ON HOLOMORPHICALLY PROJECTIVE MAPPINGS
OF \( e \)-KÄHLER MANIFOLDS

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Abstract. In this paper we study fundamental equations of holomorphically projective mappings of \( e \)-Kähler spaces (i.e. classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

1. Introduction

First we study the general dependence of holomorphically projective mappings of classical, pseudo- and hyperbolic Kähler manifolds (shortly \( e \)-Kähler) in dependence on the smoothness class of the metric. We present well known facts, which were proved by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro etc., see [2, 3, 6, 7, 8, 9, 11, 12, 15, 16, 17, 18, 19]. In these results no details about the smoothness class of the metric were stressed. They were formulated “for sufficiently smooth” geometric objects.

2. Kähler manifolds

In the following definition we introduce generalizations of Kähler manifolds.

**Definition 1.** An \( n \)-dimensional (pseudo-)Riemannian manifold \((M,g)\) is called an \( e \)-Kähler manifold \( K_n \), if beside the metric tensor \( g \), a tensor field \( F (\not= \text{Id}) \) of type \((1,1)\) is given on the manifold \( M_n \), called a structure \( F \), such that the following conditions hold:

\[
F^2 = e \text{Id} ; \quad g(X,FX) = 0 ; \quad \nabla F = 0 ,
\]

where \( e = \pm 1 \), \( X \) is an arbitrary vector of \( TM_n \), and \( \nabla \) denotes the covariant derivative in \( K_n \).

If \( e = -1 \), \( K_n \) is a (pseudo-)Kähler space (also elliptic Kähler space) and \( F \) is a complex structure. As \( A \)-spaces, these spaces were first considered by P. A. Shirokov, see [14]. Independently they were studied by E. Kähler [5].


Key words and phrases: holomorphically projective mappings, smoothness class, Kähler space, hyperbolic Kähler space.

DOI: 10.5817/AM2012-5-333
If \( e = +1 \), \( K_n \) is a hyperbolic Kähler space (also para Kähler space, see [1]) and \( F \) is a product structure. The spaces \( K^+_n \) were considered by P. K. Rashevskij [13]. The \( e \)-Kähler spaces introduced here are called shortly “Kähler” in the literature [10] [16]. By our definition we want to give a unified notation for all classes.

3. Holomorphically projective mapping theory
for \( K_n \to \bar{K}_n \) of class \( C^1 \)

Assume the \( e \)-Kähler manifolds \( K_n = (M, g, F) \) and \( \bar{K}_n = (\bar{M}, \bar{g}, \bar{F}) \) with metrics \( g \) and \( \bar{g} \), structures \( F \) and \( \bar{F} \), Levi-Civita connections \( \nabla \) and \( \bar{\nabla} \), respectively. Here \( K_n, \bar{K}_n \in C^1 \), i.e. \( g, \bar{g} \in C^1 \) which means that their components \( g_{ij}, \bar{g}_{ij} \in C^1 \).

Likewise, as in [11] we introduce the following notations.

**Definition 2.** A curve \( \ell \) in \( K_n \) which is given by the equation \( \ell = \ell(t), \lambda = d\ell/dt \), \((\neq 0), t \in I \), where \( t \) is a parameter is called analytically planar, if under the parallel translation along the curve, the tangent vector \( \lambda \) belongs to the two-dimensional distribution \( D = \text{Span} \{\lambda, F\lambda\} \) generated by \( \lambda \) and its conjugate \( F\lambda \), that is, it satisfies
\[
\nabla_t \lambda = a(t)\lambda + b(t)F\lambda,
\]
where \( a(t) \) and \( b(t) \) are some functions of the parameter \( t \).

Particularly, in the case \( b(t) = 0 \), an analytically planar curve is a geodesic.

**Definition 3.** A diffeomorphism \( f : K_n \to \bar{K}_n \) is called a holomorphically projective mapping of \( K_n \) onto \( \bar{K}_n \) if \( f \) maps any analytically planar curve in \( K_n \) onto an analytically planar curve in \( \bar{K}_n \).

Assume a holomorphically projective mapping \( f : K_n \to \bar{K}_n \). Since \( f \) is a diffeomorphism, we can suppose local coordinate charts on \( M \) or \( \bar{M} \), respectively, such that locally, \( f : \bar{K}_n \to K_n \) maps points onto points with the same coordinates, and \( M = \bar{M} \).

A manifold \( K_n \) admits a holomorphically projective mapping onto \( \bar{K}_n \) if and only if the following equations [10] [16]:
\[
\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + e\psi(FX)FY + e\psi(FY)FX
\]

hold for any tangent fields \( X, Y \) and where \( \psi \) is a differential form. If \( \psi \equiv 0 \) than \( f \) is affine or trivially holomorphically projective. Beside these facts it was proved [10] [16] that \( \bar{F} = \pm F \); for this reason we can suppose that \( \bar{F} = F \). In local form:
\[
\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i + e\psi_i \delta^h_j + e\psi_j \delta^h_i,
\]
where \( \Gamma^h_{ij} \) and \( \bar{\Gamma}^h_{ij} \) are the Christoffel symbols of \( K_n \) and \( \bar{K}_n \), \( \psi_i, F^h_i \) are components of \( \psi, F \) and \( \delta^h_i \) is the Kronecker delta, \( \psi_i = \psi_\alpha F^\alpha_i, \delta^h_i = F^h_i \).

Here and in the following we will use the conjugation operation of indices in the way
\[
A_{...i...} = A_{...k...} F^k_i.
\]
On holomorphically projective mappings of $\varepsilon$-Kähler manifolds

Equations (2) are equivalent to the following equations
\[
\nabla_Z \bar{g}(X, Y) = 2\psi(Z)\bar{g}(X, Y) + \psi(X)\bar{g}(Y, Z) + \psi(Y)\bar{g}(X, Z)
- e\psi(FX)\bar{g}(FY, Z) - e\psi(FY)\bar{g}(FX, Z).
\]

In local form:
\[
\bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi_j\bar{g}_{ik} - e\psi_i\bar{g}_{jk} - e\psi_j\bar{g}_{ik},
\]
where “,$" denotes the covariant derivative on \( K_n \). It is known that
\[
\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n + 2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|,
\]
\( \partial_i = \partial/\partial x^i \).

Domashev, Kurbatova and Mikeš [3, 6, 16] proved that equations (2) and (3) are equivalent to
\[
\nabla_Z a(X, Y) = \lambda(X)g(Y, Z) + \lambda(Y)g(X, Z)
- e\lambda(FX)g(FY, Z) - e\lambda(FY)g(FX, Z).
\]

In local form:
\[
a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_i g_{jk} - e\lambda_j g_{ik},
\]
where
\[
(a) \quad a_{ij} = e^{2\psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad (b) \quad \lambda_i = -e^{2\psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_i \alpha.
\]

From (4) follows \( \lambda_i = \partial_i \lambda = \partial_i \left( \frac{1}{4} a_{\alpha\beta} g^{\alpha\beta} \right) \). On the other hand [10]:
\[
\bar{g}_{ij} = e^{2\psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \bar{g}}{\det g} \right|,
\]
\( \|\bar{g}_{ij}\| = \|g^{i\alpha} g^{j\beta} a_{\alpha\beta}\|^{-1} \).

The above formulas are the criterion for holomorphically projective mappings \( K_n \to K_n \), globally as well as locally.

4. Holomorphically projective mapping theory

for \( K_n \to K_n \) of class \( C^2 \)

Let \( K_n \) and \( \bar{K}_n \in C^2 \) be \( \varepsilon \)-Kähler manifolds, then for holomorphically projective mappings \( K_n \to \bar{K}_n \) the Riemann and the Ricci tensors transform in this way
\[
(a) \quad \bar{R}^{h}_{ijk} = R^{h}_{ijk} + \delta^{h}_{k} \psi_{ij} - \delta^{h}_{j} \psi_{ik} - e\delta^{h}_{j} \psi_{ij} + e\delta^{h}_{j} \psi_{ik} + 2e\delta^{h}_{i} \psi_{jk};
(b) \quad \bar{R}_{ij} = R_{ij} - (n + 2)\psi_{ij},
\]
where \( \psi_{ij} = \psi_{i,j} - \psi_{i}\psi_{j} + \psi_{i}\psi_{j} \) (\( \psi_{ij} = \psi_{ji} = -e\psi_{i,j} \)).

The tensor of holomorphically projective curvature, which is defined in the following form
\[
P^{h}_{ijk} = R^{h}_{ijk} + \frac{1}{n + 2} \left( \delta^{h}_{k} R_{ij} - \delta^{h}_{j} R_{ik} - e\delta^{h}_{j} R_{ij} + e\delta^{h}_{j} R_{ik} + 2e\delta^{h}_{i} R_{jk} \right),
\]
is invariant with respect to holomorphically projective mappings, i.e. \( \bar{P}^{h}_{ijk} = P^{h}_{ijk} \).
Theorem 1. The integrability conditions of equations \([4]\) have the following form
\[
a_{ia}R_{jkl}^\alpha + a_{ja}R_{ikl}^\alpha = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k}
\]
(9)
\[- eg_{ik}\lambda_{j,l} - eg_{jk}\lambda_{i,l} + eg_{il}\lambda_{j,k} + eg_{jl}\lambda_{i,k}.
\]
We make the remark that the formulas introduced above, \((7), (8) \) and \((9)\), are not valid when \(K_n \notin C^2 \) or \(K_\bar{n} \notin C^2\).

After contraction with \(g^{jk}\) we get:
\[
a_{ia}R_{k}^{\alpha} + a_{\alpha \beta}R_{i\beta}^{\alpha} = e\lambda_{i,k} - (n - 1)\lambda_{i,k},
\]
where \(R_{i\beta}^{\alpha} = g^{\beta k}R_{i\beta k}^\alpha; \ R_{i\beta}^\alpha = g^{\alpha j}R_{i\alpha j} \) and \(\mu = \alpha_{\alpha \beta}g^{\alpha \beta}\).

Let \(\lambda_i \) be a gradient-like covector, from equation \((11)\) follows \(a_{ia}R_{j}^{\alpha} = a_{ja}R_{i}^{\alpha}\).

From \((10)\) follows that the vector field \(\lambda_i (\equiv \lambda_\alpha F_i^{\alpha})\) is a Killing vector field, i.e. \(\lambda_{i,j} + \lambda_{j,i} = 0\).

5. Holomorphically projective mappings
between \(K_n \in C^r (r > 2)\) and \(K_\bar{n} \in C^2\)

We prove the following theorem

**Theorem 1.** If \(K_n \in C^r (r > 2)\) admits holomorphically projective mappings onto \(K_\bar{n} \in C^2\), then \(K_n \in C^r\).

The proof of this theorem follows from the following lemmas.

**Lemma 1** (see [4]). Let \(\lambda^h \in C^1\) be a vector field and \(\varrho\) a function. If
\[
\partial_i \lambda^h - \varrho \delta_i^h \in C^1
\]
(12)
then \(\lambda^h \in C^2\) and \(\varrho \in C^1\).

In a similar way we can prove the following: if \(\lambda^h \in C^r (r \geq 1)\) and \(\partial_i \lambda^h - \varrho \delta_i^h \in C^{r+1}\) then \(\lambda^h \in C^{r+1}\) and \(\varrho \in C^r\).

**Lemma 2.** If \(K_n \in C^3\) admits a holomorphically projective mapping onto \(K_\bar{n} \in C^2\), then \(K_n \in C^3\).

**Proof.** In this case equations \([4]\) and \([11]\) hold. According to the assumptions \(g_{ij} \in C^3\) and \(g_{ij} \in C^2\). By a simple check-up we find \(\Psi \in C^2, \ \psi_i \in C^1, \ a_{ij} \in C^2, \ \lambda_i \in C^1\) and \(R_{ijk}^h, \ R_{ij, k}^h, \ R_{ij}^h, \ R_i^h \in C^1\).

From the above-mentioned conditions we easily convince ourselves that we can write equation \((11)\) in the form \((12)\), where
\[
\lambda^h = g^{\alpha \lambda} \lambda_\alpha \in C^1, \ \varrho = \mu/n \text{ and } f_i^h = \frac{1}{n} (\lambda^\alpha \Gamma_{\alpha i}^h - g^{\alpha \gamma}a_{\alpha \gamma}R_i^\alpha + g^{\alpha \gamma}a_{\alpha \beta}R_{i \gamma}^\alpha \beta) \in C^1.
\]
From Lemma 1 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (4) twice we convince ourselves that $a_{ij} \in C^3$. From this and formula (6) follows that also $\Psi \in C^3$ and $\tilde{g}_{ij} \in C^3$.

Further we notice that for holomorphically projective mappings between e-Kähler manifolds $K_n$ and $\tilde{K}_n$ of class $C^3$ holds the following third set of equations [6, 8, 9, 15, 16]:

$$\mu_{,k} = 2\lambda_{,\alpha} R^{\alpha}_{\beta k}.$$ (13)

If $K_n \in C^r$ and $\tilde{K}_n \in C^2$, then by Lemma 2, $\tilde{K}_n \in C^3$ and (13) holds. Because the system (4), (11) and (13) is closed, we can differentiate equations (4) $(r - 1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\tilde{g}_{ij} \in C^r \equiv \tilde{K}_n \in C^r$.

**Remark.** Moreover, in this case from equation (13) follows that the function $\mu \in C^{r-1}$.

**Acknowledgement.** The paper was supported by the project FAST-S-11-47 of the Brno University of Technology.

**References**


