ON COMPLETE SPACELIKE HYPERSURFACES WITH
\( R = aH + b \) IN LOCALLY SYMMETRIC LORENTZ SPACES

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Abstract. In this note, we investigate \( n \)-dimensional spacelike hypersurfaces \( M^n \) with \( R = aH + b \) in locally symmetric Lorentz space. Two rigidity theorems are obtained for these spacelike hypersurfaces.

1. Introduction

Let \( M^{n+1} \) be an \((n+1)\)-dimensional Lorentz space, i.e. a pseudo-Riemannian manifold of index 1. When the Lorentz space \( M^{n+1}_1 \) is of constant curvature \( c \), we call it a Lorentz space form, denoted by \( M^{n+1}_1(c) \). A hypersurface \( M^n \) of a Lorentz space is said to be spacelike if the induced metric on \( M^n \) from that of the Lorentz space is positive definite. Since Goddard’s conjecture (see [7]), several papers about spacelike hypersurfaces with constant mean curvature in de Sitter space \( S^{n+1}_1(1) \) have been published. For a more complete study of spacelike hypersurfaces in general Lorentzian space with constant mean curvature, we refer to [2]. For the study of spacelike hypersurface with constant scalar curvature in de Sitter space \( S^{n+1}_1(1) \), there are also many results such as [4, 9, 14, 15]. There are some results about spacelike hypersurfaces with constant scalar curvature in general Lorentzian space, such as [8] and [13].

It is natural to study complete spacelike hypersurfaces in the more general Lorentz spaces, satisfying the assumptions \( R = aH + b \), where \( R \) is the normalized scalar curvature at a point of space-like hypersurface, \( H \) is the mean curvature and \( a, b \in \mathbb{R} \) are constants. First of all, we recall that Choi et al. [6, 12] introduced the class of \((n + 1)\)-dimensional Lorentz spaces \( M^{n+1}_1 \) of index 1 which satisfy the following two conditions for some fixed constants \( c_1 \) and \( c_2 \):

(i) for any spacelike vector \( u \) and any timelike vector \( v \),

\[ K(u, v) = -\frac{c_1}{n}, \]

(ii) for any spacelike vectors \( u \) and \( v \),

\[ K(u, v) \geq c_2. \]

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Theorem 1.1. Let 

where 

If 

Convention. When \( M_1^{n+1} \) satisfies conditions (i) and (ii), we shall say that \( M_1^{n+1} \) satisfies condition \((\ast)\).

We compute the scalar curvature at a point of Lorentz space \( M_1^{n+1} \),

\[
\bar{R} = \sum_A \epsilon_A \bar{R}_{AA} = -2 \sum_{i=1}^n \bar{R}_{n+1 i i n + 1} + \sum_{ij} \bar{R}_{ijji} = -2c_1 + \sum_{ij} \bar{R}_{ijji} ,
\]

where \( \bar{R}_{n+1 i i n + 1} = -K(e_i, e_{n+1}) = \frac{e_i}{n} \), for \( i = 1, \ldots, n \).

It is known that \( \bar{R} \) is constant when the Lorentz space \( M_1^{n+1} \) is locally symmetric, so \( \sum_{ij} \bar{R}_{ijji} \) is constant. In this note, we shall prove the following main results:

**Theorem 1.1.** Let \( M^n \) be a complete spacelike hypersurface with bounded mean curvature in locally symmetric Lorentz space \( M_1^{n+1} \) satisfying the condition \((\ast)\). If \( R = aH + b \), \( (n-1)^2 a^2 + 4 \sum_{ij} \bar{R}_{ijji} - 4n(n-1)b \geq 0 \), and \( a \geq 0 \), then the following properties hold.

1. If \( \sup H^2 < \frac{4(n-1)}{n^2} c \), where \( c = \frac{a_n}{n} + 2c_2 \), then \( c > 0 \), \( S = nH^2 \) and \( M^n \) is totally umbilical.

2. If \( \sup H^2 = \frac{4(n-1)}{n^2} c \), then \( c \geq 0 \) and either \( S = nH^2 \) and \( M^n \) is totally umbilical, or \( \sup S = nc \).

3-a) If \( c < 0 \), then either \( S = nH^2 \) and \( M^n \) is totally umbilical, or \( \sup H^2 < \sup S \leq S^+ \).

3-b) If \( c \geq 0 \) and \( \sup H^2 \geq c > \frac{4(n-1)}{n^2} c \), then either \( S = nH^2 \) and \( M^n \) is totally umbilical, or \( \sup H^2 < \sup S \leq S^+ \).

3-c) If \( c \geq 0 \) and \( c > \sup H^2 > \frac{4(n-1)}{n^2} c \), then either \( S = nH^2 \) and \( M^n \) is totally umbilical, or \( S^- \leq \sup S \leq S^+ \).

4) \( S = \frac{n}{2(n-1)} [n^2 \sup H^2 + (n-2) \sup |H| \sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc \), if and only if \( M \) is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Here \( S^+ = \frac{n}{2(n-1)} [n^2 \sup H^2 + (n-2) \sup |H| \sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc \), and \( S^- = \frac{n}{2(n-1)} [n^2 \sup H^2 - (n-2) \sup |H| \sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc \).

**Theorem 1.2.** Let \( M^n \) \( (n > 1) \) be a complete spacelike hypersurface in locally symmetric Lorentz space \( M_1^{n+1} \) satisfying the condition \((\ast)\). If \( c = \frac{a_n}{n} + c_2 > 0 \), \( c_2 > 0 \) and

\[
W^2 = \text{tr}(W)W ,
\]

where \( W \) is the shape operator with respect to \( e_{n+1} \), then \( M^n \) must be totally geodesic.

**Remark 1.3.** The Lorentz space form \( M_1^{n+1}(c) \) satisfies the condition \((\ast)\), where \(-\frac{c_1}{n} = c_2 = \text{const} \).
2. Preliminaries

Let $M^n$ be a spacelike hypersurface of Lorentz space $M^{n+1}_1$. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \ldots, e_n, e_{n+1}\}$ in $M^{n+1}_1$ such that, restricted to $M^n$, $e_1, \ldots, e_n$ are tangent to $M^n$ and $e_{n+1}$ is the unit timelike normal vector. Denote by $\{\omega_A\}$ the corresponding dual coframe and by $\{\omega_{AB}\}$ the connection forms of $M^{n+1}_1$. Then the structure equations of $M^{n+1}_1$ are given by

\begin{align*}
  d\omega_A &= -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad \epsilon_i = 1, \quad \epsilon_{n+1} = -1, \\
  d\omega_{AB} &= -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{CD} \epsilon_C \epsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D,
\end{align*}

where $A, B, C, \cdots = 1, \ldots, n+1$ and $i, j, l, \cdots = 1, \ldots, n$. The components $\bar{R}_{CD}$ of the Ricci tensor and the scalar curvature $\bar{R}$ of $M^{n+1}_1$ are given by

\begin{align*}
  \bar{R}_{CD} &= \sum_B \epsilon_B \bar{R}_{BCDB}, \\
  \bar{R} &= \sum_A \epsilon_A \bar{R}_{AA}.
\end{align*}

The components $\bar{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\bar{R}$ are defined by

\begin{align*}
  \sum_E \epsilon_E \bar{R}_{ABCD;E} \omega_E &= d\bar{R}_{ABCD} - \sum_E \epsilon_E (\bar{R}_{EBCD} \omega_{EA} \\
  &\quad + \bar{R}_{AECDB} \omega_{EB} + \bar{R}_{ABED} \omega_{EC} + \bar{R}_{ABCE} \omega_{ED}).
\end{align*}

We restrict these forms to $M^n$, then $\omega_{n+1} = 0$ and the Riemannian metric of $M^n$ is written as $ds^2 = \sum_i \omega_i^2$. Since

\begin{align*}
  0 &= d\omega_{n+1} = -\sum_i \omega_{n+1; i} \wedge \omega_i,
\end{align*}

by Cartan’s lemma we may write

\begin{align*}
  \omega_{n+1; i} &= \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.
\end{align*}

From these formulas, we obtain the structure equations of $M^n$:

\begin{align*}
  d\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\
  d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \\
  R_{ijkl} &= \bar{R}_{ijkl} - (h_{il} h_{jk} - h_{ik} h_{jl}),
\end{align*}
where $R_{ijkl}$ are the components of curvature tensor of $M^n$. Components $R_{ij}$ of Ricci tensor and scalar curvature $R$ of $M^n$ are given by

$$
R_{ij} = \sum_k \tilde{R}_{kijk} - \left( \sum_k h_{kk} \right) h_{ij} + \sum_k h_{ik} h_{jk},
$$

(11)

$$
n(n - 1)R = \sum_{ij} \tilde{R}_{ijji} + S - n^2 H^2.
$$

(12)

We call

$$
B = \sum_{i,j,\alpha} h_{ij} \omega_i \otimes \omega_j \otimes \epsilon_{n+1}
$$

(13)

the second fundamental form of $M^n$. The mean curvature vector is $h = \frac{1}{n} \sum_i h_{ii} \epsilon_{n+1}$. We denote $S = \sum_{ij} (h_{ij})^2$, $H^2 = |h|^2$ and $W = (h_{ij})^n_{i,j=1}$. We call that $M^n$ is maximal if its mean curvature vector vanishes, i.e. $h = 0$.

Let $h_{ijk}$ and $h_{ijkl}$ denote the covariant derivative and the second covariant derivative of $h^n_{ij}$. Then we have $h_{ijk} = h_{ikj} + \tilde{R}_{(n+1)ijk}$ and

$$
h_{ijkl} - h_{ijlk} = - \sum m h_{im} R_{mjkl} - \sum m h_{mj} R_{mikl}.
$$

(14)

Restricting the covariant derivative $\tilde{R}_{ABCD;E}$ on $M^n$, then $\tilde{R}_{(n+1)ijk;l}$ is given by

$$
\tilde{R}_{(n+1)ijk;l} = \tilde{R}_{(n+1)ijkl} + \tilde{R}_{(n+1)i(n+1)k} h_{jl} + \tilde{R}_{(n+1)i(j(n+1)h_{kl}} + m \tilde{R}_{mi} h_{kl},
$$

(15)

where $\tilde{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\tilde{R}_{(n+1)ij}$ as a tensor on $M^n$ so that

$$
\tilde{R}_{(n+1)ijk;l} = g \tilde{R}_{(n+1)ijk} - \sum_l \tilde{R}_{(n+1)ijkl} \omega_l - \sum_l \tilde{R}_{(n+1)il} \omega_{lj} - \sum_l \tilde{R}_{(n+1)il} \omega_{lk}.
$$

(16)

The Laplacian $\triangle h_{ij}$ is defined by $\triangle h_{ij} = \sum_k h_{ikkk}$. Using Gauss equation, Codazzi equation Ricci identity and (2), a straightforward calculation will give

$$
\frac{1}{2} \triangle S = \sum_{ijk} h_{ijk}^2 + \sum_{ij} h_{ij} \triangle h_{ij}
$$

$$
= \sum_{ijk} h_{ijk}^2 + \sum_{ij} (nH)_{ij} h_{ij} + \sum_{ijk} (\tilde{R}_{(n+1)ijk;k} + \tilde{R}_{(n+1)kik;j}) h_{ij}
$$

$$
- \sum_{ij} nH h_{ij} \tilde{R}_{(n+1)ij(n+1)} + S \sum_k \tilde{R}_{(n+1)k(n+1)k})
$$

$$
- 2 \sum_{ijkl} (h_{kl} h_{ij} \tilde{R}_{ijlk} + h_{ij} h_{ij} \tilde{R}_{lkjk}) - nH \sum_{ijkl} h_{ii} h_{ij} h_{ij} + S^2.
$$

(17)

Set $\Phi_{ij} = h_{ij} - H \delta_{ij}$, it is easy to check that $\Phi$ is traceless and $|\Phi|^2 = S - nH^2$. In this note we consider the spacelike hypersurface with $R = aH + b$ in locally
symmetric Lorentz space $M_1^{n+1}$, where $a, b$ are real constants. Following Cheng-Yau [5], we introduce a modified operator acting on any $C^2$-function $f$ by

$$L(f) = \sum_{ij} (nH\delta_{ij} - h_{ij}) f_{ij} + \frac{n-1}{2} a \Delta f.$$  

We need the following algebraic Lemmas.

**Lemma 2.1** ([11]). Let $M^n$ be an $n$-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $F: M^n \to \mathbb{R}$ be a smooth function which is bounded above on $M^n$. Then there exists a sequence of points $x_k \in M^n$ such that

$$\lim_{k \to \infty} F(x_k) = \sup(F),$$

$$\lim_{k \to \infty} |\nabla F(x_k)| = 0,$$

$$\lim_{k \to \infty} \sup \max \{|(\nabla^2(F)(x_k))(X,X): |X| = 1\} \leq 0.$$  

**Lemma 2.2** ([1, 10]). Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta \geq 0$ is constant. Then

$$\left| \sum_i \mu_i^2 \right| \leq \frac{n-2}{\sqrt{n(n-1)}} \beta^2,$$

and equality holds if and only if at least $n-1$ of $\mu_i$’s are equal.

3. PROOF OF THE THEOREMS

First, we give the following lemma.

**Lemma 3.1.** Let $M^n$ be a complete spacelike hypersurface in locally symmetric Lorentz space $M_1^{n+1}$ satisfying the condition (*). If $R = aH + b$, $a, b \in \mathbb{R}$ and $(n-1)^2 a^2 + 4 \sum_{ij} \hat{R}_{ijji} - 4n(n-1)b \geq 0$.

(1) We have the following inequality,

$$L(nH) \geq |\Phi|^2 \left( |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\Phi| + nc - nH^2 \right).$$

where $c = 2c_2 + \frac{c_1}{n}$.

(2) If the mean curvature $H$ is bounded, then there is a sequence of points $\{x_k\} \in M$ such that

$$\lim_{k \to \infty} nH(x_k) = \sup(nH), \quad \lim_{k \to \infty} |\nabla nH(x_k)| = 0,$$

$$\lim_{k \to \infty} \sup (L(nH)(x_k)) \leq 0.$$  

**Proof.** (1) Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ and $\Phi_{ij} = \lambda_i \delta_{ij} - H \delta_{ij}$. Let $\mu_i = \lambda_i - H$ and denote $\Phi^2 = \sum_i \mu_i^2$. From [12],
and the relation \( R = aH + b \), we have

\[
L(nH) = \sum_{ij} (nH\delta_{ij} - h_{ij})(nH)_{ij} + \frac{(n-1)a}{2} \triangle(nH)
\]

\[
= nH \triangle(nH) - \sum_{ij} h_{ij}(nH)_{ij} + \frac{1}{2} \triangle(n(n-1)R - n(n-1)b)
\]

\[
= \frac{1}{2} \triangle[(nH)^2 + n(n-1)R] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij}
\]

\[
= \frac{1}{2} \triangle \left[ \sum_{ij} \bar{R}_{ijji} + S \right] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij}
\]

\[
= \frac{1}{2} \triangle S - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij}.
\]

From (17) and \( M^n \) is locally symmetric, we have

\[
L(nH) = \sum_{ijk} h^2_{ijk} - n^2|\nabla H|^2 - nH \sum_i \lambda_i^3 + S^2
\]

\[
- \left( \sum_{ij} nH\lambda_i \bar{R}_{(n+1)ii(n+1)} + S \sum_{k} \bar{R}_{(n+1)k(n+1)k} \right) - 2 \sum_{ijkl} \left( \lambda_k \lambda_i \bar{R}_{kiik} + \lambda_i^2 \bar{R}_{ikik} \right).
\]

Firstly, we estimate (I):

From Gauss equation, we have

\[
(22) \quad \sum_{ij} \bar{R}_{ijji} + S - n^2 H^2 = n(n-1)R = n(n-1)(aH + b),
\]

Taking the covariant derivative of the above equation, we have

\[
(23) \quad 2 \sum_{ijk} h_{ij} h_{ijk} = 2n^2 H H_k + n(n-1)a H_k.
\]

Therefore

\[
(24) \quad 4S \sum_{ijk} h^2_{ijk} \geq 4 \sum_k \left( \sum_{ij} h_{ij} h_{ijk} \right)^2 = \left[ 2n^2 H + n(n-1)a \right]^2 |\nabla H|^2.
\]

Since we know

\[
[2n^2 H + n(n-1)a]^2 - 4n^2 S = 4n^4 H^2 + n^2(n-1)^2 a^2 + 4n^3(n-1)a H
\]

\[
- 4n^2 \left[ n^2 H^2 + n(n-1)R - \sum_{ij} \bar{R}_{ijji} \right]
\]

\[
= n^2 \left[ (n-1)^2 a^2 + 4 \sum_{ijji} \bar{R}_{ijji} - 4n(n-1)b \right] \geq 0.
\]
if follows that
\begin{equation}
\sum_{ijk} h^2_{ijk} \geq n^2 |\nabla H|^2.
\end{equation}

Secondly, we estimate (II):

It is easy to know that
\begin{equation}
\sum_{i} \lambda_i^3 = nH^3 + 3H \sum_i \mu_i^2 + \sum_i \mu_i^3.
\end{equation}

By applying Lemma \[2.2\] to real numbers $\mu_1, \ldots, \mu_n$, we get
\begin{align*}
S^2 - nH \sum_{i} \lambda_i^3 &= (|\Phi|^2 + nH^2)^2 - n^2 H^4 - 3nH^2 |\Phi|^2 - nH \sum_{i} \mu_i^3 \\
\geq |\Phi|^4 - nH^2 |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||\Phi|^3.
\end{align*}

Finally, we estimate (III):

Using curvature condition (\star), we get
\begin{equation}
L(nH) \geq |\Phi|^2 \left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\Phi| + nc - nH^2 \right).
\end{equation}

(2) Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. By definition, $L(nH) = \sum_i (nH - \lambda_i)(nH)_{ii} + \frac{(n-1)a}{2} \sum_i (nH)_{ii}$. If $H \equiv 0$ the result is obvious. Let suppose that $H$ is not identically zero. By changing the orientation of $M^n$ if necessary, we may assume that $\sup H > 0$. From
\begin{align*}
(\lambda_i)^2 &\leq S = n^2 H^2 + n(n-1)R - \sum_{ij} \bar{R}_{ijji} \\
&= n^2 H^2 + n(n-1)(aH + b) - \sum_{ij} \bar{R}_{ijji} \\
&= (nH + \frac{(n-1)a}{2})^2 - \frac{1}{4} (n-1)^2 a^2 - \sum_{ij} \bar{R}_{ijji} + n(n-1)b \\
&\leq (nH + \frac{(n-1)a}{2})^2,
\end{align*}

we have
\begin{equation}
|\lambda_i| \leq |nH + \frac{(n-1)a}{2}|.
\end{equation}

Since $H$ is bounded and Eq. (30), we know that $S$ is also bounded. From the
Eq. (10),
\begin{align}
R_{ijji} &= \bar{R}_{ijji} - h_{ii}h_{jj} + (h_{ij})^2 \\
&= c_2 - \lambda_i \lambda_j \geq c_2 - S.
\end{align}

This shows that the sectional curvatures of $M^n$ are bounded from below because
$S$ is bounded. Therefore we may apply Lemma 2.1 to the function $nH$, and obtain
a sequence of points $\{x_k\} \in M^n$ such that
\begin{equation}
\lim_{k \to \infty} nH(x_k) = \sup(nH), \quad \lim_{k \to \infty} |\nabla(nH)(x_k)| = 0,
\end{equation}
\begin{equation}
\lim_{k \to \infty} \sup(nH_{ii}(x_k)) \leq 0.
\end{equation}

Since $H$ is bounded, taking subsequences if necessary, we can arrive to a sequence
$\{x_k\} \in M^n$ which satisfies (33) and such that $H(x_k) \geq 0$ (by changing the
orientation of $M^n$ if necessary). Thus from (31) we get
\begin{align}
0 &\leq nH(x_k) + \frac{(n-1)a}{2} - |\lambda_i(x_k)| \leq nH(x_k) + \frac{(n-1)a}{2} - \lambda_i(x_k) \\
&\leq nH(x_k) + \frac{(n-1)a}{2} + |\lambda_i(x_k)| \leq 2(nH(x_k) + \frac{(n-1)a}{2}).
\end{align}

Using once the fact that $H$ is bounded, from (34) we infer that $\{nH(x_k) -
\lambda_i^{n+1}(x_k)\}$ is non-negative and bounded. By applying $L(nH)$ at $x_k$, taking the
limit and using (33) and (34) we have
\begin{align}
\lim_{k \to \infty} \sup(L(nH))(x_k)
&\leq \sum_i \lim_{k \to \infty} \sup(nH + \frac{(n-1)a}{2} - \lambda_i(x_k))nH_{ii}(x_k) \leq 0.
\end{align}

\begin{remark}
When $a = 0$, then $R = b$ is constant, the inequality (20) appeared in [3, 8, 13].
\end{remark}

**Proof of Theorem 1.1.** According to Lemma 3.1 (2), there exists a sequence of
points $\{x_k\}$ in $M^n$ such that
\begin{equation}
\lim_{k \to \infty} nH(x_k) = \sup(nH), \quad \lim_{k \to \infty} \sup(L(nH)(x_k)) \leq 0.
\end{equation}

From Gauss equation, we have that
\begin{align}
|\Phi|^2 &= S - nH^2 = n(n-1)H^2 + n(n-1)(aH + b) - \sum_{ij} \bar{R}_{ijji}.
\end{align}
Notice that \( \lim_{k \to \infty} (nH)(x_k) = \sup(nH) \), \( a \geq 0 \) and \( \sum_{ij} \bar{R}_{ijij} \) is constant, we have
\[
(38) \quad \lim_{k \to \infty} |\Phi|^2(x_k) = \sup |\Phi|^2.
\]
Evaluating (20) at the points \( x_k \) of the sequence, taking the limit and using (36), we obtain that
\[
0 \geq \lim_{k \to \infty} \sup \left( L(nH)(x_k) \right)
\]
\[
(39) \quad \geq \sup |\Phi|^2 \left( \sup |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup |H| \sup |\Phi| + nc - n \sup H^2 \right).
\]
Consider the following polynomial given by
\[
(40) \quad P_{\sup H}(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup |H|x + nc - n \sup H^2.
\]
(1) If \( \sup H^2 < \frac{4(n-1)}{n^2} c \) holds, then we have \( c > 0 \) and \( P(\sup |\Phi|) > 0 \). From (39), we know that \( \sup |\Phi| = 0 \), that is \( |\Phi| = 0 \). Thus, we infer that \( S = nH^2 \) and \( M^n \) is totally umbilical.
(2) If \( \sup H^2 = \frac{4(n-1)}{n^2} c \) holds, then we have \( c \geq 0 \) and \( P(\sup |\Phi|) = (|\Phi| - \frac{n-2}{\sqrt{n}} \sqrt{c})^2 \geq 0 \). If \( (|\Phi| - \frac{n-2}{\sqrt{n}} \sqrt{c})^2 > 0 \), from (39) we have, \( \sup |\Phi| = 0 \), that is \( |\Phi| = 0 \). Thus, we infer that \( S = nH^2 \) and \( M^n \) is totally umbilical. If \( \sup |\Phi| = \frac{n-2}{\sqrt{n}} \sqrt{c} \), we have that \( S = nc \).
(3) If \( \sup H^2 > \frac{4(n-1)}{n^2} c \), we know that \( P(x) \) has two real roots \( x_{\sup H}^{-} \) and \( x_{\sup H}^{+} \) given by
\[
x_{\sup H}^{-} = \sqrt{\frac{n}{4(n-1)}} \left\{ (n-2) \sup |H| - \sqrt{n^2 \sup H^2 - 4(n-1)c} \right\}
\]
\[
x_{\sup H}^{+} = \sqrt{\frac{n}{4(n-1)}} \left\{ (n-2) \sup |H| + \sqrt{n^2 \sup H^2 - 4(n-1)c} \right\}
\]
It is easy to know that \( x_{\sup H}^{+} \) is always positive. In this case, we also have that
\[
(41) \quad P_{\sup H}(x) = (\sup |\Phi| - x_{\sup H}^{-})(\sup |\Phi| - x_{\sup H}^{+})
\]
From (39) and (41), we have that
\[
(42) \quad 0 \geq \sup |\Phi|^2 (\sup |\Phi| - x_{\sup H}^{-}) (\sup |\Phi| - x_{\sup H}^{+}).
\]
(3-a) If \( c < 0 \), we know that \( x_{\sup H}^{-} < 0 \). Therefore, from (42), we have, \( \sup |\Phi| = 0 \), in this case \( M^n \) is totally umbilical, or \( 0 < \sup |\Phi| \leq x_{\sup H}^{+} \), i.e.
\[
n \sup H^2 < \sup S \leq S^{+}.
\]
(3-b) If \( c \geq 0 \) and \( \sup(H)^2 \geq c > \frac{4(n-1)}{n^2} c \), we know that \( x_{\sup H}^{-} < 0 \). Therefore, from (42), we have, \( \sup |\Phi| = 0 \), in this case \( M^n \) is totally umbilical, or \( 0 < \sup |\Phi| \leq x_{\sup H}^{+} \), i.e.
\[
n \sup H^2 < \sup S \leq S^{+}.
\]
(3-c) If $c \geq 0$ and $c > \sup (H)^2 > \frac{4(n-1)}{n^2} c$, then we have $x^-_{\sup H} > 0$. Therefore, from (39), we have that $\sup |\Phi| = 0$, in this case $M^n$ is totally umbilical or $x^-_{\sup H} \leq \sup |\Phi| \leq x^+_{\sup H}$, i.e.

$$S^- \leq \sup S \leq S^+.$$  

(4) If $S \equiv \frac{n}{2(n-1)} [n^2 \sup H^2 + (n-2) \sup |H| \sqrt{n^2 \sup H^2 - 4(n-1)c} - nc]$ holds, from Gauss equation, we have $S = nH^2 + n(n-1)(aH + b) - \sum_{ij} \bar{R}_{ijji}$. Since $S$ is constant, then $H$ is also constant. We know that these inequalities in the proof of Lemma 2.2 and (27) are equalities and $S > nH^2$. Hence, we have $H^2 \geq \frac{4(n-1)}{n^2} c$ from (1) in Theorem 1.1. Thus, we can infer that $n - 1$ of the principal curvatures $\lambda_i$ are equal. Since $S$ and $H$ are constant, we know that principal curvatures are constant on $M^n$. Thus, $M^n$ is an isoparametric hypersurface with two distinct principal curvatures one of which is simple. This proves Theorem 1.1. □

**Proof of Theorem 1.2.** From (3), we have that

$$\sum_{k} h_{ik} h_{jk} = nH h_{ij}, \quad \text{for} \quad i, j \in \{1, \ldots, n\},$$

and

$$\sum_{ij} h_{ij}^2 = n^2 H^2, \quad \text{i.e.} \quad S = n^2 H^2.$$  

Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $R_{ij} = \nu_i \delta_{ij}$. From (11) and (43), we have $R_{ii} = \sum_k \bar{R}_{kiik} \geq (n-1)c_2 > 0$, that is, $\nu_i \geq (n-1)c_2 > 0$, so we know that $\text{Ric} = (R_{ij}) \geq (n-1)c_2 I$, we see by the Bonnet-Myers theorem that $M^n$ is bounded and hence compact.

From (12) and (44), we have that $n(n-1)R = \sum_{ij} \bar{R}_{ijji}$ is constant, then from Lemma 3.1 for $\alpha = 0$, we have the following inequality

$$L(nH) \geq |\Phi|^2 \left( |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi| + nc - nH^2 \right).$$

Since $L$ is self-adjoint and $M^n$ is compact, we have

$$0 \geq \int_{M^n} |\Phi|^2 \left( |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi| + nc - nH^2 \right).$$

Since $n^2 |H|^2 = S$ and $|\Phi|^2 = S - nH^2 = n(n-1)H^2$, we have

$$nc - nH^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H |\Phi|$$

$$= nc - nH^2 + n(n-1)H^2 - n(n-2)H^2 = nc > 0.$$  

so we know that $|\Phi|^2 = 0$, that is, $S = nH^2$. From Eq. (44), we know that $n^2 H^2 = nH^2$, so we have $H = 0$, i.e. $S = nH^2 = 0$, so $M^n$ is totally geodesic. This proves Theorem 1.2. □

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