GLOBAL BEHAVIOR OF THE DIFFERENCE EQUATION

\[ x_{n+1} = \frac{ax_{n-3}}{b+cx_{n-1}x_{n-3}}, \]  

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Abstract.

In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation

\[ x_{n+1} = \frac{ax_{n-3}}{b+cx_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots \]

where \(a, b, c\) are positive real numbers and the initial conditions \(x_{-3}, x_{-2}, x_{-1}, x_0\) are real numbers.

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [3, 5, 8, 9, 11, 12, 14, 15, 18] and the references therein.

In [4], the authors discussed the global behavior of the difference equation

\[ x_{n+1} = \frac{Ax_{n-2r-1}}{B+Cx_{n-2l}x_{n-2k}}, \quad n = 0, 1, \ldots \]

where \(A, B, C\) are nonnegative real numbers and \(r, l, k\) are nonnegative integers such that \(l \leq k\) and \(r \leq k\).

In [2] we have discussed global asymptotic stability of the difference equation

\[ x_{n+1} = \frac{A+Bx_{n-1}}{C+Dx_n^2}, \quad n = 0, 1, \ldots \]

where \(A, B\) are nonnegative real numbers and \(C, D > 0\).

We have also discussed in [1] the global behavior of the solutions of the difference equation

\[ x_{n+1} = \frac{Bx_{n-2k-1}}{C+D\prod_{i=1}^{k} x_{n-2i}}, \quad n = 0, 1, \ldots \]


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In [17], D. Simsek et al. introduced the solution of the difference equation
\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots \]
where \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty) \).

Also in [16], D. Simsek et al. introduced the solution of the difference equation
\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots \]
with positive initial conditions.

R. Karatas et al. [10] discussed the positive solutions and the attractivity of the difference equation
\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, \ldots \]
where the initial conditions are nonnegative real numbers.

In [6], E.M. Elsayed discussed the solutions of the difference equation
\[ x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots \]
where the initial conditions are nonzero real numbers.

In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation
\[ x_{n+1} = \frac{ax_{n-3}}{b + cx_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots \]
where \( a, b, c \) are positive real numbers and the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0 \) are real numbers.

2. Solution of equation (1.1)

In this section, we establish the solutions of equation (1.1).

From equation (1.1), we can write
\[ x_{2n+1} = \frac{ax_{2n-3}}{b + cx_{2n-1}x_{2n-3}}, \quad n = 0, 1, \ldots \]
(2.1)
\[ x_{2n+2} = \frac{ax_{2n-2}}{b + cx_{2n}x_{2n-2}}, \quad n = 0, 1, \ldots \]
(2.2)

Using the substitution \( y_{2n-1} = \frac{1}{x_{2n-1}x_{2n-3}} \), equation (2.1) is reduced to the linear nonhomogeneous difference equation
\[ y_{2n+1} = \frac{b}{a} y_{2n-1} + \frac{c}{a}, \quad y_{-1} = \frac{1}{x_{-1}x_{-3}}, \quad n = 0, 1, \ldots \]
(2.3)
Note that for the backward orbits, the product reciprocals \( v_{2k-1} = \frac{1}{x_{2k-1}x_{2k-3}} \) satisfy the equation

\[
v_{2k+1} = \frac{a}{b} v_{2k-1} - \frac{c}{b}, \quad v_{-1} = \frac{1}{x_{-1}x_{-3}} = -\frac{c}{b}, \quad k = 0, 1, \ldots
\]

Therefore,

\[
x_{2n-1}x_{2n-3} = -\frac{b}{c} \sum_{r=0}^{n} (\frac{b}{c})^r.
\]

By induction on \( n \) we can show that for any \( n \in \mathbb{N} \), if \( x_{2n-1}x_{2n-3} = -\frac{b}{c} \sum_{r=0}^{n} (\frac{b}{c})^r \), then \( x_{-1}x_{-3} = -\frac{b}{c} \).

The same argument can be done for equation (2.2) and will be omitted.

Now we are ready to give the following lemma.

**Lemma 2.1.** The forbidden set \( F \) of equation (1.1) is

\[
F = \bigcup_{n=0}^{\infty} \{ (u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-3} = -\left( \frac{b}{c} \sum_{r=0}^{n} (\frac{b}{c})^r \right) \frac{1}{u_{-1}} \} \cup \bigcup_{m=0}^{\infty} \{ (u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-2} = -\left( \frac{b}{c} \sum_{r=0}^{m} (\frac{b}{c})^r \right) \frac{1}{u_0} \}.
\]

Clear that the forbidden set \( F \) is a sequence of hyperbolas contained entirely in the interiors of the 2\(^{nd} \) and the 4\(^{th} \) quadrant of the planes \( u_0u_{-2} \) and \( u_{-1}u_{-3} \) of the four dimensional Euclidean space

\[
\mathbb{R}^4 = \{ (u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-i} \in \mathbb{R}, \ i = 0, 1, 2, 3 \}.
\]

That is the forbidden set is a sequence of hyperbolas contained entirely in the set

\[
\{ (u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-1}u_{-3} < 0 \} \cup \{ (u_0, u_{-1}, u_{-2}, u_{-3}) : u_0u_{-2} < 0 \}.
\]

We define \( \alpha_i = x_{-2+i}x_{-4+i}, i = 1, 2, \ldots \)

**Theorem 2.2.** Let \( x_{-3}, x_{-2}, x_{-1} \) and \( x_0 \) be real numbers such that \( (x_0, x_{-1}, x_{-2}, x_{-3}) \notin F \). If \( a \neq b \), then the solution \( \{ x_{n} \}_{n=0}^{\infty} \) of equation (1.1) is

\[
x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-1} \left( \frac{1}{a} \right)^{2j+1} \theta_1 + c, & n = 1, 5, 9, \ldots \\
  x_{-2} \prod_{j=0}^{n-2} \left( \frac{1}{a} \right)^{2j+1} \theta_2 + c, & n = 2, 6, 10, \ldots \\
  x_{-1} \prod_{j=0}^{n-3} \left( \frac{1}{a} \right)^{2j+1} \theta_3 + c, & n = 3, 7, 11, \ldots \\
  x_0 \prod_{j=0}^{n-4} \left( \frac{1}{a} \right)^{2j+1} \theta_4 + c, & n = 4, 8, 12, \ldots 
\end{cases}
\]

where \( \theta_i = \frac{a-b-c\alpha_i}{\alpha_i}, \alpha_i = x_{-2+i}x_{-4+i}, \) and \( i = 1, 2, \ldots \)

**Proof.** We can write the given solution as

\[
x_{4m+1} = x_{-3} \prod_{j=0}^{m} \left( \frac{1}{a} \right)^{2j+1} \theta_1 + c, \quad x_{4m+2} = x_{-2} \prod_{j=0}^{m} \left( \frac{1}{a} \right)^{2j+1} \theta_2 + c, \\
x_{4m+3} = x_{-1} \prod_{j=0}^{m} \left( \frac{1}{a} \right)^{2j+1} \theta_3 + c, \quad x_{4m+4} = x_0 \prod_{j=0}^{m} \left( \frac{1}{a} \right)^{2j+1} \theta_4 + c, \quad m = 0, 1, \ldots
\]
It is easy to check the result when $m = 0$. Suppose that the result is true for $m > 0$.

Then

$$x_{4(m+1)+1} = \frac{ax_{4m+1}}{b + cx_{4m+1}x_{4m+3}} = \frac{ax_{3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_{1} + c}{b + cx_{3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_{1} + c} x_{3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_{1} + c$$

$$= \frac{ax_{3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_{1} + c}{b + cx_{3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_{1} + c} x_{3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_{1} + c$$

Similarly we can show that

$$x_{4(m+1)+2} = x_{2} \prod_{j=0}^{m+1} \left( \frac{b}{a} \right)^{2j} \theta_{2} + c$$

$$x_{4(m+1)+3} = x_{3} \prod_{j=0}^{m+1} \left( \frac{b}{a} \right)^{2j+1} \theta_{1} + c$$

and

$$x_{4(m+1)+4} = x_{0} \prod_{j=0}^{m+1} \left( \frac{b}{a} \right)^{2j+2} \theta_{2} + c$$

This completes the proof. \[\square\]
3. Global Behavior of Equation (1.1)

In this section, we investigate the global behavior of equation (1.1) with \( a \neq b \), using the explicit formula of its solution.

We can write the solution of equation (1.1) as

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \beta(j, t, i),
\]

where \( \beta(j, t, i) = \frac{(\frac{b}{a})^{2j+t} \theta_i + c}{(\frac{b}{a})^{2j+t+1} \theta_i + c} \), \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \).

In the following theorem, suppose that \( a > b \) for all \( i \in \{1, 2\} \).

**Theorem 3.1.** Let \( \{x_n\}_{n=-3}^{\infty} \) be a solution of equation (1.1) such that \( (x_0, x_{-1}, x_{-2}, x_{-3}) \notin F \). Then the following statements are true.

1. If \( a < b \), then \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.
2. If \( a > b \), then \( \{x_n\}_{n=-3}^{\infty} \) converges to a period-4 solution.

**Proof.**

1. If \( a < b \), then \( \beta(j, t, i) \) converges to \( \frac{a}{b} < 1 \) as \( j \to \infty \), for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). So, for every pair \( (t, i) \in \{0, 1\} \times \{1, 2\} \) we have for a given \( 0 < \epsilon < 1 \) that, there exists \( j_0(t, i) \in \mathbb{N} \) such that, \( |\beta(j, t, i)| < \epsilon \) for all \( j \geq j_0(t, i) \). If we set \( j_0 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_0(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\[
|x_{4m+2t+i}| = |x_{-4+2t+i}| \prod_{j=0}^{m} \beta(j, t, i) |
\]

\[
= |x_{-4+2t+i}| \prod_{j=0}^{j_0-1} \beta(j, t, i) | \prod_{j=j_0}^{m} \beta(j, t, i) |
\]

\[
< |x_{-4+2t+i}| \prod_{j=0}^{j_0-1} \beta(j, t, i) |e^{m-j_0}|
\]

As \( m \) tends to infinity, the solution \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.

2. If \( a > b \), then \( \beta(j, t, i) \to 1 \) as \( j \to \infty \), \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). This implies that, for every pair \( (t, i) \in \{0, 1\} \times \{1, 2\} \) there exists \( j_1(t, i) \in \mathbb{N} \) such that, \( \beta(j, t, i) > 0 \) for all \( j \geq j_1(t, i) \). If we set \( j_1 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_1(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \beta(j, t, i)
\]

\[
= x_{-4+2t+i} \prod_{j=0}^{j_1-1} \beta(j, t, i) \exp \left( \sum_{j=j_1}^{m} \ln(\beta(j, t, i)) \right).
\]
We shall test the convergence of the series \( \sum_{j=j_1}^{\infty} |\ln (\beta(j, t, i))| \).
Since for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we have
\[
\lim_{j \to \infty} \left| \frac{\ln (\beta(j + 1, t, i))}{\ln (\beta(j, t, i))} \right| = \frac{0}{0},
\]
using L’Hospital’s rule we obtain
\[
\lim_{j \to \infty} \left| \frac{\ln (\beta(j + 1, t, i))}{\ln (\beta(j, t, i))} \right| = \left( \frac{b}{a} \right)^2 < 1.
\]
It follows from the ratio test that the series \( \sum_{j=j_1}^{\infty} |\ln (\beta(j, t, i))| \) is convergent.
This ensures that there are four positive real numbers \( \nu_{ti}, \ t \in \{0, 1\} \) and \( i \in \{1, 2\} \) such that
\[
\lim_{m \to \infty} x_{4m + 2t + i} = \nu_{ti} , \quad t \in \{0, 1\} \quad \text{and} \quad i \in \{1, 2\}
\]
where
\[
\nu_{ti} = x_{-4 + 2t + i} \prod_{j=0}^{\infty} \left( \frac{b}{a} \right)^{2j + t} \theta_i + c , \quad t \in \{0, 1\} \quad \text{and} \quad i \in \{1, 2\}.
\]

\[
\text{Fig. 1: } x_{n+1} = \frac{2x_{n-3}}{3 + x_{n-1}x_{n-3}} \quad \text{Fig. 2: } x_{n+1} = \frac{3x_{n-3}}{1 + 2x_{n-1}x_{n-3}}
\]

**Example 1.** Figure 1 shows that if \( a = 2, b = 3, c = 1 \) (\( a < b \)), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.1) with initial conditions \( x_{-3} = 0.2, x_{-2} = 2, x_{-1} = -2 \) and \( x_0 = 0.4 \) converges to zero.

**Example 2.** Figure 2 shows that if \( a = 3, b = 1, c = 2 \) (\( a > b \)), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.1) with initial conditions \( x_{-3} = 0.2, x_{-2} = 2, x_{-1} = -2 \) and \( x_0 = 0.4 \) converges to a period-4 solution.

4. **Case \( a = b = c \)**

In this section, we investigate the behavior of the solution of the difference equation
\[
(4.1) \quad x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}} , \quad n = 0, 1, \ldots
\]
Lemma 4.1. The forbidden set $G$ of equation (4.1) is

$$G = \bigcup_{n=0}^{\infty} \{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-3} = -\left(\frac{1}{n+1}\right) \frac{1}{u_{-1}} \} \cup \bigcup_{m=0}^{\infty} \{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-2} = -(\frac{1}{m+1}) \frac{1}{u_0}\}.$$  

Theorem 4.2. Let $x_{-3}, x_{-2}, x_{-1}$ and $x_0$ be real numbers such that $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin G$. Then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (4.1) is

$$x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-1} \frac{1+(2j)\alpha_1}{1+(2j+1)\alpha_1}, & n = 1, 5, 9, \ldots \\
  x_{-2} \prod_{j=0}^{n-2} \frac{1+(2j)\alpha_2}{1+(2j+1)\alpha_2}, & n = 2, 6, 10, \ldots \\
  x_{-1} \prod_{j=0}^{n-3} \frac{1+(2j+1)\alpha_1}{1+(2j+2)\alpha_1}, & n = 3, 7, 11, \ldots \\
  x_0 \prod_{j=0}^{n-4} \frac{1+(2j+1)\alpha_2}{1+(2j+2)\alpha_2}, & n = 4, 8, 12, \ldots 
\end{cases}$$  

(4.2)

Proof. The proof is similar to that of Theorem 2.2 and will be omitted. \qed

We can write the solution of equation (4.1) as

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i),$$

where $\gamma(j, t, i) = \frac{1+(2j+t)\alpha_i}{1+(2j+t+1)\alpha_i}$, $t \in \{0, 1\}$ and $i \in \{1, 2\}$.

In the following theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1, 2\}$.

Theorem 4.3. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (4.1) such that $(x_0, x_{-1}, x_{-2}, x_{-3}) \notin G$. Then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

Proof. It is clear that $\gamma(j, t, i) \to 1$ as $j \to \infty$, $t \in \{0, 1\}$ and $i \in \{1, 2\}$. This implies that, for every pair $(t, i) \in \{0, 1\} \times \{1, 2\}$ there exists $j_2(t, i) \in \mathbb{N}$ such that, $\gamma(j, t, i) > 0$ for all $j \geq j_2(t, i)$. If we set $j_2 = \max_{0 \leq i \leq 1, 1 \leq i \leq 2} j_2(t, i)$, then for all $t \in \{0, 1\}$ and $i \in \{1, 2\}$ we get

$$x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i)$$

$$= x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j, t, i) \exp \left( - \sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j, t, i)} \right).$$

We shall show that $\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j, t, i)} = \sum_{j=j_2}^{\infty} \ln \frac{1+(2j+t+1)\alpha_i}{1+(2j+t)\alpha_i} = \infty$, by considering the series $\sum_{j=j_2}^{\infty} \frac{\alpha_i}{1+\alpha_i(2j+t)}$. As

$$\lim_{j \to \infty} \frac{1}{\gamma(j, t, i)} = \lim_{j \to \infty} \frac{\ln ((1 + \alpha_i(2j + t + 1))/(1 + \alpha_i(2j + t)))}{\alpha_i/(1 + \alpha_i(2j + t))} = 1,$$
using the limit comparison test, we get \(\sum_{j=j_2}^{\infty} \ln \frac{1}{\gamma(j, t, i)} = \infty\). Therefore,

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j, t, i) \exp \left( -\sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j, t, i)} \right),
\]

converges to zero as \(m \to \infty\). □

References

[6] Elsayed, E.M., On the difference equation \(x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}\), Int. J. Contemp. Math. Sciences 3 (33) (2008), 1657–1664.
[10] Karatas, R., Cinar, C., Simsek, D., On the positive solution of the difference equation \(x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}\), Int. J. Contemp. Math. Sciences 1 (10) (2006), 495–500.
[16] Simsek, D., Cinar, C., Karatas, R., Yalcinkaya, I., On the recursive sequence \(x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}\), Int. J. Pure Appl. Math. 28 (1) (2006), 117–124.
[17] Simsek, D., Cinar, C., Yalcinkaya, I., On the recursive sequence \(x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}\), Int. J. Contemp. Math. Sciences 1 (10) (2006), 475–480.


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