SOME NEW MODIFIED COSINE SUMS AND
$L^1$-CONVERGENCE OF COSINE TRIGONOMETRIC SERIES

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Abstract. In this paper we introduce some new modified cosine sums and then using these sums we study $L^1$-convergence of trigonometric cosine series.

1. Introduction and preliminaries

Let

\begin{equation}
\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx
\end{equation}

be cosine trigonometric series and satisfy condition $a_k \to 0$, $k \to \infty$. The partial sum of series (1) we denote by $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$ and let be $f(x) = \lim_{n \to \infty} S_n(x)$.

A sequence $(a_k)$ is said to belong to the class $S$, or briefly $a_k \in S$, if $a_k \to 0$ as $k \to \infty$, and there exists a sequence of numbers $(A_k)$ such that

$A_k \downarrow 0$, \\
$\sum_{k=1}^{\infty} A_k < \infty$, \\
and

$|\Delta a_k| \leq A_k$, \\
for all $k$, where $\Delta a_k = a_k - a_{k+1}$.

This class of sequences was defined by Sidon in [18] and by Telyakovskiı in [21], therefore the class $S$ is sometimes called the Sidon-Telyakovskiı class. The class $S$ is generalized later by Tomovski in [22] and by Leindler in [16].

Tomovski defined the class $S_r$, $r = 1, 2, \ldots$ as follows: $\{a_k\}_{k=1}^{\infty} \in S_r$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}_{k=1}^{\infty}$ such that

\begin{equation}
|\Delta a_k| \leq A_k \leq \frac{1}{r} A_{k-1},
\end{equation}

for all $k$, where $\Delta a_k = a_k - a_{k+1}$.

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\[ \sum_{k=1}^{\infty} k^r A_k < \infty \text{ and } |\Delta a_k| \leq A_k \text{ for all } k. \] There was noticed that from \( A_k \downarrow 0 \) and \( \sum_{k=1}^{\infty} k^r A_k < \infty \) it follows \( k^{r+1} A_k = o(1), k \to \infty. \) It is clear that \( S_{r+1} \subset S_r \) for all \( r = 1, 2, \ldots \) and for \( r = 0 \) we get the class \( S_0 \equiv S. \)

Garret and Stanojević [3] have introduced modified cosine sums

\[ f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx. \]

Garret and Stanojević [4], Ram [17], Singh and Sharma [20], and Kaur and Bhatia [11], [6], [10] studied the \( L^1 \)-convergence of this cosine sum under different sets of conditions on the coefficients \( a_n. \)

Kumari and Ram [15] introduced new modified cosine and sine sums as

\[ h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left( \frac{a_j}{j} \right) k \cos kx, \]
\[ g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left( \frac{a_j}{j} \right) k \sin kx \]

and have studied their \( L^1 \)-convergence under the condition that the coefficients \( a_n \) belong to different classes of sequences. They deduced some results about \( L^1 \)-convergence of cosine and sine series as corollaries, as well.

N. Hooda, B. Ram and S. S. Bhatia [5] introduced new modified cosine sums as

\[ R_n(x) = \frac{1}{2} \left( a_1 + \sum_{k=0}^{n} \Delta^2 a_k \right) + \sum_{k=1}^{n} (a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j) \cos kx \]

and studied the \( L^1 \)-convergence of these cosine sums.

K. Kaur [9] introduced new modified sine sums as

\[ K_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx, \]

and studied the \( L^1 \)-convergence of this modified sine sum with semi-convex coefficients. Also, Kaur at al. [12] introduced a new class of numerical sequences as follows:

**Definition 1.** If \( a_k = o(1) \) as \( k \to \infty, \) and

\[ \sum_{k=1}^{\infty} k |\Delta^2 a_{k-1} - \Delta^2 a_{k+1}| < +\infty \quad (a_0 = 0) \]

then we say that \( \{a_k\} \) belongs to the class \( K. \)

In their paper they proved the following result regarding to \( L^1 \)-convergence of the modified sums \( K_n(x). \)

**Theorem 1.** Let the sequence \( \{a_k\} \) belong to the class \( K, \) then \( K_n(x) \) converges to \( f(x) \) in the \( L^1 \)-norm.
Later on, Singh and Kaur [19] defined new modified generalized sine sums
\[ K_{nr}(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} (\Delta^r a_{k-1} - \Delta^r a_{k+1}) \tilde{S}_{r}^{-1}(x), \]
and a new class of sequences:

**Definition 2.** Let \( \alpha \) be a positive real number. If \( a_k = o(1) \) as \( k \to \infty \), and
\[ \sum_{k=1}^{\infty} k^{\alpha} |\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}| < +\infty \quad (a_0 = 0) \]
then we say that \( \{a_k\} \) belongs to the class \( K^{\alpha} \).

They proved the following generalization of Theorem 1:

**Theorem 2.** Let the sequence \( \{a_k\} \) belong to the class \( K^{\alpha} \), then \( K_{nr}(x) \) converges to \( f(x) \) in the \( L^1 \)-norm.

Some new modified sums are presented in [13] by present author (see also [14]) as follows
\[ H_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta [(a_{j-1} - a_{j+1}) \sin jx], \]
and also we have proved a new result as below.

**Theorem 3.** Let \( (a_n) \) be a semi-convex null sequence, then \( H_n(x) \) converges to \( f(x) \) in \( L^1 \)-norm.

The interested reader can find some new results in very recently published papers, [7] where the complex form of the sums \( K_n(x) \) is introduced, and paper [8] in which it is studied the \( L^1 \)-convergence of sine trigonometric series by using a newly introduced modified cosine trigonometric sums under a new class of coefficient sequences (see [8] for details therein).

We recall that with regard to the \( L^1 \)-convergence of Ress-Stanojević cosine sums \( f_n(x) \) to a cosine trigonometric series, belonging to the class \( S \), Ram [17] proved the following theorem:

**Theorem 4.** If \( \{1,1\} \) belongs to the class \( S \), then \( \|f - f_n\|_{L^1} = o(1), \ n \to \infty \).

In order to make an advanced study, on this treating topic, now we shall introduce new modified cosine sums as
\[ G_n(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \Delta^2 (a_{k_3} \cos k_3 x), \]
where \( \Delta^2 a_k = \Delta (\Delta a_k) = a_k - 2a_{k+1} + a_{k+2} \).

**Remark 1.** The advantage of introducing of the above modified cosine sums is the following: We have verified that the sums \( G_n(x) \) converge in \( L^1 \)-norm to \( f(x) \), without a new class of null-sequences being defined, in contrary what the other authors previously did in their papers (as examples serve classes \( K, K^{\alpha} \), etc.).
The purpose of this paper is to prove analogous statement with Theorem 4 using new modified cosine sums $G_n(x)$ instead of $g_n(x)$ and the $L^1$-convergence of the series \([1.1]\) will be derived as a corollary.

As usual $D_n(x)$ will denote the real Dirichlet kernel, i.e.

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx.$$ 

For the proof of main result we need the following lemma.

**Lemma 1** ([2]). If $|c_k| \leq 1$, then

$$\int_0^\pi \left| \sum_{k=0}^{n} c_k \frac{\sin ((k+1/2)x)}{2 \sin \frac{x}{2}} \right| dx \leq C(n+1),$$

where $C$ is a positive absolute constant.

2. **Main results**

We establish the following result.

**Theorem 5.** Let \([1.1]\) belong to the class $S_2$, then $\|f - G_n\|_{L^1} = o(1)$, as $n \to \infty$.

**Proof.** We have

$$G_n(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \Delta^2 (a_{k_3} \cos k_3 x)$$

$$= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \left[ \Delta (a_{k_2} \cos k_2 x) - \Delta (a_{k_2+1} \cos (k_2 + 1)x) \right]$$

$$+ \cdots + \Delta (a_n \cos nx) - \Delta (a_{n+1} \cos (n + 1)x) \right]$$

$$= \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \left[ \Delta (a_{k_2} \cos k_2 x) - \Delta (a_{n+1} \cos (n + 1)x) \right]$$

$$= \frac{a_0}{2} + \sum_{k_1=1}^{n} \left[ a_{k_1} \cos k_1 x - a_{k_1+1} \cos (k_1 + 1)x + \cdots + a_n \cos nx \right.$$  

$$- a_{n+1} \cos (n + 1)x \Delta (a_{n+1} \cos (n + 1)x) \sum_{k_1=1}^{n} (n - k_1 + 1)$$

$$= S_n(x) - na_{n+1} \cos (n + 1)x - \frac{1}{2} n(n+1) \Delta (a_{n+1} \cos (n + 1)x)$$

$$+ \frac{1}{2} n(n+1)a_{n+2} \cos (n + 2)x.$$

(2.1)
From \( A_k \downarrow 0 \) and \( \sum_{k=1}^{\infty} k^2 A_k < \infty \) follows \( k^3 A_k = o(1) \), \( k \to \infty \), which gives \( k^2 A_k = o(1) \), \( k \to \infty \). Therefore from

\[
0 \leq n^2 |a_n| = n^2 \left| \sum_{k=n}^{\infty} \Delta a_k \right| \leq \left| \sum_{k=n}^{\infty} k^2 \Delta a_k \right| \leq \sum_{k=n}^{\infty} k^2 A_k = o(1), \quad n \to \infty
\]

follow

(2.2) \[ n^2 a_n = o(1), \quad na_n = o(1), \quad n \to \infty. \]

Also, \( \cos(n+1)x \) and \( \cos(n+2)x \) are finite in \([0, \pi]\) therefore from (2.1) and (2.2) we get

\[
\lim_{n \to \infty} G_n(x) = \lim_{n \to \infty} S_n(x) = f(x).
\]

On the other side, using Abel’s transformation we have

\[
f(x) - G_n(x) = \lim_{m \to \infty} \left( \sum_{k=n+1}^{m-1} \Delta a_k D_k(x) + a_m D_m(x) - a_{n+1} D_{n}(x) \right)
\]

\[+ \frac{1}{2} n(n+3)a_{n+1} \cos(n+1)x - \frac{1}{2} n(n+1)a_{n+2} \cos(n+2)x\]

\[= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1} D_{n}(x) + \frac{1}{2} n(n+3)a_{n+1} \cos(n+1)x \]

\[+ \frac{1}{2} n(n+1)a_{n+2} \cos(n+2)x.\]

Therefore

\[
\int_{0}^{\pi} |f(x) - G_n(x)| dx \leq \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + |a_{n+1}| \int_{0}^{\pi} |D_{n}(x)| dx
\]

\[+ \frac{1}{2} n(n+3)|a_{n+1}| \int_{0}^{\pi} | \cos(n+1)x | dx \]

\[+ \frac{1}{2} n(n+1)|a_{n+2}| \int_{0}^{\pi} | \cos(n+2)x | dx
\]

(2.3) \[:= \sum_{\nu=1}^{4} B_{\nu}(n). \]

Since \( a_k \in S_2 \subset S_0 = S \) then \( \sum_{k=n+1}^{\infty} (k+1) \Delta A_k = o(1) \) as \( n \to \infty \), therefore from this fact, Lemma 1, and using Abel’s transformation we have

\[
B_1(n) = \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx \leq \sum_{k=n+1}^{\infty} \Delta A_k \int_{0}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x) \right| dx
\]

(2.4) \[= O\left( \sum_{k=n+1}^{\infty} (k+1) \Delta A_k \right) = o(1), \quad n \to \infty. \]
By well-known Zygmund’s theorem (see [20, p. 458]), for \( n \) sufficiently large, the following relation holds
\[
\int_0^\pi |D_n(x)| \, dx \sim \log n,
\]
therefore from the last relation and (2.2) we have
\[
B_2(n) = |a_{n+1}| \log n \leq n|a_{n+1}| = o(1), \quad n \to \infty.
\]
Moreover, from fact that integrals \( \int_0^\pi \left| \cos(n+1)x \right| \, dx, \int_0^\pi \left| \cos(n+2)x \right| \, dx \) are bounded, and from relation (2.2) we conclude that
\[
B_3(n) = O\left(n(n+3)|a_{n+1}|\right) = o(1), \quad n \to \infty
\]
and similarly
\[
B_4(n) = O\left(n(n+1)|a_{n+2}|\right) = o(1), \quad n \to \infty.
\]
Finally, from (2.3)–(2.7) it follows that
\[
\|f - G_n\|_{L^1} = o(1), \quad n \to \infty.
\]
The proof of the Theorem 5 is completed. \( \Box \)

**Corollary 1.** Let (1.1) belong to the class \( S_2 \), then \( \|f - S_n\|_{L^1} = o(1) \) as \( n \to \infty \).

**Proof.** From Theorem 5 and relations (2.6), (2.7), we have
\[
\|f - S_n\|_{L^1} = \|f - G_n + G_n - S_n\|_{L^1}
\leq \|f - G_n\|_{L^1} + \|G_n - S_n\|_{L^1}
\leq \|f - G_n\|_{L^1} + \frac{1}{2}n(n+3)|a_{n+1}| \int_0^\pi \left| \cos(n+1)x \right| \, dx
+ \frac{1}{2}n(n+1)|a_{n+2}| \int_0^\pi \left| \cos(n+2)x \right| \, dx = o(1)
\]
as \( n \to \infty \), which completely proves the corollary. \( \Box \)

**Remark 2.** A closer examination of the proofs of Theorem 5 and Corollary 1 reveals that condition \( a_k \in S_2 \) can be replaced by conditions \( a_k \in S \) and \( n^2|a_n| = o(1) \). This enables us to formulate Theorem 5 and Corollary 1 in the following form:

**Theorem 6.** Let \( (a_k) \) belong to the class \( S \) and \( n^2|a_n| = o(1) \), then \( \|f - G_n\|_{L^1} = o(1) \) as \( n \to \infty \).

**Corollary 2.** Let \( (a_k) \) belong to the class \( S \) and \( n^2|a_n| = o(1) \), then \( \|f - S_n\|_{L^1} = o(1) \) as \( n \to \infty \).

We would like to finalize this paper with a comment. We have noticed during this study that, if someone tries to introduce some modified sums of the form
\[
T_{n,m}(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \ldots \sum_{k_m=k_{m-1}}^{n} \Delta^{m-1}(a_{k_m}/k_m) k_1 \cos k_1 x,
\]
where $m \in \mathbb{N}$, $m > 3$, $\Delta a_k = a_k - a_{k+1}$, $\Delta^{m-1} a_k = \Delta \left( \Delta^{m-2} a_k \right)$, which is a natural extension of our results, then several difficulties in the proof of the counterpart of Theorem 5 will be appeared. This is why we are focused only on the case $m = 3$.

References


