THE $G$-GRADED IDENTITIES
OF THE GRASSMANN ALGEBRA

Lucio Centrone

Abstract. Let $G$ be a finite abelian group with identity element $1_G$ and $L = \bigoplus_{g \in G} L^g$ be an infinite dimensional $G$-homogeneous vector space over a field of characteristic 0. Let $E = E(L)$ be the Grassmann algebra generated by $L$. It follows that $E$ is a $G$-graded algebra. Let $|G|$ be odd, then we prove that in order to describe any ideal of $G$-graded identities of $E$ it is sufficient to deal with $G'$-grading, where $|G'| \leq |G|$, $\dim_F L^1 G' = \infty$ and $\dim_F L^g < \infty$ if $g' \neq 1_G$. In the same spirit of the case $|G|$ odd, if $|G|$ is even it is sufficient to study only those $G$-gradings such that $\dim_F L^g = \infty$, where $o(g) = 2$, and all the other components are finite dimensional. We also compute graded cocharacters and codimensions of $E$ in the case $\dim L^1 G = \infty$ and $\dim L^g < \infty$ if $g \neq 1_G$.

1. Introduction

All algebras we refer to are to be considered associative and unitary over a field of characteristic 0 unless explicitly written. Let $F$ be a field and $X = \{x_1, x_2, \ldots\}$ be a countable infinite set of variables and let $F\langle X \rangle$ be the free associative algebra freely generated by $X$. If $A$ is an $F$-algebra, we say that $f(x_1, \ldots, x_n) \in F\langle X \rangle$ is a polynomial identity of $A$ if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. If $A$ has a non-trivial polynomial identity we say that $A$ is a polynomial identity algebra or PI-algebra and we denote by $T(A)$ the set of all polynomial identities satisfied by $A$. It is well known that $T(A)$ is an ideal of $F\langle X \rangle$ invariant under all endomorphisms of $F\langle X \rangle$, i.e., it is a $T$-ideal called the $T$-ideal of $A$. We say that the variety generated by the algebra $A$ is the class

$$
\mathcal{V} = \mathcal{V}(A) = \{ B \text{ associative algebra} \mid T(A) \subseteq T(B) \}.
$$

The Grassmann algebra $E$, generated by an infinite dimensional vector space and its identities, plays an important role in the structure theory of Kemer on varieties of associative algebras with polynomial identities [11][10]. More precisely, Kemer proved that any associative PI-algebra over a field $F$ of characteristic zero satisfies the same identities (is PI-equivalent) of the Grassmann envelope of a finite
This work has been generalized by the author for any infinite field of characteristic $p > 2$ (see [2]). In [1] Anisimov constructed an algorithm to compute the exact value of the graded codimension of $E$ for any $\mathbb{Z}_p$-grading of $E$, where $p$ is a prime number.

In this paper we consider a finite abelian group $G$ with identity element $1_G$ and an infinite dimensional $G$-homogeneous vector space $L$ over the field $F$ which generates the infinite dimensional Grassmann algebra $E = E(L)$. The latter inherits the structure of a $G$-graded algebra, hence we are allowed to study its $G$-graded identities. Let $|G|$ be odd, then we prove that in order to describe any ideal of $G$-graded identities of $E$ it is sufficient to deal with a $G'$-grading, where $|G'| \leq |G|$, $\dim_F L^{1_{G'}} = \infty$ and $\dim_F L^g < \infty$ if $g' \neq 1_{G'}$. In the same spirit of the case $|G|$ odd, if $|G|$ is even it is sufficient to study only those $G$-gradings such that $\dim_F L^g = \infty$ and $o(g) = 2$, where $o(g)$ stands for the order of the group element $g$. Finally we give a complete description of $T_G(E)$, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$ in some particular cases. We also compute graded cocharacters and codimensions of $E$ in the case $\dim L^{1_G} = \infty$ and $\dim L^g < \infty$ if $g \neq 1_G$.

2. **Free algebras, graded PI-algebras**

We introduce the key tools for the study of graded polynomial identities. We start off with the following definition.

**Definition 2.1.** Let $G$ be a group and $A$ be an algebra over a field $F$. We say that the algebra $A$ is $G$-graded if $A = \bigoplus_{g \in G} A^g$ and for all $g, h \in G$, one has $A^g A^h \subseteq A^{gh}$.

It is easy to note that if $a$ is any element of $A$ it can be uniquely written as a finite sum $a = \sum_{g \in G} a_g$, where $a_g \in A^g$. We shall call the subspaces $A^g$
the \( G \)-homogeneous components of \( A \). Accordingly, an element \( a \in A \) is called \( G \)-homogeneous if exists \( g \in G \) such that \( a \in A^g \). If \( B \subseteq A \) is a subspace of \( A \), \( B \) is \( G \)-graded if and only if \( B = \bigoplus_{g \in G} (B \cap A^g) \). Analogously one can define \( G \)-graded algebras, subalgebras, ideals, etc.

Let \( \{ X^g \mid g \in G \} \) be a family of disjoint countable sets of indeterminates. Set \( X = \bigcup_{g \in G} X^g \) and denote by \( F\langle X \mid G \rangle \) the free associative algebra freely generated by \( X \). An indeterminate \( x \in X \) is said to be of homogeneous \( G \)-degree \( g \), written \( \| x \| = g \), if \( x \in X^g \). We always write \( x^g \) if \( x \in X^g \). The homogeneous \( G \)-degree of a monomial \( m = x_{i_1}x_{i_2} \ldots x_{i_k} \) is defined to be \( \| m \| = \| x_{i_1} \| \cdot \| x_{i_2} \| \ldots \cdot \| x_{i_k} \| \). For every \( g \in G \), denote by \( F\langle X \mid G^g \rangle \) the subspace of \( F\langle X \mid G \rangle \) spanned by all monomials having homogeneous \( G \)-degree \( g \). Notice that \( F\langle X \mid G \rangle^g F\langle X \mid G \rangle^{g'} \subseteq F\langle X \mid G \rangle^{g + g'} \) for all \( g, g' \in G \). Thus

\[
F\langle X \mid G \rangle = \bigoplus_{g \in G} F\langle X \mid G^g \rangle
\]

is a \( G \)-graded algebra. The elements of the \( G \)-graded algebra \( F\langle X \mid G \rangle \) are called \( G \)-graded polynomials or, simply, graded polynomials.

**Definition 2.2.** If \( A \) is a \( G \)-graded algebra, a \( G \)-graded polynomial

\[
f(x_1, \ldots, x_n)
\]

is said to be a graded polynomial identity of \( A \) if

\[
f(a_1, a_2, \ldots, a_n) = 0
\]

for all \( a_1, a_2, \ldots, a_n \in \bigcup_{g \in G} A^g \) such that \( a_k \in A^{\| x_k \|}, k = 1, \ldots, n \). We shall write \( f \equiv 0 \) in order to say that \( f \) is a graded polynomial identity for \( A \).

Given an algebra \( A \) graded by a group \( G \), we define

\[
T_G(A) := \{ f \in F\langle X \mid G \rangle \mid f \equiv 0 \text{ on } A \},
\]

the set of \( G \)-graded polynomial identities of \( A \).

**Definition 2.3.** An ideal \( I \) of \( F\langle X \mid G \rangle \) is said to be a \( T_G \)-ideal if it is invariant under all \( F \)-endomorphisms \( \varphi : F\langle X \mid G \rangle \to F\langle X \mid G \rangle \) such that \( \varphi (F\langle X \mid G^g \rangle) \subseteq F\langle X \mid G \rangle^g \) for all \( g \in G \).

Hence \( T_G(A) \) is a \( T_G \)-ideal of \( F\langle X \mid G \rangle \). On the other hand, it is easy to check that all \( T_G \)-ideals of \( F\langle X \mid G \rangle \) are of this type. We shall denote by \( \langle S \rangle^{T_G} \) the \( T_G \)-ideal generated by the set \( S \), i.e., the smallest \( T_G \)-ideal containing \( S \). In this case we say \( S \) is a basis for \( \langle S \rangle^{T_G} \) or the elements of \( \langle S \rangle^{T_G} \) follow from those of \( S \).

From now on all the groups are assumed to be finite abelian. The theory of \( G \)-graded PI-algebras passes through the representation theory of the symmetric group. More precisely we study the following spaces.

**Definition 2.4.** Let

\[
P^G_n = \text{span} \{ x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \ldots x_{\sigma(n)}^{g_n} \mid g_i \in G, \sigma \in S_n \},
\]

then the elements in \( P^G_n \) are called multilinear polynomials of degree \( n \) of \( F\langle X \mid G \rangle \).
It turns out that $P_n^G$ is a left $S_n$-module under the natural left action of the symmetric group $S_n$. As a consequence the factor module $P_n^G(A) := P_n^G/(P_n^G \cap T_G(A))$ is an $S_n$-module, too. We observe that $P_n^G(A)$ affords a representation of the symmetric group $S_n$ which naturally carries on a character of $S_n$ (or $S_n$-character). Let us denote the $S_n$-character of $P_n^G(A)$ by $\chi_n^G(A)$, and by $c_n^G(A)$ its dimension over $F$. We say that

$$
\left(\chi_n^G(A)\right)_{n \in \mathbb{N}} \text{ is the } G\text{-graded cocharacter sequence of } A
$$

$$
\left(c_n^G(A)\right)_{n \in \mathbb{N}} \text{ is the } G\text{-graded codimension sequence of } A.
$$

Now, for $l_{g_1}, \ldots, l_{g_r} \in \mathbb{N}$ let us consider the blended components of the multilinear polynomials in the indeterminates labeled as follows: $x_{1}^{g_1}, \ldots, x_{l_{g_1}}^{g_1}$, then $x_{l_{g_1} + 1}^{g_2}, \ldots, x_{l_{g_1} + l_{g_2}}^{g_2}$ and so on. We denote this linear space by $P_{l_{g_1}, \ldots, l_{g_r}}^G$. Of course, this is a left $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-module. We shall denote by $\chi_{l_{g_1}, \ldots, l_{g_r}}^G(A)$ the character of the module $P_{l_{g_1}, \ldots, l_{g_r}}^G(A)/(P_{l_{g_1}, \ldots, l_{g_r}}^G(A) \cap T_G(A))$ and by $c_{l_{g_1}, \ldots, l_{g_r}}^G(A)$ its dimension.

Since the ground field $F$ is infinite, a standard Vandermonde-argument yields that a polynomial $f$ is a $G$-graded polynomial identity for $A$ if and only if its homogeneous components (with respect to the ordinary $\mathbb{N}$-grading), are identities as well. Moreover, since $\text{char}(F) = 0$, the well known multilinearization process shows that the $T_G$-ideal of a $G$-graded algebra $A$ is determined by its multilinear polynomials, i.e. by the various $P_{l_{g_1}, \ldots, l_{g_r}}^G(A)$. We remark that, given the cocharacter $\chi_{l_{g_1}, \ldots, l_{g_r}}^G(A)$, the graded cocharacter $\chi_n^G(A)$ is known as well. More precisely, the following is due to Di Vincenzo (see [3] Theorem 2).

**Proposition 2.5.** Let $A$ be a $G$-graded algebra with graded cocharacter sequences $\chi_{l_{g_1}, \ldots, l_{g_r}}^G(A)$. Then

$$
\chi_n^G(A) = \sum_{(l_{g_1}, \ldots, l_{g_r})} \chi_{l_{g_1}, \ldots, l_{g_r}}^G(A)^{\uparrow S_n},
$$

where $\chi_{l_{g_1}, \ldots, l_{g_r}}^G(A)^{\uparrow S_n}$ stands for the induced $S_n$-character of the $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-module $P_{l_{g_1}, \ldots, l_{g_r}}^G(A)$.

Moreover

$$
c_n^G(A) = \sum_{(l_{g_1}, \ldots, l_{g_r})} \left(\begin{array}{c} n \\ l_{g_1}, \ldots, l_{g_r} \end{array}\right) c_{l_{g_1}, \ldots, l_{g_r}}^G(A).
$$

Let us consider the free algebra $F\langle Y \cup Z \rangle$ (where $Y$ is the set of all indeterminates of $G$-degree 1$_G$ and $Z$ is the set of all the remaining indeterminates). The $Y$-proper polynomials (see [4] Section 2) are the elements of the unitary $F$-subalgebra $B$ of $F\langle X \rangle$ generated by the elements of $Z$ and by all non-trivial commutators. More precisely, a polynomial $f \in F\langle Y \cup Z \rangle$ is $Y$-proper if all the $y \in Y$ occurring in $f$
appear in commutators only. Notice that if \( f \in F(Z) \), then \( f \) is \( Y \)-proper. It is well known (see, for instance, Lemma 1 Section 2 in \([7]\)) that all the graded polynomial identities of a superalgebra \( A \) follow from the \( Y \)-proper ones. This means that the set \( T_{\mathbb{Z}_2}(A) \cap B \) generates the whole \( T_{\mathbb{Z}_2}(A) \) as a \( T_{\mathbb{Z}_2} \)-ideal. Similarly, for any finite abelian group \( G \), all the \( G \)-graded polynomial identities of a \( G \)-graded algebra \( A \) follow from the \( Y \)-proper ones. This means that the set \( T_G(A) \cap B \) generates the whole \( T_G(A) \) as a \( T_G \)-ideal. Let us define \( B(A) := B/(T_G(A) \cap B) \). We shall refer to \( B(A) \) as \( Y \)-proper relatively-free algebra of \( A \).

We shall denote by \( \Gamma_n^G \) the set of multilinear \( Y \)-proper polynomials of \( P_n^G \). It is not difficult to see that \( \Gamma_n^G \) is a left \( S_n \)-submodule of \( P_n^G \) and the same holds for \( \Gamma_n^G \cap T_G(A) \). Hence the factor module

\[
\Gamma_n^G(A) := \Gamma_n^G/(\Gamma_n^G \cap T_G(A))
\]

is an \( S_n \)-submodule of \( P_n^G(A) \). We denote the \( S_n \)-character of the factor module \( \Gamma_n^G/(\Gamma_n^G \cap T_G(A)) \) by \( \xi_n(A) \) and by \( \gamma_n(A) \) its dimension over \( F \). We say:

\[
(\xi_n(A))_{n \in \mathbb{N}} \text{ is the } G \text{-graded proper cocharacter sequence of } A;
\]

\[
(\gamma_n(A))_{n \in \mathbb{N}} \text{ is the } G \text{-graded proper codimension sequence of } A.
\]

We shall denote by \( \Gamma_{m_1,\ldots,m_r}^G \) the set of multilinear \( Y \)-proper polynomials of \( P_{m_1,\ldots,m_r}^G \). We observe that \( \Gamma_{m_1,\ldots,m_r}^G \) is a left \( S_{m_1} \times \cdots \times S_{m_r} \)-submodule of \( P_{m_1,\ldots,m_r}^G \) and the same holds for \( \Gamma_{m_1,\ldots,m_r}^G \cap T_G(A) \). Hence the factor module

\[
\Gamma_{m_1,\ldots,m_r}^G(A) := \Gamma_{m_1,\ldots,m_r}^G/(\Gamma_{m_1,\ldots,m_r}^G \cap T_G(A))
\]

is an \( S_{m_1} \times \cdots \times S_{m_r} \)-submodule of \( P_{m_1,\ldots,m_r}^G(A) \). We denote the \( S_{m_1} \times \cdots \times S_{m_r} \)-character of the factor module \( \Gamma_{m_1,\ldots,m_r}^G/(\Gamma_{m_1,\ldots,m_r}^G \cap T_G(A)) \) by \( \xi_{m_1,\ldots,m_r}^G(A) \), and by \( \gamma_{m_1,\ldots,m_r}(A) \) its dimension over \( F \). When we refer to \( A \) without any ambiguity, we shall use \( \gamma_{m_1,\ldots,m_r} \) instead of \( \gamma_{m_1,\ldots,m_r}(A) \).

Let \( L \) be an infinite dimensional vector space over \( F \), a field of characteristic zero, then we indicate by \( E = E(L) \) the Grassmann algebra generated by \( L \). Let \((G,\cdot)\) be a finite abelian group and suppose \( E \) is \( G \)-graded. In this section we want to study the \( G \)-graded identities of \( E \) in the case when \( L \) is a \( G \)-homogeneous space.

Let \( B_L = \{ e_1, e_2, \ldots \} \) be a linear basis of \( L \), where for any \( i \in \mathbb{N} \), \( e_i \) is a \( G \)-homogeneous element, so \( B_E = \{ e_{i_1} e_{i_2} \ldots e_{i_n} \mid n \in \mathbb{N}, i_1 < i_2 < \cdots < i_n \} \) is a basis of \( E \) as a vector space over \( F \). Notice that the existence of a homogeneous \( G \)-grading is equivalent to the existence of a map

\[
\| \| : B_L \to G.
\]

We have that the \( G \)-degree of the element \( e_{i_1} e_{i_2} \ldots e_{i_n} \) is

\[
\| e_{i_1} e_{i_2} \ldots e_{i_n} \| = \| e_{i_1} \| \| e_{i_2} \| \ldots \| e_{i_n} \| .
\]

In this case we say that the set

\[
\{ e_{i_1}, e_{i_2}, \ldots, e_{i_n} \}
\]

is the support of \( e_{i_1} e_{i_2} \ldots e_{i_n} \) and the non-negative integer \( n \) is its length.
For our purposes we shall pass from a fixed $G$-grading of $E$ to the grading associated to the quotient group $G/H$, for some special subgroups $H$ of $G$. More generally, let $A = \bigoplus_{g \in G} A^g$ be a $G$-graded algebra and let $H < G$, then for every coset $gH \in G/H$, we define $A^{gH} = \bigoplus_{f \in gH} A^f$. In particular, if $T$ is a transversal set for $H$ in $G$, then

$$A = \bigoplus_{t \in T} A^{tH}.$$  

We observe that for every $g, g' \in G$ $A^{gH}A^{g'H} \subseteq A^{gg'H}$, so $A$ inherits a structure of $G/H$-graded algebra and we shall call it quotient grading of $A$.

3. Graded identities of $E$

In what follows we shall denote by $Z(A)$ the center of the algebra $A$. Recall that in the case $E$ is the infinite dimensional Grassmann algebra, then $Z(E) = E^0$, where $E^0$ is the $\mathbb{Z}_2$-component of degree 0 in the canonical $\mathbb{Z}_2$-grading of $E$. In what follows we shall use the following notation: if $H$ is a normal subgroup of $G$ and no confusion occurs, we shall denote by $gH$ the coset $gH$.

In order to investigate the relations between the graded identities of $E$ with respect to $G$-gradings and to its quotient $G/H$-gradings, it is reasonable to consider the following homomorphism between free graded algebras

$$\pi: F\langle X \mid G \rangle \to F\langle Y \mid G/H \rangle,$$

where $Y$ is an infinite set of $G/H$-graded variables, such that for every $g \in G$ and for every $i \in \mathbb{N}$, $\pi(x^g_i) = y^g_i$. For any $G$-graded algebra $A$ and for any subgroup $H$ of $G$ we have the next result.

**Lemma 3.1.** Let $f(x_1, \ldots, x_n) \in F\langle X \mid G \rangle$ be a multilinear polynomial. If $\pi(f) \in T_{G/H}(A)$, then $f \in T_G(A)$.

**Proof.** Let $\varphi: x_i \mapsto a_i$ be a $G$-graded substitution, so $\|a_i\| = \|x_i\| = g_i \in G$ for some $g_i \in G$. Now we have that $a_i \in A^{g_i}$ and $a_i$ is homogeneous of degree $g_iH$ in the quotient grading. Then $\varphi$ is a $G/H$-graded substitution too. Due to the fact that $\pi(f) = f(y_1, \ldots, y_n) \in T_{G/H}(A)$, we have $0 = f(a_1, \ldots, a_n)$ and $f \in T_G(A)$. \hfill $\square$

Under opportune hypothesis, it is possible to invert this result. Above all, we give the following definition.

**Definition 3.2.** Let $G$ be a finite abelian group and suppose $E$ is $G$-graded. We say that the subgroup $H$ of $G$ has the property $\mathcal{P}$ if for any $h \in H$, $E^h$ has infinite elements of even length with pairwise disjoint support.

The interest of this property is given by the following proposition.

**Proposition 3.3.** Let $H < G$ having the property $\mathcal{P}$ and let $f \in F\langle X \mid G \rangle$ be a multilinear polynomial. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{G/H}(E)$. 
Proof. We have to prove just the only if part. Let
\[ f = f(x_1^{q_1}, x_2^{q_2}, \ldots, x_i^{q_i}, \ldots, x_{r-1}^{q_{n-1}}, \ldots, x_r^{l_{r-1}+1}, \ldots, x_r^{l_r}) \in T_G(E) \]
and let \( F = \pi(f) \). Let \( \varphi \) be any \( G/H \)-graded substitution, hence \( \varphi(y_j^h) = \sum_{h \in H} a_j^{gh} \), and by the multilinearity of \( f \), we can consider only substitutions \( \varphi \) such that \( y_j^h \mapsto a_j^{gh} \), for some \( h \in H \) and for any \( j \). Now observe that if every homogeneous component \( E^h \) has infinite elements of even length, then for every \( j \) and for every \( h \in H \) there exists \( b_j^{h^{-1}} \) of even length such that \( \| b_j^{h^{-1}} \| = h^{-1} \). For every \( h \in H \), \( w_j^g = a_j^{gh} b_j^{h^{-1}} \) is a homogeneous element of degree \( g \) in the \( G \)-grading of \( E \). Let us consider a new substitution \( \psi \) such that \( x_j^g \mapsto w_j^g \). This is a \( G \)-graded substitution. Now, since \( f \in T_G(E) \), \( 0 = f(w_1^{q_1}, \ldots, w_i^{q_i}, \ldots) = \prod_{h \in H} b_j^{h^{-1}} \cdot F(a_j^{gh}) \) because the \( b_j^{h^{-1}} \)'s are in \( Z(E) \) and this implies \( F(a_j^{gh}) = 0 \).

We consider now some subgroups of \( G \) having the property \( \mathcal{P} \).

Lemma 3.4. Let \( H = \langle g \rangle \) for some \( g \in G \). If \( L^g \) is infinite dimensional and \( |H| = n \) is odd, then \( H \) has the property \( \mathcal{P} \).

Proof. Let \( \{v_1, v_2, \ldots\} \) be a linear basis of \( L^g \) and let \( h = g^t \) for some \( t \in \mathbb{N} \). Notice that
\[ \| v_{i_1} \cdots v_{i_l} \| = g^t, \]
hence \( v_{i_1} \cdots v_{i_l} \in E^h \) if and only if \( g^t = g^l \) that is if and only if \( l \equiv t \mod n \). Now, if \( t \) is even, then the elements of \( E^h \), \( v_1 \cdots v_t \), \( v_{t+1} \cdots v_{2t} \), \( v_{2t+1} \cdots v_{(k+1)t} \), \ldots have pairwise disjoint supports of even length. Similarly, if \( t \) is odd, an infinite subset of elements of \( E^h \) having pairwise disjoint supports is given by \( v_1 \cdots v_t, v_{t+1} \cdots v_{t+n}, v_{t+n+1} \cdots v_{2(t+n)}, \ldots, v_{k(t+n)+1} \cdots v_{(k+1)(t+n)}, \ldots \) and we are done. \( \square \)

Proposition 3.5. Let \( G \) be a finite abelian group and
\[ H = \langle g \mid \dim_F L^g = \infty \text{ and } o(g) \text{ is odd} \rangle. \]
Then \( H \) has the property \( \mathcal{P} \).

Proof. For any \( h \in H \) there exist distinct elements \( b_1, \ldots, b_s \in H \) such that \( h = b_1^{t_1} \cdots b_s^{t_s} \) for some positive integers \( t_1, \ldots, t_s \). Then by Lemma 3.4 and its proof, for every \( i = 1, \ldots, s \), \( E^{b_i} \) has infinite elements
\[ w_1^i, w_2^i, \ldots, w_m^i, \ldots \]
of even length with pairwise disjoint supports, moreover these elements belong to the Grassmann algebra \( E_i \) generated by the subspace \( L^{b_i} \). We set
\[ u_m = w_m^1 w_m^2 \cdots w_m^s, \]
for \( m \geq 1 \) and clearly \( \{u_1, u_2, \ldots, u_m, \ldots\} \) is the required subset of \( E^h \). \( \square \)

As a consequence of Propositions 3.5 and 3.3, we have the following.
Theorem 3.6. Let $G$ be a finite abelian group of odd order and let

$$H = \langle g \mid \dim_F L^g = \infty \rangle.$$ 

Then the following properties hold:

1. For any multilinear polynomial $f(x_1, \ldots, x_n) \in F\langle X \rangle$ one has
   $$f \in T_G(E) \text{ if and only if } \pi(f) \in T_{G/H}(E).$$

2. In the quotient grading of $E$, $L^g$ is infinite dimensional if and only if $
   \overline{g} = 1_{G/H}.$

If $G$ is any finite abelian group, we have the following result.

Proposition 3.7. Let $G$ be a finite abelian group and $g \in G$ such that $\dim_F L^g = \infty$. Let $H = \langle g \rangle$ and $|H| = n$, an even number. Then $K = \langle g^{2t} \rangle$ has the property $P$.

Proof. Let $\{e_1, e_2, \ldots\}$ be a linear basis of $L^g$ and let $k = g^{2t} \in K$, then the elements of $E^k e_1 \ldots e_2t, e_{2t+1} \ldots e_{4t}, \ldots, e_{s(2t)+1} \ldots e_{(s+1)(2t)}, \ldots$ have pairwise disjoint supports of even length.

Now let us consider the following subsets of $G$:

$$\mathcal{I} = \{g \in G \mid \dim_F L^g = \infty\},$$

$$\mathcal{I}_1 = \{g \in \mathcal{I} \mid o(g) \text{ is odd}\},$$

$$\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1$$

and

$$\mathcal{I}_3 = \{g^2 \mid g \in \mathcal{I}_2\} - \mathcal{I}_1.$$ 

We have the following.

Theorem 3.8. Let $G$ be a finite abelian group and let $H = \langle g \mid g \in \mathcal{I}_1 \cup \mathcal{I}_3 \rangle$. Then the following properties hold:

1. For any multilinear polynomial $f = f(x_1, \ldots, x_n) \in F\langle X \rangle$ one has
   $$f \in T_G(E) \text{ if and only if } \pi(f) \in T_{G/H}(E).$$

2. In the quotient grading of $E$, if $L^g$ is infinite dimensional, then $\overline{g^2} = 1 \in G/H$.

Proof. (1) Let $h \in H$, then there exist $a_1, \ldots, a_r \in \mathcal{I}_1, b_1, \ldots, b_s \in \mathcal{I}_3$ and positive integers be such that $h = a_1^{m_1} \ldots a_r^{m_r} b_1^{m_{r+1}} \ldots b_s^{m_{r+s}}$. Let $a_{r+1}, \ldots, a_{r+s} \in \mathcal{I}_2$ such that $b_i = a_{r+i}^2$, then $\dim_F L^{a_i} = \infty$ for any $i = 1, \ldots, r+s$. Let us denote by $E_i$ the Grassmann algebra generated by the subspace $L^{a_i^{m_i}}$. As in the proof of Proposition 3.5 for any $i = 1, \ldots, r+s$, $E_i$ contains infinitely many elements

$$w_1^i, w_2^i, \ldots, w_m^i,$$

of even length with pairwise disjoint supports. Moreover, for all $m \geq 1$ we have that

$$\|w_m^i\| = a_i^{m_i} \text{ if } i = 1, \ldots, r \text{ and } \|w_m^i\| = b_i^{m_i} \text{ for } i = r+1, \ldots, r+s.$$ 

We consider in $E^h$ the elements $u_m = w_1^m \ldots w_r^m \ldots w_r^{m+s}, m \geq 1$; clearly the elements $\{u_m | m \geq 1\}$ have pairwise disjoint supports and they have even length. Now $H$ has the property $P$ and the assertion comes by Proposition 3.3.
Let $\bar{g} = gH \in G/H$ be such that $L^\bar{g} = \bigoplus_{h \in H} L^g h$ is infinite dimensional. Since $G$ is finite there exists $g' \in gH$ such that $L^{g'}$ is infinite dimensional. If $o(g')$ is odd, then $g' \in H$ and so $gH = g'H = 1_{G/H}$. If $o(g')$ is even, then $g'^2 \in H$ and so $(gH)^2 = (g'H)^2 = 1_{G/H}$.

\[\square\]

4. Graded codimensions and cocharacters of $E$

We shall study graded codimensions and graded cocharacters for $E$ in the case $\dim_F L^1_G$ is infinite and all the other homogeneous components of $L$ have finite dimension. We shall use the language of the representation theory of symmetric groups (see the book [13] by Sagan for more details).

**Theorem 4.1.** Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Suppose that $L^{g_1}$ has infinite dimension. Let

\[l_{g_1}, l_{g_2}, \ldots, l_{g_r} \in \mathbb{N}\]

such that

\[l_{g_1} + l_{g_2} + \cdots + l_{g_r} = m.\]

Then $P_{l_{g_1}, \ldots, l_{g_r}} \subseteq T_G(E)$ or for any $f \in P_{l_{g_1}, l_{g_2}, \ldots, l_{g_r}}$ one has

\[f(x_1^{q_{g_1}}, \ldots, x_l^{q_{g_1}}, \ldots, x_{i-1}^{q_{g_i}}, \ldots, x_l^{q_{g_i}}, \ldots) \in T_G(E)\]

if and only if $f(x_1, \ldots, x_m) \in T(E)$.

**Proof.** It is sufficient to prove that if $P_{l_{g_1}, \ldots, l_{g_r}} \nsubseteq T_G(E)$, then any element of $P_{l_{g_1}, \ldots, l_{g_r}} \cap T_G(E)$ is an ordinary polynomial identity for $E$. Let us suppose

\[P_{l_{g_1}, \ldots, l_{g_r}} \nsubseteq T_G(E),\]

then there exists a graded monomial with a non-zero graded evaluation of elements $a_1, \ldots, a_m$ of the basis $B_L$ of $E$. Any other monomial of $P_{l_{g_1}, \ldots, l_{g_r}}$ is non-zero with respect to the same evaluation. Since $L^{g_1}$ is infinite dimensional, we can always suppose $a_1, \ldots, a_m$ are of even length multiplying them by some $e_i$'s of degree $g_1$. Now let us consider the elements of the basis of $L^{g_1}$ which are not involved in the expression of the given elements $a_1, \ldots, a_m$, to say $v_i$'s. Clearly, the latter generate an infinite dimensional Grassmann algebra $E'$, hence $T(E) = T(E')$. Let $f = f(x_1^{q_{g_1}}, \ldots, x_l^{q_{g_1}}, \ldots, x_{i-1}^{q_{g_i}}, \ldots, x_l^{q_{g_i}}, \ldots) \in T_G(E)$ and let $\varphi$ be any substitution such that $x_i \mapsto v_i \in E'$ for any $i$. Let us consider a new substitution $\psi$ such that $x_i^{q_{g_i}} \mapsto v_i a_i$. This is a $G$-graded substitution on $E$. Now, since $f \in T_G(E)$,

\[0 = f(v_1 a_1, \ldots, v_m a_m) = a_1 \cdots a_m f(v_1, \ldots, v_m)\]

because the $a_i$'s are in $Z(E)$ and this implies $f(v_1, \ldots, v_m) = 0$ because the supports of $v_1, \ldots, v_m$ are distinct from those of $a_1, \ldots, a_m$ and $a_1 \cdots a_m \neq 0$ by hypothesis, then $f \in T(E') = T(E)$ and we are done. \[\square\]

**Theorem 4.2.** Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Let $L$ be a $G$-homogeneous vector space over $L$ such that $\dim_F L^{g_1} = \infty$ and $\dim_F L^{g_i} = k_i < \infty$, if $i \neq 1$. If $E = E(L)$ is the Grassmann algebra generated by $L$, then $T_G(E)$ is generated as a $T_G$-ideal by the following polynomials:
(1) \([u_1, u_2, u_3]\) for any choice of the G-degree of the variables \(u_1, u_2, u_3\),
(2) monomials of \(P_{0, t_2, \ldots, t_r}\) such that \(\sum_{i=2}^r t_i = 1 + \sum_{i=2}^r k_i\),
(3) monomials of \(P_{0, t_2, \ldots, t_r}\) such that \(\sum_{i=2}^r t_i < 1 + \sum_{i=2}^r k_i\)
and \(P_{0, t_2, \ldots, t_r} \subseteq T_G(E)\).

**Proof.** In light of Theorem 4.1 we have that \(T_G(E)\) is generated by the polynomials from (1) of the claim and by all monomials of \(P_{1, \ldots, l_r}\) such that \(P_{1, \ldots, l_r} \subseteq T_G(E)\). Notice that a graded monomial \(w\) is surely a graded polynomial identity when the sum of the numbers of its indeterminates of \(G\)-degree different from \(g_1\) is strictly greater than \(\sum_{i=2}^r k_i\). Moreover, for any \(l_1, \ldots, l_r \in \mathbb{N}\), any monomial in \(P_{1, \ldots, l_r}\) is in the \(T_G\)-ideal generated by the monomials in \(P_{0, t_2, \ldots, t_r}\). Now we have just to observe that the monomials in \(P_{0, t_2, \ldots, t_r}\) follow from the monomials in \(P_{0, t_2-1, \ldots, t_r}, \ldots, P_{0, t_2, \ldots, l_1-1, \ldots, t_r}\) for \(l_i \geq 1\) due to the Young rule. Hence if \(l_2, \ldots, l_r \in \mathbb{N}\) are such that \(l_2 + \cdots + l_r > 1 + \sum_{i=2}^r k_i\), then \(P_{0, t_2, \ldots, t_r}\) is in the \(T_G\)-ideal generated by the monomials of \(P_{0, t_2, \ldots, t_r}\) such that \(t_2 + \cdots + t_r = 1 + \sum_{i=2}^r k_i\) and the claim follows.

We have the following corollary which proof repeats verbatim the one of Proposition 5 of [6].

**Corollary 4.3.** Let \(G = \{g_1, \ldots, g_r\}\) be a finite abelian group with \(g_1 = 1_G\). If \(L^{g_1}\)
has infinite dimension and \(l_{g_1}, l_{g_2}, \ldots, l_{g_r} \in \mathbb{N}\) are such that \(l_{g_1} + l_{g_2} + \cdots + l_{g_r} = m\),
then
\[
c_{l_{g_1}, \ldots, l_{g_r}}(E) = 0 \quad \text{or} \quad c_{l_{g_1}, \ldots, l_{g_r}}(E) = 2^{m-1}
\]
and in the latter case, \(P_{l_{g_1}, \ldots, l_{g_r}}(E)\) and \(P_m(E)\) are isomorphic \(S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}\)-modules.

If \(G\) is a finite abelian group and \(L\) is a vector space with basis \(B_L = \{e_1, e_2, \ldots\}\), let
\[
\varphi: B_L \to G
\]
be any map. As we said before, \(\varphi\) induces a \(G\)-grading on \(E\). Let us consider now the set
\[
S(\varphi) = \{(l_{g_1}, l_{g_2}, \ldots, l_{g_r}) \in \mathbb{N}^r \mid P_{l_{g_1}, l_{g_2}, \ldots, l_{g_r}} \subseteq T_G(E)\}.
\]

We note that if \(L^{1_G}\) is the only homogeneous subspace of \(L\) such that \(\dim_F L^{1_G} = \infty\), then \(S(\varphi) \neq \emptyset\).

\(S(\varphi)\) allows us to give the complete description of the sequence of the graded cocharacters and codimensions of \(E\). In fact, we have the following proposition.

**Proposition 4.4.** Let \(G = \{g_1, \ldots, g_r\}\) be a finite abelian group and \(L\) be a \(G\)-homogeneous vector space with linear basis \(\{e_1, e_2, \ldots\}\). Let \(\varphi: B_L \to G\) be a map such that \(|\varphi^{-1}(1_G)| = \infty\) and consider \(E\), the \(G\)-graded Grassmann algebra obtained by \(\varphi\). Then
\[
\chi^{G}_{l_{g_1}, \ldots, l_{g_r}}(E) = 2^{|G|-1} \sum_{\alpha_1=0}^{l_{g_1}-1} \sum_{\alpha_2=0}^{l_{g_2}-1} \cdots \sum_{\alpha_r=0}^{l_{g_r}-1} \lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}
\]
if \((l_{g_1}, \ldots, l_{g_r}) \notin S(\varphi)\), where \(\lambda_{a_i}\) is the hook partition of leg \(a_i\) and arm \(l_{g_i} - a_i + 1\).
Moreover
\[
c_n^G(E) = 2^{n-1} \sum_{(l_{g_1}, \ldots, l_{g_r}) \notin S(\varphi) \atop l_{g_1} + \ldots + l_{g_r} = n} \binom{n}{l_{g_1}, \ldots, l_{g_r}}.
\]

**Proof.** By Corollary 4.3 the spaces \(P_n(E)\) and \(P_{l_{g_1}, \ldots, l_{g_r}}^G(E)\) are \(S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}\)-isomorphic modules. Hence the result follows using the decomposition of \(\chi_n(E) = \sum_{i=0}^{n-1} (n - i, 1^i)\) and the representation theory of symmetric groups. More precisely, it follows by Branching Rule that when we restrict the irreducible representation \(\nu_i = (n - i, 1^i)\) of \(S_n\) to its subgroup \(S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}\) then its \(S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}\)-irreducible components are \(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}\) for some \(\lambda_{a_i} = (l_{g_1} - a_i, 1^{a_i})\). By Frobenius Reciprocity Law the multiplicity of \(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}\) in the decomposition of \(\nu_i\) equals the multiplicity of \(\nu_i\) in the induced representation \((\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r})^\uparrow_{S_n}\). We argue only for \(r = 2\) because the other cases are treated similarly. By the Littlewood-Richardson Rule, if \(c_{a,b}^i\) is the multiplicity of \(\nu_i\) in the induced representation \((\lambda_a \otimes \lambda_b)^\uparrow_{S_n}\), \(c_{a,b}^i\) is the number of semistandard tableau \(T\) such that \(T\) has shape \(\nu_i/\lambda_a\), content \(\lambda_b\) and the row word of \(T\) is a reverse lattice permutation. Since \(\nu_i\) and \(\lambda_a\) are both hook partitions, then the skew shape \(\nu_i/\lambda_a\) has at most two connected components. The first one is a row of length \(n - i - (l_{g_1} - a) = l_{g_2} - (i - a)\), the second is a column of height \(i - a\). By the previous conditions on the semistandard tableau \(T\), we obtain that the entries in the column constitute a standard tableau \(T'\). If 1 does not appear in \(T'\) then \(1 + b = l_{g_2} - (i - a)\), on the other hand if one entry of \(T'\) is 1 then \(b = l_{g_2} - (i - a)\). Therefore \(c_{a,b}^i\) is non-zero if and only if either \(i - a + b = l_{g_2} - 1\) or \(i - a + b = l_{g_2}\), in both cases one has \(c_{a,b}^i = 1\) since the semistandard tableau \(T\) is uniquely determined. Then there exist exactly two hook partitions in the decomposition of \((\lambda_a \otimes \lambda_b)^\uparrow_{S_n}\). Repeating this process, we have that the total multiplicity of the hook partitions appearing in the decomposition of \((\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r})^\uparrow_{S_n}\) is \(2^{r-1} = 2^{|G|-1}\). Due to the fact that all of these partitions are components of \(\chi_n(E)\), we have that the multiplicity of \(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}\) in \(\chi_{l_{g_1}, \ldots, l_{g_r}}\) is exactly \(2^{|G|-1}\).

Finally we have just to use Proposition 2.5 while Corollary 4.3 says that
\[
c_{s_{g_1}, \ldots, l_{g_r}}^G(A) = c_n(E) = 2^{n-1}
\]
and the assertion follows. \(\square\)

In light of the previous results, we can give a new proof of an Anisimov’s result (see [1]). Let \(p\) be a prime odd number and let \(G = \mathbb{Z}_p\), then we have the following:

**Proposition 4.5.** If there exists \(k \in G\), \(k \neq 0\) such that \(\dim F L^k = \infty\), then for any \(m \in \mathbb{N}\),
\[
c_m(E) = p^m 2^{m-1}.
\]
If for any \( k \in \mathbb{Z}_p - \{0\} \) \( \dim_F L^k < \infty \), then for any \( m \in \mathbb{N} \),
\[
c_m(E) = 2^{m-1} \sum_{(m_0, \ldots, m_{p-1}) \notin S(\varphi)} \binom{m}{m_0, \ldots, m_{p-1}} c_{m_0, \ldots, m_{p-1}}(E).
\]

**Proof.** If exists \( k \in \mathbb{Z}_p - \{0\} \) such that \( \dim_F L^k = \infty \), then \( \langle k \rangle \) has the property \( \mathcal{P} \). In particular, \( \mathbb{Z}_p \) has this property. The quotient grading on \( E \) is the trivial one and in light of Proposition 3.3 every \( G \)-graded polynomial identity of \( E \) is an ordinary polynomial identity of \( E \). Then, for any \( m \in \mathbb{N} \), we have
\[
c_m(E) = \sum_{m_0 + \cdots + m_{p-1} = m} \binom{m}{m_0, \ldots, m_{p-1}} c_{m_0, \ldots, m_{p-1}}(E)
\]
\[
= 2^{m-1} \sum_{m_0 + \cdots + m_{p-1} = m} \binom{m}{m_0, \ldots, m_{p-1}} = p^m 2^{m-1}.
\]

If for any \( k \in \mathbb{Z}_p - \{0\} \) \( \dim_F L^k < \infty \), then \( \dim_F L^0 = \infty \). By Corollary 4.3 and Proposition 4.4 for any \( m \in \mathbb{N} \), we have
\[
c_m(E) = \sum_{(m_0, \ldots, m_{p-1}) \notin S(\varphi)} \binom{m}{m_0, \ldots, m_{p-1}} c_{m_0, \ldots, m_{p-1}}(E)
\]
\[
= 2^{m-1} \sum_{(m_0, \ldots, m_{p-1}) \notin S(\varphi)} \binom{m}{m_0, \ldots, m_{p-1}}.
\]

and we are done. \( \square \)

Notice that the case \( p = 2 \) has been completely solved in [3] and in [6].

5. **Two Examples of Gradings by Groups of Order 4**

5.1. **\( \mathbb{Z}_4 \)-grading on \( E \).** The group \( G = \mathbb{Z}_4 = \{0, 1, 2, 3\} \) is the first cyclic group such that its order is not prime. In light of Theorem 3.8 we have that \( T_G(E) \) “behaves” as \( T_{\mathbb{Z}_2}(E) \) in the quotient grading if \( \dim_F L^1 = \infty \) or \( \dim_F L^3 = \infty \). Moreover, because of Theorem 4.1 the only cases to be studied are the ones for which \( \dim_F L^2 = \infty \). We study a particular case of \( G \)-grading when \( \dim_F L^2 = \infty \).

Let \( L \) be a vector space with basis \( B_L = \{e_1, e_2, \ldots\} \) and let us consider the following map:
\[
\varphi: B_L \rightarrow G
\]
such that \( \varphi(e_1) = 1, \varphi(e_2) = 3 \) and \( \varphi(e_i) = 2 \) for any \( i \neq 1, 2 \). Then \( \varphi \) induces a \( G \)-grading on \( E \) such that \( \dim L^2 = \infty \). In particular, it is easy to see that:

- \( E^0 = \text{span}\{e_1^k e_2^l e_3 \mid k \equiv 2 \text{mod } 2 \} \); 
- \( E^1 = \text{span}\{e_1^k e_2^l e_3 \mid k \equiv 1 \text{mod } 2 \} \);
• $E^2 = \text{span}\{e_1^k e_2^t e_1, \ldots e_1 s \equiv 1 \mod 2 \text{ and } (k, t) \in \{(1, 1), (0, 0)\}\}$;
• $E^3 = \text{span}\{e_1^k e_2^t e_1, \ldots e_1 s \equiv 0 \mod 2 \text{ and } (k, t) = (0, 1)\} \\
\text{or } s \equiv 1 \mod 2 \text{ and } (k, t) = (1, 0)$. \\

From the previous description of the $G$-graded homogeneous components of $E$ one easily has the following.

**Proposition 5.1.** The following monomials are $G$-graded polynomial identities of $E$:

$$x_1^1 x_2^1 x_3^1, \ x_1^3 x_2^2 x_3, \ x_1^1 x_2^2 x_3, \ x_1^2 x_2^1 x_3, \ x_1^3 x_2^2 x_3, \ x_1^1 x_2^3 x_3, \ x_1^1 x_2 x_3^1.$$  

**Proof.** We argue only for the monomial $x_1^1 x_2^1 x_3^1$ because the other cases are treated similarly. From the previous observations it follows that if we want to evaluate one variable of $G$-homogeneous degree 1, we shall deal with a word which contains at least one of the basis elements $e_1, e_2$. Now the proposition follows because any evaluation of three variables of $G$-degree 1 repeats twice one between $e_1$ or $e_2$ and we are done. \qed

We have not only monomial graded identities.

**Proposition 5.2.** The following polynomials are $G$-graded polynomial identities of $E$:

$$x_1^2 x_2^2 + x_2^2 x_1^2, \ [x_1^1, x_2^1], \ [x_1^3, x_2^3], \ [x_1^0, x_2^0],$$

for any $g \in G$.

**Proof.** The fact that $x_1^2 x_2^2 + x_2^2 x_1^2$ and $[x_1^0, x_2^0]$ are graded identities follows directly from the description of $E^0, E^2$. For, the elements of $E^0$ have even length so they are in the center of $E$. On the other hand, the elements of $E^2$ have odd length.

Let us argue for $[x_1^1, x_2^1]$. If we evaluate the variable $x_1^1$ with a $G$-degree 1 element of $E$ of odd length, we are dealing with a word containing $e_1$. Hence the evaluation of $x_1^1$ lies in the center of $E$ and the commutator vanishes, otherwise $e_1$ appears twice. We argue analogously for $[x_1^3, x_2^3]$ and we are done. \qed

We are now ready to compute $T_G(E)$. For this purpose, let

$$I_1 = \langle [u_1, u_2, u_3], \ x_1^2 x_2^2 + x_2^2 x_1^2, \ [x_1^1, x_2^1], \ [x_1^3, x_2^3], \ x_1^2 x_2^2 + x_2^2 x_1^2, \ [x_1^0, x_1^1], \ x_1^3 x_2^3, \ x_1^3 x_2^3, \ x_1^1 x_2^3, \ x_1^3 x_2 x_3^3 \rangle_{T_G} \mod I,$$

for any $g \in G$. Observe that modulo $I$ the identity $[x_1^1 x_2^3, x_2^3]$ equals the polynomial $x_1^1 [x_2^3, x_2^3] + x_2^3 [x_1^1, x_2^3]$ that is

(1)  
$$x_1^1 [x_2^3, x_2^3] = -x_2^3 [x_1^1, x_2^3] \ (\text{mod } I).$$

Analogously we have

(2)  
$$x_1^1 [x_2^3, x_3^3] = -x_2^3 [x_1^1, x_3^3] \ (\text{mod } I),$$

(3)  
$$x_1^3 [x_2^3, x_3^1] = +x_2^3 [x_1^3, x_3^1] \ (\text{mod } I),$$

(4)  
$$[x_1^1, x_3^3] x_2^3 = -[x_1^1, x_3^3] x_2^3 \ (\text{mod } I).$$

Then we have the following.
Theorem 5.3. \( I_1 = T_G(E) \).

Proof. The Propositions 5.1 and 5.2 give the inclusion \( I_1 \subseteq T_G(E) \). We shall use the method of \( Y \)-proper polynomials. In light of Proposition 5.1, we have that the only non-trivial subspaces of multilinear \( Y \)-proper polynomials are: \( \Gamma_{0,1,l,1}, \Gamma_{0,1,l,0}, \Gamma_{0,0,l,0}, \Gamma_{0,0,l,0} \), and \( \Gamma_{0,0,l,2} \) for any \( l \in \mathbb{N} \). Let us argue only for \( \Gamma_{0,1,l,1} \) because the other cases are treated similarly. Let \( w \) be any non-zero element in \( \Gamma_{0,1,l,1} \), then \( w \) can be written as a linear combination of the following polynomials:

\[
\begin{align*}
x^1 x_1^2 \ldots x_l^2 x^3, \\
x_1^2 \ldots x_l^2 [x^1, x^3], \\
x^1 x_1^2 \ldots \widehat{x_i^2} \ldots x_l^2 [x_i^2, x^3], \\
x_1^2 \ldots \widehat{x_i^2} \ldots x_l^2 x^3 [x^1, x_i^2].
\end{align*}
\]

The Equations (1) and (2) give us

\[
x^1 x_1^2 \ldots \widehat{x_i^2} \ldots x_l^2 [x_i^2, x^3] + \alpha x_1^2 \ldots x_l^2 [x^1, x^3] \equiv x_1^2 \ldots \widehat{x_i^2} \ldots x_l^2 x^3 [x_i^2, x^1].
\]

Analogously it can be shown \( x^1 x_1^2 \ldots \widehat{x_i^2} \ldots x_l^2 [x_i^2, x^3] \) is a linear combination of \( x_1^2 \ldots x_l^2 [x^1, x^3] \) and \( x^1 x_1^2 \ldots x_l^2 x^3 \). Finally, any non-trivial polynomial of \( \Gamma_{0,1,l,1} \) is a linear combination of the following polynomials:

\[
\begin{align*}
w_1 &= x^1 x_1^2 \ldots x_l^2 x^3, \\
w_2 &= x_1^2 \ldots x_l^2 [x^1, x^3].
\end{align*}
\]

Now it suffices to show that \( w_1, w_2 \) are linearly independent modulo \( T_G(E) \). Suppose by contradiction they are linearly dependent, then there exist \( \alpha_1, \alpha_2 \in F \) such that \( \sum_{i=1}^2 \alpha_i w_i \in T_G(E) \). Let us consider the following substitution \( \varphi \):

\[
\begin{align*}
\varphi(x^1) &= e_2 e_3, \\
\varphi(x^3) &= e_1 e_4, \\
\varphi(x_i^2) &= e_{i+4} \quad \text{for any} \quad i = 1, \ldots, l.
\end{align*}
\]

Then

\[
\varphi(w_2) = 0
\]

but

\[
\varphi(w_1) = e_2 e_3 e_5 \ldots e_{l+4} e_1 e_4 \neq 0,
\]

a contradiction and the proof is complete.

According to [6], it seems that \( f \) is a multilinear \( \mathbb{Z}_4 \)-graded identity of \( E \) if and only if \( \gamma(f) \) is a \( \mathbb{Z}_2 \)-graded identity of \( E_2 \) for some special function \( \gamma \).
5.2. $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings on $E$. Theorem 3.8 is useful in order to reduce the order of the grading group if $G$ has non-trivial squares. This is not the case of finite powers of $\mathbb{Z}_2$. In this section we shall deal with some special cases of $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $E$.

Let us suppose firstly that $L$ is a $G$-homogeneous vector space over $F$ such that
\[ \dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \dim_F L^{(1,1)} = \infty, \quad \dim_F L^{(0,0)} < \infty, \]
and let $E = E(L)$ be the Grassmann algebra generated by $L$. Let $B_1 = \{e_1, e_2, \ldots\}$ be a basis of $L^{(1,0)}$, $B_2 = \{e'_1, e'_2, \ldots\}$ be a basis of $L^{(0,1)}$ and $B_3 = \{e''_1, e''_2, \ldots\}$ be a basis of $L^{(1,1)}$ as vector spaces. Let us consider the map
\[ \varphi: B_1 \cup B_2 \cup B_3 \to G \]
associated to the $G$-grading over $E$. It is such that $\varphi(e_i) = (1, 0)$ for any $i = 1, 2, \ldots$, $\varphi(e'_j) = (0, 1)$ for any $j$ and $\varphi(e''_s) = (1, 1)$ for any $s$. We have the following.

Lemma 5.4. $G$ has the property $\mathcal{P}$.

Proof. The pairwise disjoint sets of elements $\{e_{2k+1}e'_{2k+1} \mid k \geq 0\}$, $\{e_{2k}e''_{2k} \mid k \geq 1\}$, $\{e'_k e''_{k+1} \mid k \geq 1\}$, and $\{e''_{6k+3} e''_{6k+5} \mid k \geq 1\}$ belong respectively to $E^{(1,1)}$, $E^{(0,1)}$, $E^{(1,0)}$, $E^{(0,0)}$ and the proof is complete. \qed

In light of the Proposition 3.3, we have the following result.

Theorem 5.5. Let $L$ be a $G$-homogeneous vector space over $F$ such that
\[ \dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \dim_F L^{(1,1)} = \infty, \quad \dim_F L^{(0,0)} < \infty, \]
and let $E = E(L)$ the Grassmann algebra generated by $L$. Let $f$ be a multilinear polynomial in $F(X \mid G)$. Then $f \in T_G(E)$ if and only if $\pi(f) \in T(E)$. \qed

Proof. By Lemma 5.4 $G$ has the property $\mathcal{P}$ and we are done because of Proposition 3.3

Suppose now that $L$ is a $G$-homogeneous vector space over $F$ such that
\[ \dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \infty, \quad \dim_F L^{(0,0)} < \infty, \quad \dim_F L^{(1,1)} < \infty \]
and let $E = E(L)$ the Grassmann algebra generated by $L$. Let $B_1 = \{e_1, e_2, \ldots\}$ be a basis of $L^{(1,0)}$, $B_2 = \{f_1, f_2, \ldots\}$ be a basis of $L^{(0,1)}$ as vector spaces. Let us consider the map
\[ \varphi: B_1 \cup B_2 \to G \]
associated to the $G$-grading over $E$. It is such that $\varphi(e_i) = (1, 0)$ for any $i = 1, 2, \ldots$, $\varphi(f_j) = (0, 1)$ for any $j$. We have the analog of Lemma 5.4. Let $H = \langle g \rangle$, where $g = (1, 1)$. Notice that $H \equiv \mathbb{Z}_2$. Then we obtain the next result.

Lemma 5.6. $H$ has the property $\mathcal{P}$.

Theorem 5.7. Let $L$ be a $G$-homogeneous vector space over $F$ such that
\[ \dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \infty, \quad \dim_F L^{(0,0)} < \infty, \quad \dim_F L^{(1,1)} < \infty \]
and let $E = E(L)$ the Grassmann algebra generated by $L$. Let $f$ be a multilinear polynomial in $F(X \mid G)$. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{\mathbb{Z}_2}(E)$, where in the $\mathbb{Z}_2$-grading, the dimension of the $G$-homogeneous underlying vector space $L^0$ is finite.
Proposition 5.8. The following monomials are $G$-graded polynomial identities of $E$: \[
\bigcup_{r+s \geq k+1} W_{r,s}.
\]

Proof. In light of the fact that any element of $G$-degree $(1,0)$ and $(1,1)$ may contain at least one element among $\{e_1, \ldots, e_k\}$, we have that if $\varphi$ is any graded substitution of $w_{r,s}$, one of the $k$ basis elements repeats at least twice and the Proposition follows.

We have not only monomial graded identities.

Proposition 5.9. The following polynomials are $G$-graded polynomial identities of $E$: 
\[
\begin{align*}
&x_1^{(0,1)} x_2^{(0,1)} + x_2^{(0,1)} x_1^{(0,1)}, & x_1^{(0,1)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(0,1)}, & x_1^{(1,0)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(1,0)}, \\
&[x_1^{(0,0)}, x_2^g], & [x_1^{(1,1)}, x_2^g],
\end{align*}
\]
for any $g \in G$. 

Proof. It follows directly from the description of the various $E^g$. 

We are now ready to compute $T_G(E)$. For this purpose, let $I_2$ the $T_G$ ideal generated by 
\[ \{u_1, u_2, u_3\}, \bigcup_{r+s=k+1} W_{r,s}, \ x_1^{(0,1)} x_2^{(0,1)} x_0^{(0,1)}, \ x_1^{(0,1)} x_2^{(1,0)} + x_2^{(0,1)} x_1^{(0,1)}, \ x_1^{(0,1)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(0,1)}, \ x_1^{(1,0)} x_2^{(0,1)} + x_2^{(1,0)} x_1^{(1,0)}, \ x_1^{(0,0)} x_2^{(0,0)} + x_1^{(1,1)} x_2^{(1,1)} \] for any $g \in G$.

We have the following:

**Theorem 5.10.** $I_2 = T_G(E)$.

**Proof.** The Propositions 5.8 and 5.9 give the inclusion $I_2 \subseteq T_G(E)$. We shall use the method of $Y$-proper polynomials once again. The only non-trivial subspaces of $Y$-proper polynomials are $\Gamma_{0,t,r,s}$, such that $r+s \leq k$. Due to the anticommutativity of the variables of $G$-degree $(0,1), (1,0)$ and the commutativity of the variables of $G$-degree $(1,1)$, as in the previous proposition, we can write any polynomial in $\Gamma_{0,t,r,s}$ as linear combination of polynomials
\[ x_1^{(0,1)} \ldots x_t^{(0,1)} x_{t+1}^{(1,0)} \ldots x_{t+r}^{(1,0)} x_{t+r+1}^{(1,1)} \ldots x_{t+s}^{(1,1)} \]
such that $r+s \leq k$ which are clearly linearly independent modulo $T_G(E)$. The conclusion follows as in the proof of Theorem 5.3 \hfill \Box

Again, according to [9], it seems that $f$ is a multilinear $Z_2 \times Z_2$-graded identity of $E$ if and only if $\gamma(f)$ is a $Z_2$-graded identity of $E_2$, for some special function $\gamma$.

**References**


IMECC, Universidade Estadual de Campinas,
Rua Sérgio Buarque de Holanda 651,
Campinas (SP), Brazil

E-mail: centrone@ime.unicamp.br