ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 163–172

COMPACT SPACE-LIKE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN LOCALLY SYMMETRIC LORENTZ SPACES

Yaning Wang and Ximin Liu

Abstract. A new class of $(n+1)$-dimensional Lorentz spaces of index 1 is introduced which satisfies some geometric conditions and can be regarded as a generalization of Lorentz space form. Then, the compact space-like hypersurface with constant scalar curvature of this spaces is investigated and a gap theorem for the hypersurface is obtained.

1. Introduction

Let $\mathbb{N}_p^{n+p}$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p. It is called a semi-definite space of index p. When we refer to index p, we mean that there are only p negative eigenvalues of semi-Riemannian metric of $\mathbb{N}_p^{n+p}$ and the other eigenvalues are positive. In particular, $\mathbb{N}_1^{n+1}$ is called a Lorentz space when $p=1$. When the Lorentz space $\mathbb{N}_1^{n+1}$ is of constant curvature c, we call it Lorentz space form, denote it by $\mathbb{N}_1^{n+1}(c)$, with de Sitter space $\mathbb{S}_1^{n+1}(1)$ and anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ as its special cases. A hypersurface M of a Lorentz space is said to be space-like if the induced metric from that of the ambient space is positive definite.

The authors in [3] introduced a class of Lorentz spaces $\overline{M}$ of index 1. Let $\overline{\nabla}$, $\overline{K}$ and $\overline{R}$ denote the semi-Riemannian connection, sectional curvature and curvature tensor on $\overline{M}$, respectively. For constant c_1, c_2 and c_3, they considered Lorentz spaces which satisfy the following conditions:

(1) for any space-like vector u and any time-like vector v, $\overline{K}(u,v) = -\frac{c_1}{n}$,

(2) for any space-like vector u and v, $\overline{K}(u,v) \geq c_2$,

(3)
$$|\overline{\nabla}\,\overline{R}| \leq \frac{c_3}{n}\,.$$

2010 *Mathematics Subject Classification*: primary 53C42; secondary 53C15.

Key words and phrases: space-like hypersurface, constant scalar curvature, second fundamental form, locally symmetric Lorentz space.

This work is supported by NSFC (No. 10931005) and Natural Science Foundation of Guangdong Province of China (No. S2011010000471).

Received November 11, 2011, revised June 13, 2012. Editor O. Kowalski.

DOI: 10.5817/AM2012-3-163

When $\overline{M}$ satisfies conditions (1) and (2), they say that $\overline{M}$ satisfies condition $(*)$. When $\overline{M}$ satisfies conditions (1) – (3), they say that $\overline{M}$ satisfies condition $(**)$.

Also they give some examples as following.

Example 1.1. The semi-Riemannian product manifold $H_1^k(-\frac{c_1}{n}) \times M^{n+1-k}(c_2)$, $c_1 > 0$. Its sectional curvature is given by

$$\overline{K}(u_1, u_b) = \overline{K}(u_a, u_b) = -\frac{c_1}{n}\,, \quad \overline{K}(u_a, u_r) = 0\,, \quad \overline{K}(u_r, u_s) = c_2\,,$$

where $a, b = 2, \dots, k$; $r, s = k+1, \dots, n+1$, u_1 and u_a, u_r denote time-like and space-like vectors respectively.

Example 1.2. The semi-Riemannian product manifold $R_1^k \times S^{n+1-k}(1)$. Its sectional curvature is given by

$$\overline{K}(u_1, u_a) = \overline{K}(u_a, u_b) = 0, \quad \overline{K}(u_1, u_r) = 0\,, \quad \overline{K}(u_r, u_s) = 1\,,$$

where $a, b = 2, \dots, k$; $r, s = k+1, \dots, n+1$. In particular, $R_1^1 \times S^n(1)$ is called Einstein Static Universe. Notice that it is not a Lorentz space form.

The authors in [2, 8] investigated complete space-like hypersurfaces M in a Lorentz space satisfying condition $(**)$. They estimate the square norm of the second fundamental form of M under some conditions. Baek-Cheng-Suh in [3] studied complete space-like hypersurfaces with constant mean curvature satisfying the condition $(*)$. Later, Xu and Chen in [9] generalized the related results in [3] by investigating complete space-like submanifolds with constant mean curvature in locally symmetric semi-Riemannian spaces. Recently, Liu and Wei in [4] obtained a gap theorem for complete space-like hypersurface with constant scalar curvature in locally symmetric Lorentz spaces.

Now we consider Lorentz spaces which satisfy another condition:

(4) for any space-like vectors u and v, $\overline{K}(u, v) \leq c_2$.

When $\overline{M}$ satisfies conditions (1) and (4), we shall say that $\overline{M}$ satisfies conditions $(\overline{*})$. When $\overline{M}$ satisfies conditions (1), (3) and (4), we shall say that $\overline{M}$ satisfies condition $(\overline{**})$. In this paper, we mainly discuss the compact space-like hypersurfaces with constant scalar curvature in a locally symmetric Lorentz spaces satisfying the condition $(\overline{*})$. It is worthy to point out that both Example 1.1 and 1.2 satisfy the condition $(\overline{*})$.

Remark 1.3. It is easy to see that a Lorentz space form $N_1^{n+1}(s)$ satisfies both conditions $(**)$ and $(\overline{**})$, where $-\frac{c_1}{n} = c_2 = s$.

Remark 1.4. If a Lorentz space $\overline{M}$ is locally symmetric, then the condition (3) holds naturally, because $\overline{\nabla}\,\overline{R} = 0$ in this situation.

Remark 1.5. As discussed in section 4, our theorem extend the results in [6] under some geometric conditions.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional Lorentz space of index 1. Throughout the paper, manifolds are assumed to be connected and geometric objects are assumed to be of class C^∞. For any point $p \in \overline{M}$, we choose a local field of semi-orthonormal frames $\{e_A\} = \{e_1, e_2, \ldots, e_{n+1}\}$ on a neighborhood of p, where $e_1, \ldots, e_n$ are space-like and e_{n+1} is time-like. We use the following convention on the range of indices throughout the paper

$$A, B, \ldots = 1, \ldots, n+1\,; \quad i, j, \ldots = 1, 2, \ldots, n\,.$$

Let $\{\omega_A\} = \{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ denote the dual frame fields of $\{e_A\}$ on $\overline{M}$. The metric tensor $\overline{g}$ of $\overline{M}$ satisfies $\overline{g}(e_A, e_B) = \epsilon_A \delta_{AB}$, where $\epsilon_1 = \ldots = \epsilon_n = 1$ and $\epsilon_{n+1} = -1$. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ satisfy the following structure equations

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \qquad \omega_{AB} + \omega_{BA} = 0\,, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \omega_C \wedge \omega_D\,. \tag{2.2}$$

The components $\overline{R}_{CD}$ of the Ricci tensor and the scalar curvature $\overline{R}$ are given respectively by

$$\overline{R}_{CD} = \sum_B \epsilon_B \overline{R}_{BCDB}\,, \tag{2.3}$$

and

$$\overline{R} = \sum_A \epsilon_A \overline{R}_{AA}\,. \tag{2.4}$$

The components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\overline{R}$ are defined by

$$\begin{aligned} &\sum_E \epsilon_E \overline{R}_{ABCD;E} \\ &= d\overline{R}_{ABCD} - \sum_E \epsilon_E (\overline{R}_{EBCD}\omega_{EA} + \overline{R}_{AECD}\omega_{EB} + \overline{R}_{ABED}\omega_{EC} + \overline{R}_{ABCE}\omega_{ED})\,. \end{aligned} \tag{2.5}$$

Restricting the forms $\{\omega_A\}$ to a space-like hypersurface M in $\overline{M}$, we have

$$\omega_{n+1} = 0\,, \tag{2.6}$$

and the induced metric g of M is given by $g = \sum_i \omega_i \otimes \omega_i$. It is well known that by Cartan's Lemma we get

$$\omega_{(n+1)i} = \sum_j h_{ij}\omega_j\,, \qquad h_{ij} = h_{ji}\,, \tag{2.7}$$

where h_{ij} are the coefficients of the second fundamental form of M. Then we denote by $H = \frac{1}{n}\sum_i h_{ii}$ and $S = \sum_{ij} h_{ij}^2$ the mean curvature and squared norm of the second fundamental form of M, respectively.

The structure equations of M are given by

$$d\omega_i = -\sum_i \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0\,, \tag{2.8}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,. \tag{2.9}$$

The Gauss equation is given by

$$R_{ijkl} = \overline{R}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk})\,. \tag{2.10}$$

The Ricci tensor and normalized scalar curvature of M are given respectively by

$$R_{ij} = \sum_k \overline{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}\,, \tag{2.11}$$

and

$$n(n-1)R = \sum_{j,k} \overline{R}_{kjjk} - n^2H^2 + S\,. \tag{2.12}$$

Let $\overline{M}$ be a locally symmetric Lorentz space satisfying the condition $(\overline{*})$. We know that the scalar curvature $\overline{R}$ of $\overline{M}$ is a constant. By using the structure equations of $\overline{M}$, we have

$$\overline{R} = \sum_A \epsilon_A \overline{R}_{AA} = -2\sum_i \overline{R}_{(n+1)ii(n+1)} + \sum_{i,j} \overline{R}_{ijji} = -2c_1 + \sum_{i,j} \overline{R}_{ijji}\,, \tag{2.13}$$

which means that $\sum_{i,j} \overline{R}_{ijji}$ is a constant. We assume from now that the scalar curvature R of M is constant. Together with the above equation and (2.12), we define a constant P by

$$n(n-1)P = n^2H^2 - S = \sum_{ij} \overline{R}_{ijji} - n(n-1)R\,. \tag{2.14}$$

By taking exterior differentiation of (2.7) and defining h_{ijk} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k (h_{kj}\omega_{ki} + h_{ik}\omega_{kj})\,, \tag{2.15}$$

we have the following Codazzi equation

$$h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}. \tag{2.16}$$

Similarly, we define h_{ijkl} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l (h_{ljk}\omega_{li} + h_{ilk}\omega_{lj} + h_{ijl}\omega_{lk})\,. \tag{2.17}$$

By taking exterior differentiation of (2.15), we have Ricci formula for the second fundamental form of M

$$h_{ijkl} - h_{ijlk} = -\sum_r (h_{ir}R_{rjkl} + h_{jr}R_{rikl})\,. \tag{2.18}$$

Restricting (2.5) on M, $\overline{R}_{(n+1)ijk;l}$ is given by

$$\begin{aligned}\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl} &+ \overline{R}_{(n+1)i(n+1)k}h_{jl} \\ &+ \overline{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \overline{R}_{mijk}h_{ml}\,,\end{aligned} \tag{2.19}$$

where $\overline{R}_{(n+1)ijkl}$ denote the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on M by

$$\begin{aligned}\sum_l \overline{R}_{(n+1)ijkl}\omega_l = d\overline{R}_{(n+1)ijk} &- \sum_l \overline{R}_{(n+1)ljk}\omega_{li} \\ &- \sum_l \overline{R}_{(n+1)ilk}\omega_{lj} - \sum_l \overline{R}_{(n+1)ijl}\omega_{lk}\,.\end{aligned} \tag{2.20}$$

Remark 2.1. If $\overline{M}$ is a Lorentz space form of index 1, by a straightforward calculation we check that the sum of the last three terms of right-hand side of (2.19) goes to zero. Then we have $\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl}$, which is the same as in the case that the ambient space is a space form.

It is well known that the Laplacian Δh_{ij} is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}\,. \tag{2.21}$$

By using Codazzi equation and Ricci formula, we get

$$\begin{aligned}\Delta h_{ij} &= \sum_k h_{ikjk} + \sum_k \overline{R}_{(n+1)ijkk} = \sum_k h_{kijk} + \sum_k \overline{R}_{(n+1)ijkk} \\ &= \sum_k \Big(h_{kikj} - \sum_l (h_{kl}R_{lijk} + h_{il}R_{lkjk}) + \overline{R}_{(n+1)ijkk}\Big)\,.\end{aligned} \tag{2.22}$$

From the Codazzi equation $h_{ikjk} = h_{kkij} + \overline{R}_{(n+1)kikj}$, we have

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_k \left(\overline{R}_{(n+1)ijkk} + \overline{R}_{(n+1)kikj}\right) - \sum_{k,l} \left(h_{kl}R_{lijk} + h_{il}R_{lkjk}\right).$$

Together with Gauss equation and above equation and (2.19), we have

$$\begin{aligned}\Delta h_{ij} = &\sum_k h_{kkij} + \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) \\ &- \sum_{k,l} \left(2h_{kl}\overline{R}_{lijk} + h_{jl}\overline{R}_{lkik} + h_{il}\overline{R}_{lkjk}\right) + Sh_{ij} \\ &- \sum_k \left(h_{kk}\overline{R}_{(n+1)ij(n+1)} + h_{ij}\overline{R}_{(n+1)k(n+1)k}\right) - nH\sum_l h_{il}h_{jl}\,.\end{aligned} \tag{2.23}$$

Thus

$$\begin{aligned}\frac{1}{2}\Delta S =&\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\\ =&\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}\big(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\big)h_{ij}\\ &- \Big(nH\sum_{i,j} h_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_{k}\overline{R}_{(n+1)k(n+1)k}\Big) + S^2\\ &- \sum_{i,j,k,l} 2\big(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}\big) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij}\,.\end{aligned} \tag{2.24}$$

3. Estimates of Laplacian and Key lemmas

Let $\overline{M}$ be a locally symmetric Lorentz space, i.e., $\overline{R}_{ABCD;E} = 0$. We also may choose a canonical bases $\{e_1, e_2, \dots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$, thus

$$\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j} = 0\,. \tag{3.1}$$

Noticing that $\overline{M}$ satisfies condition $(\overline{*})$, we have

$$\begin{aligned}&- \Big(nH\sum_{i,j} h_{ij}\overline{R}_{(n+1)ij(n+1)} + S\sum_{k}\overline{R}_{(n+1)k(n+1)k}\Big)\\ =&- \Big(nH\sum_{i} \lambda_i\overline{R}_{(n+1)ii(n+1)} + S\sum_{i}\overline{R}_{(n+1)i(n+1)i}\Big)\\ =&\, c_1(S - nH^2)\,.\end{aligned} \tag{3.2}$$

Also we have

$$\begin{aligned}&- \sum_{i,j,k,l} 2(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk})\\ =&- 2\sum_{j,k}(\lambda_j\lambda_k - \lambda_k^2)\overline{R}_{kjjk} \le c_2\sum_{j,k}(\lambda_j - \lambda_k)^2 = 2c_2(nS - n^2H^2)\,.\end{aligned} \tag{3.3}$$

Substituting (3.1), (3.2) and (3.3) in to (2.24), it yields that

$$\frac{1}{2}\Delta S \le \sum_{i,k} h_{iik}^2 + \sum_{i}\lambda_i(nH)_{ii} + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH\sum_{i}\lambda_i^3)\,. \tag{3.4}$$

Lemma 3.1 ([7]). *Let $\{\mu_1, \mu_2, \dots, \mu_n\}$ be real numbers satisfying $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = A$, where A is a constant no less than zero. Then we have*

$$\Big|\sum_{i}\mu_i^3\Big| \le \frac{n-2}{\sqrt{n(n-1)}}A^{\frac{3}{2}}\,,$$

and the equality holds if and only if at least $n-1$ of the μ_i are equal, i.e.,

$$\mu_1 = \mu_2 = \ldots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}A}, \quad \mu_n = \sqrt{\frac{n-1}{n}A}\,.$$

Lemma 3.2. *Let M be a space-like hypersurface with constant normalized scalar curvature R in locally symmetric $(n+1)$-dimensional Lorentz space satisfying the condition $(\circledast)$. If $h_{ijk} \geq 0$, then*

$$\sum_{i,j,k} h_{ijk}^2 \leq n^2 |\nabla H|^2 .$$

Proof. Notice that the following equation holds:

$$\begin{aligned} n^2|\nabla H|^2 &= \sum_k \big(\sum_{i,j} h_{ijk}\big)^2 = \sum_{i,j,k,l,m} h_{ijk}h_{lmk} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i\neq l,j,k,m} h_{ijk}h_{lmk} + \sum_{i,j\neq m,k} h_{ijk}h_{imk} . \end{aligned}$$

Then the proof follows from the above equation. □

Next we will use the well known self-adjoint operator □ introduced in [1] to the function nH and using (2.14), we have

$$\begin{aligned} \square(nH) &:= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\Delta(n(n-1)P) + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{ii} . \end{aligned} \tag{3.5}$$

By (2.14), we know that P is a constant, so we have $\frac{1}{2}\Delta(n(n-1)P) = 0$. Then substituting (3.4) to (3.5), we obtain

$$\square(nH) \leq \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + (2nc_2 + c_1)(S - nH^2) + (S^2 - nH\sum_i \lambda_i^3) . \tag{3.6}$$

Lemma 3.3. *Let M be a compact space-like hypersurface of dimension n with constant scalar curvature in a locally symmetric Lorentz space which satisfies condition $(\circledast)$ and $h_{ijk} \geq 0$. Then we have the following inequality*

$$\square(nH) \leq \frac{n-1}{n}(S - nP)\phi_P(S) ,$$

where $\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S + \frac{n-2}{n}\sqrt{(n(n-1)P + S)(s - nP)}$ and $c = 2c_2 + \frac{c_1}{n}$.

Proof. We denote

$$\mu_i = \lambda_i - H, \quad B = \sum_i \mu_i^2 .$$

It is obvious to see that

$$\sum_i \mu_i = 0 , \qquad B = S - nH^2 , \qquad \sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3HB + nH^3 .$$

By using Lemma 3.1, we have

$$
\begin{aligned}
-nH\sum_i \lambda_i^3 =& -n^2H^4 - 3nH^2B - nH\sum_i \mu_i^3 \\
\leq& 2n^2H^4 - 3nSH^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^{\frac{3}{2}}\,.
\end{aligned} \tag{3.7}
$$

Substituting (3.7) to (3.6) and together with the Lemma 3.2, we get

$$
\square(nH) \leq B\Big(nc - nH^2 + B + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|B^{\frac{1}{2}}\Big)\,. \tag{3.8}
$$

It follows from (2.14) that

$$
B = S - nH^2 = \frac{n-1}{n}(S - nP)\,. \tag{3.9}
$$

Putting the above equation into (3.8), we get

$$
\square(nH) \leq \frac{n-1}{n}(S-nP)\phi_H(S)\,, \tag{3.10}
$$

where

$$
\phi_H(S) = nc - 2nH^2 + S + \frac{n(n-2)}{\sqrt{n(n-1)}}\|H\|\sqrt{S - nH^2}\,. \tag{3.11}
$$

Putting (3.9) into (3.11), we have

$$
\phi_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S + \frac{n-2}{n}\sqrt{(n(n-1)P + S)(S - nP)}\,.
$$

Finally, (3.10) becomes

$$
\square(nH) \leq \frac{n-1}{n}(S-nP)\phi_P(S)\,, \tag{3.12}
$$

then we complete the proof. □

4. Main theorems and proofs

Theorem 4.1. *Let M be a compact space-like hypersurface of dimension n (where $n > 2$) with constant scalar curvature in a locally symmetric Lorentz space of dimension $n+1$ which satisfies condition ($\overline{*}$) and $h_{ijk} \geq 0$. If $0 \leq c \leq P$ or $c \leq P < \frac{2}{n}c$ or $P > \frac{1}{n-1}c, c < 0$, then the norm square of the second fundamental form S satisfies*

$$
S \geq \frac{n}{(n-2)(nP - 2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big)\,,
$$

where P is given by (2.14) and $c = 2c_2 + \frac{c_1}{n}$.

Proof. Since □ is a self-adjoint operator and M is compact, then we have

$$
\int_M \square(nH) * 1 = 0\,. \tag{4.1}
$$

We notice that $S - nP \geq 0$ holds naturally by (3.9) because $S \geq nH^2$. By taking integration on both sides of (3.12), we get $\phi_P(S) \geq 0$. By directly calculation we see that $\phi_P(S) \geq 0$ is equivalent to

$$S \geq \frac{n}{n-2}\big(2(n-1)P - nc\big)$$

or

$$\frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big) \leq S < \frac{n}{n-2}\big(2(n-1)P - nc\big)\,.$$

By solving the above inequalities, we complete the proof. □

Theorem 4.2. *Let M be a compact space-like hypersurface of dimension n (where $n > 2$) with constant scalar curvature in a locally symmetric Lorentz space of dimension $n+1$ which satisfies condition $(\circledast)$ and $h_{ijk} \geq 0$ and $0 \leq c \leq P$ or $c \leq P < \frac{2}{n}c$ or $P > \frac{1}{n-1}c, c < 0$. If the norm square of the second fundamental form S satisfies*

$$nP \leq S \leq \frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big), \tag{4.2}$$

then

(i) *$S = nP$ and M is totally umbilical, or*

(ii) *$S = \frac{n}{(n-2)(nP-2c)}\big(n(n-1)P^2 - 4c(n-1)P + nc^2\big)$ and M has two distinct principal curvatures.*

Proof. Together with (4.2) and the definition of P, we see that the right-hand side term of (3.12) is non-positive. As in proof of Theorem 4.1, we take integration on both sides of (3.12) and notice (4.1), we have $(S - nP)\phi_P(S) = 0$. In particular, we notice that $\phi_P(S) = 0$ if and only if the equality holds in Lemma 3.1, thus we prove the theorem. □

Remark 4.3. Let $\overline{M}$ in Theorem 4.2 be a Lorentz space form with constant sectional curvature s. In particular, we assume that $s = 1$ such that $\overline{M}$ is nothing but a de Sitter space. As seen in Remark 1.3, we have $-\frac{c_1}{n} = c_2 = 1$. Thus c defined in Lemma 3.3 is 1. Then our theorem is just like Liu's corollary in [6].

Finally, we discuss the compact space-like surface in a locally symmetric Lorentz spaces of dimension 3, i.e., the version of $n = 2$ of the Theorem 4.1. We using the convention of the ranges of the indexes as following

$$i, j, k = 1, 2\,, \qquad A, B, C = 1, 2, 3\,.$$

Theorem 4.4. *Let M be a compact space-like surface with constant scalar curvature in a locally symmetric 3-Minkowski space which satisfies condition $(\circledast)$ and $h_{ijk} \geq 0$. Then*

$$P \leq c\,,$$

where P is given by (2.14) and $c = 2c_2 + \frac{c_1}{n}$ and h_{ijk} is defined by (2.15).

Proof. We notice that when $n = 2$, (3.12) becomes $\square(2H) \leq (S - 2P)(c - P)$. Taking integration on both sides of the inequality, then we complete the proof. □

Corollary 4.5. *Let M be a compact space-like surface with constant scalar curvature in a locally symmetric 3-Minkowski space which satisfies condition ($\overline{*}$) and $h_{ijk} \geq 0$. If $P \geq c$, then*

(i) *$S = 2P$ and M is totally umbilical, or*

(ii) *$P = c$.*

The proof is the same as the proof of Theorem 4.2.

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments for the improvement of this paper.

References

[1] Chen, S. Y., Yau, S. T., *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195–204.

[2] Choi, S. M., Kwon, J. H., Suh, Y. J., *Complete spacelike hypersurfaces in a Lorentz manifold*, Math. J. Toyama Univ. **22** (1999), 53–76.

[3] Jin, O. B., Cheng, Q. M., Young, J. S., *Complete spacelike hypersurfaces in locally symmetric Lorentz space*, J. Geom. Phys. **49** (2004), 231–247.

[4] Liu, J. C., Wei, L., *A gep theorem for complete spacelike hypersurface with constant scalar curvature in locally symmetric Lorentz space*, Turkish J. Math. **34** (2010), 105–114.

[5] Liu, X., *Space-like hypersurfaces of constant scalar in the de Sitter space*, Atti Sem. Mat. Fis. Univ. Modena **48** (2000), 99–106.

[6] Liu, X. M., *Complete spacelike hypersurfaces with constant scalar curvature*, Manuscripta Math. **105** (2001), 367–377.

[7] Okumura, M., *Hypersurfaces and a piching problem on the second fundamental thesor*, J. Math. Soc. Japan **19** (1967), 205–214.

[8] Suh, Y. J., Choi, Y. S., Yang, H. Y., *On spacelike hypersurfaces with constant mean curvature in Lorentz manifold*, Houston J. Math. **28** (2002), 47–70.

[9] Xu, S. L., Chen, D. M., *Complete space-like submanifolds in locally symmetric semi-definite spaces*, Anal. Theory Appl. **20** (2004), 383–390.

[10] Yau, S. T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[11] Zheng, Y. F., *Spacelike hypersurfaces with constant scalar curvature in the de Sitter space*, Differential Geom. Appl. **6** (1996), 51–54.

Department of Mathematics,
South China University of Technology,
Guangzhou 510641, GuangDong, P. R. China
E-mail: wyn051@163.com.cn

School of Mathematical Sciences,
Dalian University of Technology,
Dalian 116024, LiaoNing, P. R. China
E-mail: ximinliu@dlut.edu.cn