$. Since $0 = \nabla (\varphi F) = \varphi \nabla F + F \nabla \varphi$, we have

$$2\varphi \left( \nabla w, \nabla F \right) = 2F \left( \nabla w, \nabla \varphi \right) \geq -2F |\nabla w| |\nabla \varphi| \geq -2\frac{c_1}{R} \varphi^\frac{1}{2} F |\nabla w|.$$  

By (2.4), we yield

$$2\varphi \left( \nabla w, \nabla F \right) \geq 2F \left( \nabla w, \nabla \varphi \right) \geq -2F |\nabla w| |\nabla \varphi| \geq -2\frac{c_1}{R} \varphi^\frac{1}{2} F |\nabla w|.$$  

By the similar argument as Davies [2] or as Negrin [7], we put $\mu = \frac{|\nabla w|^2}{F}$. Then (2.16) can be read as

$$\frac{2\varphi t \left[ (\beta - 1)\mu F - F \right]^2}{n t^2} \leq AF + \frac{4\varphi t \beta \left[ (\beta - 1)\mu F - F \right] b}{n t}$$

$$+ \varphi F t \mu \left( \frac{2\beta L^2}{N} + a(\beta - 1) \right) + 2\beta \varphi t \left( \nabla w, \nabla b \right)$$

$$+ \varphi t (ab - b_t) + 2\frac{c_1}{R} \mu^\frac{1}{2} \varphi^\frac{1}{2} F^\frac{3}{2} + a\varphi F + \frac{\varphi F}{t}.$$
Multiplying both sides of the above inequality by $\varphi t$ we arrive at

$$
\frac{2\beta[(\beta-1)t\mu - 1]^2}{n} (\varphi F)^2 \leq \frac{2c_1}{R} t\mu \varphi^2 F^2 + (At + 1)\varphi F
$$

$$
+ \left\{ \frac{4\beta[(\beta-1)t\mu - 1]b}{n} + t\mu \left( \frac{2\beta L^2}{N} + a(\beta-1) \right) + a \right\} t\varphi^2 F
$$

(2.17)

$$
+ 2\beta \varphi^2 t^2 \langle \nabla w, \nabla b \rangle + \varphi^2 t^2 (ab - b_t)
$$

Now we want to estimate the right hand side of (2.17). The first term of the right-hand side of (2.17) can be estimated as follows.

$$
2 \frac{c_1}{R} t\mu \varphi^2 F^2 \leq \frac{2}{n} \left( \frac{4\beta[(\beta-1)t\mu - 1]b}{n} + t\mu \left( \frac{2\beta L^2}{N} + a(\beta-1) \right) + a \right) t\varphi^2 F
$$

(2.18)

with $0 < \delta < 1$, and the third term of the right-hand side of (2.17) is evaluated as below.

$$
2\varphi^2 t^2 \beta \langle \nabla w, \nabla b \rangle \leq 2\varphi^2 t^2 \beta |\nabla b| (\mu F)^{\frac{1}{2}} \leq t^2 \beta |\nabla b| (\mu \varphi F + 1).
$$

(2.19)

By the definition of $\theta$, it is easy to see that

$$
B := t^2 \beta |\nabla b| \leq \theta t^2 \beta \quad \text{and} \quad C := t^2 \beta |\nabla b| + \varphi^2 t^2 (ab - b_t) \leq \theta t^2 \beta + \varphi^2 t^2 (|a| + 1)\theta.
$$

Plugging these above estimates and (2.18), (2.19) into (2.17), we obtain

$$
\frac{2\beta[(\beta-1)t\mu - 1]^2}{n} (\varphi F)^2 \leq \frac{2\delta \beta[(\beta-1)t\mu - 1]^2}{n} (\varphi F)^2 + \frac{nc_1^2 t^2 \mu}{2\delta \beta[(\beta-1)t\mu - 1]^2 R^2} (\varphi F)
$$

$$
+ \left\{ \frac{4\beta[(\beta-1)t\mu - 1]b}{n} + t\mu \left( \frac{2\beta L^2}{N} + a(\beta-1) \right) + a \right\} t\varphi^2 F
$$

(2.20)

$$
+ (At + 1)\varphi F + \mu B \varphi F + C.
$$

Now, we have two cases.

1. If $a \leq 0$ then $at\varphi^2 F \leq 0$, $|a| = -a$, and

$$
\frac{4t\beta[(\beta-1)t\mu - 1]b}{n} \leq -\frac{4t\beta[(\beta-1)t\mu - 1]\theta}{n}.
$$

By (2.20), we have

$$
(\varphi F)^2 \leq \frac{n}{2(1-\delta)\beta[(\beta-1)t\mu - 1]^2} \left\{ \frac{nc_1^2 t^2 \mu}{2\delta \beta[(\beta-1)t\mu - 1]^2 R^2} + At + 1
$$

$$
+ \left( a + \frac{\theta \beta}{\beta - 1} - \frac{4\beta \theta}{n} \right) t^2 \mu (\beta - 1) + \frac{4t\beta \theta}{n} + 2t^2 \mu \frac{\beta L^2}{N} \right\} \varphi F
$$

$$
+ \frac{n}{2(1-\delta)\beta[(\beta-1)t\mu - 1]^2} \left( \theta t^2 \beta + \varphi^2 t^2 (1-a)\theta \right).
$$
Using the fact that if \( a, b \geq 0 \) satisfying \( x^2 \leq ax + b \) then \( x \leq a + \sqrt{b} \), the above inequality implies

\[
\varphi F \leq \frac{n}{2(1 - \delta)\beta[(\beta - 1)t\mu - 1]^2} \left\{ \frac{nc^2t^2}{2} \mu + \frac{At + 1}{2(1 - \delta)\beta[(\beta - 1)t\mu - 1]^2R^2} \right. \\
+ \left( a + \frac{\theta \beta}{\beta - 1} - \frac{4\beta \theta}{n} \right) t^2 \mu(\beta - 1) + \frac{4t\beta \theta}{n} + \frac{2t^2 \mu}{N^2} \right\} \\
+ \sqrt{\frac{n(\theta t^2 \beta + \varphi^2 t^2(1 - a)\theta)}{2(1 - \delta)\beta[(\beta - 1)t\mu - 1]^2}}.
\]

(2.21)

Since \( ((\beta - 1)t\mu - 1)^2 \geq 2(1 - \beta)t\mu + 1 \geq 1 \), we have

\[
\frac{1}{2(1 - \delta)\beta((\beta - 1)t\mu - 1)^2} \leq \frac{1}{2(1 - \delta)\beta}.
\]

Therefore,

\[
\frac{1}{2(1 - \delta)\beta((\beta - 1)t\mu - 1)^2} \leq \frac{nc^2t^2}{2} \mu \frac{2}{2(1 - \delta)\beta(16\delta\beta(1 - \beta)R^2)},
\]

(2.22)

and

\[
\frac{1}{2(1 - \delta)\beta((\beta - 1)t\mu - 1)^2} \left( At + 1 + \frac{4t\beta \theta}{n} \right) \leq \frac{1}{2(1 - \delta)\beta} \left( At + 1 + \frac{4t\beta \theta}{n} \right),
\]

(2.23)

where in (2.22), we used

\[
((1 - \beta)t\mu + 1) \geq 2(1 - \beta)t\mu.
\]

Since \( ((\beta - 1)t\mu - 1)^2 \geq 2(1 - \beta)t\mu \), we have

\[
\frac{1}{2(1 - \delta)\beta[(\beta - 1)t\mu - 1]^2} \left( a + \frac{\theta \beta}{\beta - 1} - \frac{4\beta \theta}{n} \right) t^2 \mu(\beta - 1) + \frac{2t^2 \mu}{N^2} \beta L^2 \\
\leq \frac{1}{2(1 - \delta)\beta} \left( a + \frac{\theta \beta}{\beta - 1} - \frac{4\beta \theta}{n} \right) t + \frac{t\beta L^2}{(1 - \beta)N}.
\]

(2.24)

Moreover, since \( \varphi^2 \leq 1 \) and \( 0 < \delta < 1 \), we infer

\[
\sqrt{\frac{n(\theta t^2 \beta + \varphi^2 t^2(1 - a)\theta)}{2(1 - \delta)\beta[(\beta - 1)t\mu - 1]^2}} \leq \sqrt{\frac{n(\theta t^2 \beta + \varphi^2 t^2(1 - a)\theta)}{2(1 - \delta)\beta}} \frac{nt}{2(1 - \theta)\beta} \sqrt{\frac{20\beta(1 + \beta - a)}{n}}.
\]

(2.25)
Plugging (2.22), (2.24), (2.23) and (2.25) into (2.21), we obtain
\[
\varphi F \leq \frac{n}{2(1-\delta)\beta} \left\{ \frac{\theta t \beta}{2(1-\delta)\beta} + \frac{4t\beta}{n} + \frac{t\beta L^2}{(1-\beta)N} - \frac{at}{2} \right\}
\]
\[
+ \left( \frac{\theta t \beta}{2(1-\delta)\beta} + \frac{4t\beta}{2n} \right) + \frac{nt}{2(1-\delta)\beta} \sqrt{\frac{2t\beta(1+\beta-a)}{n}}.
\]
In particular, at \((x_0, T) \in B(p, R) \times [0, T]\), we have
\[
\beta \frac{\left| \nabla u \right|^2}{u^2} + \log u - \frac{u_t}{u} \leq \frac{n}{2(1-\delta)\beta} \left\{ \frac{\theta t \beta}{16\delta \beta(1-\beta)R^2} + \frac{At + 1 + 6t\beta}{T} + \frac{\beta L^2}{(1-\beta)N} - \frac{a}{2} + \frac{\theta t \beta}{2(1-\beta)} \sqrt{\frac{2t\beta(1+\beta-a)}{n}} \right\}.
\]

Hence, we complete the proof of the part (1).

2. If \(a \geq 0\) then \(a(\beta - 1)t^2\varphi^2\mu F \leq 0\), \(|a| = a\) and
\[
\frac{4t\beta(\beta - 1)t\mu - 1}{n} b \leq \frac{4t\beta(\beta - 1)t\mu - 1}{n} \theta.
\]
The inequality (2.20) implies
\[
(\varphi F)^2 \leq \frac{n}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2} \left\{ \frac{\theta t \beta}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2R^2} + \frac{At + 1}{n} \right\}
\]
\[
+ \left( \frac{\theta t \beta}{\beta - 1} - \frac{4t\beta}{n} \right) t^2 \mu (\beta - 1) + \frac{4t\beta}{n} + \frac{t\beta L^2}{(1-\beta)N} \right\} \varphi F
\]
\[
+ \frac{n}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2} \left( \theta t^2 \beta + \varphi^2 t^2 (1 + a)\theta \right).
\]

By the same argument as in the proof of the part (1), we conclude that
\[
\varphi F \leq \frac{n}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2} \left\{ \frac{\theta t \beta}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2R^2} + \frac{At + 1}{n} \right\}
\]
\[
+ \left( \frac{\theta t \beta}{\beta - 1} - \frac{4t\beta}{n} \right) t^2 \mu (\beta - 1) + \frac{4t\beta}{n} + \frac{t\beta L^2}{(1-\beta)N} \right\}
\]
\[
+ \sqrt{\frac{n(\theta t^2 \beta + \varphi^2 t^2 (1 + a)\theta)}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2}}.
\]

Since \((\beta - 1)t^2 \mu - 1 \geq 2(1 - \beta)\mu t\), we have
\[
\frac{1}{2(1-\delta)\beta[1(\beta - 1)t\mu - 1]^2} \left( \left( \frac{\theta t \beta}{\beta - 1} - \frac{4t\beta}{n} \right) t^2 \mu (\beta - 1) + \frac{t\beta L^2}{(1-\beta)N} \right)
\]
\[
\leq \frac{1}{2(1-\delta)\beta} \left( \frac{\theta t \beta}{2} \left( \frac{\theta t \beta}{\beta - 1} - \frac{4t\beta}{n} \right) + \frac{t\beta L^2}{(1-\beta)N} \right).
\]
Moreover, since \((\beta - 1)ut - 1\)^2 ≥ 1, \(\varphi^2 \leq 1\) and \(0 < \delta < 1\), we infer

\[
\frac{1}{2(1-\delta)\beta((\beta - 1)\mu t - 1)^2} \left( At + 1 + \frac{4t\beta \theta}{n} + at \right) \leq \frac{1}{2(1-\delta)\beta} \left( At + 1 + \frac{4t\beta \theta}{n} + at \right)
\]

and

\[
\sqrt{\frac{n(\theta^2 \beta + \varphi^2 t^2(1-a)\theta)}{2(1-\delta)\beta} \left( (\beta - 1)\mu t - 1 \right)^2} \leq \sqrt{\frac{n(\theta^2 \beta + \varphi^2 t^2(1+a)\theta)}{2(1-\delta)\beta}} \leq \frac{nt}{2(1-\theta)\beta} \sqrt{\frac{2\theta(1+\beta+a)}{n}}.
\]

Combining (2.27), (2.28), (2.29) and (2.26), we conclude that

\[
\varphi F \leq \frac{n}{2(1-\delta)\beta} \left\{ \frac{\text{tc}_1^2}{16\delta\beta(1-\beta)R^2} + 1 + 1 + \frac{4t\beta \theta}{n} + \frac{t\beta L^2}{(1-\beta)N} \right\}
\]

\[
\quad + \frac{\theta t}{2(1-\beta)} + \frac{4t\beta \theta}{2n} + at \right\} \leq \sqrt{\frac{n(\theta^2 \beta + \varphi^2 t^2(1+a)\theta)}{2(1-\delta)\beta}} \leq \frac{nt}{2(1-\theta)\beta} \sqrt{\frac{2\theta(1+\beta+a)}{n}}.
\]

Therefore, for all \((x_0, T) \in B(p, R) \times [0, T]\), we have

\[
\beta \frac{\left| \nabla u \right|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{n}{2(1-\delta)\beta} \left\{ \frac{nc_1^2}{16\delta\beta(1-\beta)R^2} + A + \frac{1}{T} + \frac{6\beta \theta}{n} \right\}
\]

\[
\quad + \frac{\beta L^2}{(1-\beta)N} + a + \frac{\theta \beta}{2(1-\beta)} + \sqrt{\frac{2\theta(1+\beta+a)}{n}} \}
\]

The proof of the part (2) is complete. □

3. Applications

**Theorem 3.1.** Let \((M, g)\) be a noncompact \(n\)-dimensional Riemannian manifold with \(\text{Ric}_N^V\) bounded from below by the constant \(-K := -K(2R)\), where \(R > 0\), \(K(2R) > 0\) in the geodesic ball \(B(p, 2R)\) with radius \(2R\) around \(p \in M\) and \(V\) is a smooth vector field on \(M\). Let \(a\) be a constant and the equation

\[
\frac{\partial u}{\partial t} = \Delta_V u + au \log u
\]

has a positive solution \(u\) on \(M \times [0, \infty)\). Then

1. If \(a \leq 0\), we have

\[
\beta \frac{\left| \nabla u \right|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{N + n}{2(1-\delta)\beta} \left( \frac{(N + n)c_1^2}{16\delta\beta(1-\beta)R^2} + A + \frac{1}{T} + \frac{a}{2} \right);
\]
2. If \( a \geq 0 \), we have
\[
\beta \frac{|\nabla u|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{N + n}{2(1 - \delta)\beta} \left( \frac{(N + n)c_1^2}{16\delta\beta(1 - \beta)R^2} + A + \frac{1}{t} + a \right),
\]
where \( c_1 \) and \( c_2 \) are positive constants, \( 0 < \delta < 1 \), \( \beta = e^{-2Kt} \) and \( A \) is defined by
\[
A = \frac{(n - 1 + \sqrt{nKR})c_1 + c_2 + 2c_1^2}{R^2}.
\]

**Proof.** Note that if \( \text{Ric}_V^N \geq -K \) then the Laplacian comparison can be read as follows (see [6])
\[
\Delta_V \rho \leq \sqrt{(n - 1)K} \coth \left( \frac{\sqrt{K}}{n - 1} \rho \right) \leq \sqrt{(n - 1)K} + \frac{n - 1}{\rho}.
\]
Moreover, (2.18) can be estimate by
\[
2 \frac{c_1^2}{R} t \frac{1}{\mu^2} (\varphi F)^3 \leq \frac{2\delta \beta [(\beta - 1) t \mu - 1]^2}{N + n} (\varphi F)^2 + \frac{(N + n)c_1^2 t^2 \mu}{2\delta \beta [(\beta - 1) - 1]^2 R^2} (\varphi F).
\]
Now, let
\[
A = \frac{(n - 1 + \sqrt{(n - 1)KR})c_1 + c_2 + 2c_1^2}{R^2}
\]
and using the same argument as in the proof of Theorem 1.1, we complete the proof of Theorem 3.1. \( \square \)

In particular, if \( V = -\nabla f \) where \( f \) is a smooth function on \( M \), we recover the result of Huang-Ma in [4]. Hence, our result is a generalization of Huang-Ma’s work. Moreover, let \( R \to \infty \) in Theorem 3.1, we obtain the following global gradient estimate of a general heat equation.

**Theorem 3.2.** Let \((M, g)\) be a noncompact \( n \)-dimensional Riemannian manifold with \( \text{Ric}_V^N \) bounded from below by the constant \(-K\), where \( K > 0 \) and \( V \) is a smooth vector field on \( M \). Let \( a \) be a constant and the equation
\[
\frac{\partial u}{\partial t} = \Delta_V u + au \log u
\]
has a positive solution \( u \) on \( M \times [0, \infty) \). Then

1. If \( a \leq 0 \), we have
\[
\beta \frac{|\nabla u|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{N + n}{2(1 - \delta)\beta} \left( \frac{1}{t} - \frac{a}{2} \right);
\]
2. If \( a \geq 0 \), we have
\[
\beta \frac{|\nabla u|^2}{u^2} + a \log u - \frac{u_t}{u} \leq \frac{N + n}{2(1 - \delta)\beta} \left( \frac{1}{t} + a \right),
\]
where \( \beta = e^{-2Kt} \) and \( 0 < \delta < 1 \).

Now, similarly to [4], we show a Harnack type inequality.
Theorem 3.3. Let $(M, g)$ be a noncompact $n$-dimensional Riemannian manifold with $\text{Ric}^N$ bounded from below by the constant $-K$, where $K > 0$ and $V$ is the smooth vector field on $M$. Suppose that the equation
\[
\frac{\partial u}{\partial t} = \Delta_V u
\]
has a positive solution $u$ on $M \times [0, \infty)$. Then
1. The solution $u$ satisfies
\[
\frac{u_t}{u} - e^{-2Kt} \frac{|\nabla u|^2}{u^2} + e^{2Kt} \frac{N + n}{2t} \geq 0
\]
2. For any points $(x_1, t_1)$ and $(x_2, t_2)$ in $M \times [0, +\infty)$ with $0 < t_1 < t_2$, we have the following Harnack inequality
\[
u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{N+n}{2}} e^{\phi(x_1, x_2, t_1, t_2)} + B.
\]
Here
\[
\phi(x_1, x_2, t_1, t_2) = \inf_\gamma \int_0^t \frac{1}{4} e^{2Kt} |\dot{\gamma}|^2 dt, \quad B = \frac{N+n}{2} \left(e^{2Kt_2} - e^{2Kt_1}\right)
\]
where $\gamma$ is a parameterized curve with $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$.

Proof. 1. Applying Theorem 3.2 with $a = 0$, we have
\[
\beta|\nabla u|^2 + \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta t}.
\]
Letting $\delta \to 0$ and $\beta = e^{-2Kt}$ into the inequality (3.31) we obtain
\[
\frac{u_t}{u} - e^{-2Kt} \frac{|\nabla u|^2}{u^2} + e^{2Kt} \frac{N + n}{2t} \geq 0.
\]
The proof is complete.

2. The proof can be followed by using (3.30) and the argument in [4]. We omit the details. □

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References


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