A NOTE ON FUSION BANACH FRAMES

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Abstract. For a fusion Banach frame \((\{G_n, v_n\}, S)\) for a Banach space \(E\), if \((\{v_n^*(E^*), v_n^*\}, T)\) is a fusion Banach frame for \(E^*\), then \((\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)\) is called a fusion bi-Banach frame for \(E\). It is proved that if \(E\) has an atomic decomposition, then \(E\) also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of a fusion bi-Banach frame is given. Finally, a characterization of fusion bi-Banach frames is given.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in 1952 and re-introduced in 1986 by Daubechies, Grossmann and Meyer [4]. Casazza [2] and Benedetto and Fickus [1] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. Frames are now used as a tool in many areas like data compression, sampling theory, optics, filter banks, signal detection, time-frequency analysis etc.

The concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Gröchenig [6] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Gröchenig [7] who introduced the notion of Banach frames for Banach spaces. Jain et al. [9], generalized Banach frames in Banach spaces and introduced frames of subspaces (Fusion Banach frames) for Banach spaces. They gave the following definition of a fusion Banach frame.

Definition 1.1 ([9]). Let \(E\) be a Banach space. Let \(\{G_n\}\) be a sequence of non-trivial subspaces of \(E\) and \(\{v_n\}\) be a sequence of bounded linear projections such that \(v_n(E) = G_n\), \(n \in \mathbb{N}\). We associate a Banach space \(A\) and an operator \(S : A \rightarrow E\) with the space \(E\). Then \((\{G_n, v_n\}, S)\) is called a frame of subspaces (fusion Banach frame) for \(E\) with respect to \(A\) if

(i) \(\{v_n(x)\} \in A\), for all \(x \in E\),

(ii) there exist constants \(A, B (0 < A \leq B < \infty)\) such that

\[ A\|x\|_E \leq \|\{v_n(x)\}\|_A \leq B\|x\|_E, \quad x \in E, \]

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(iii) $S$ is a bounded linear operator such that

$$S(\{v_n(x)\}) = x, \quad x \in E.$$ 

The following lemma, proved in [9], is used in the sequel

**Lemma 1.2.** Let $\{G_n\}$ be a sequence of non-trivial subspaces of $E$ and $\{v_n\}$ be a sequence of bounded linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over $E$, i.e., $\{x \in E : v_n(x) = 0\}$, for all $n \in \mathbb{N}$, then $A = \{v_n(x) : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_A = \|x\|_E, \ x \in E$.

For other related notions on frames in Banach spaces one may refer to [3, 8, 10, 11].

In the present paper, we introduce fusion bi-Banach frames for a Banach space $E$. We prove that if $E$ has an atomic decomposition, then $E$ also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of fusion bi-Banach frames is given. Finally, a characterization of fusion bi-Banach frames is obtained.

## 2. Main Results

One may observe that, if $\{(G_n, v_n), S\}$ is a fusion Banach frame for $E$ with respect to some associated Banach space $A$, then there may not exist a Banach space $A_1$ associated with $E^*$ together with an operator $T : A_1 \to E^*$ such that $\{(v_n^*(E^*), v_n^*), T\}$ is a fusion Banach frame for $E^*$ with respect to $A_1$.

In this regard, we have the following examples

**Example 2.1.** Consider the Banach space

$$E = \ell_\infty(X) = \{\{x_n\} : x_n \in X; \ \sup_{1 \leq n < \infty} \|x_n\|_X < \infty\}$$

equipped with the norm $\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_X$, $\{x_n\} \in E$, where $(X, \|\cdot\|)$ is a Banach space. For each $n \in \mathbb{N}$, define $G_n = \{\delta_n^x : x \in X\}$ and $v_n(x) = \delta_n^x$, $\ x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \ldots, 0, x, 0, \ldots)$ for all $n \in \mathbb{N}$ and $x \in X$. Then by Lemma 1.2, there exist an associated Banach space $A = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{v_n(x)\}\|_A = \|x\|_E$, $x \in E$ together with an operator $S : A \to E$ given by $S(\{v_n(x)\}) = x$, $x \in E$ such that $\{(G_n, v_n), S\}$ is a fusion Banach frame for $E$ with respect to $A$. But, there does not exist a Banach space $A_1$ associated with $E^*$ together with an operator $T : A_1 \to E^*$ such that $\{(v_n^*(E^*), v_n^*), T\}$ is a fusion Banach frame for $E^*$ with respect to $A_1$. For otherwise, $\bigcup_{n=1}^{\infty} G_n = E$, which is not true.

**Example 2.2.** Let $E$ be a Banach space defined as

$$E = c_0(X) = \{\{x_n\} : x_n \in X; \ \lim_{n\to\infty} \|x_n\|_X = 0\}$$

equipped with the norm given by

$$\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_X,$$ 

where $(X, \|\cdot\|)$ is a Banach space.
Define a sequence \( \{G_n\} \) of subspaces of \( E \) by
\[
G_{2n-1} = \{ \delta_{2n-1}^x - 2^{n-1}\delta_{2n}^x : x \in X \} \\
G_{2n} = \{ \delta_{2n}^x : x \in X \}.
\]
Also define operators \( v_n \) on \( E \) by
\[
v_{2n-1}(x) = \delta_{2n-1}^{x_{2n-1}} - 2^{n-1}\delta_{2n}^{x_{2n-1}} \\
v_{2n}(x) = \delta_{2n}^{x_{2n-1}+x_{2n}} \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}.
\]

Then by Lemma [1.2] there exist an associated Banach space \( A \) and an operator \( S: A \to E \) such that \( (\{G_n, v_n\}, S) \) is a fusion Banach frame for \( E \) with respect to \( A \).

If \( \bigcup_{n=1}^{\infty} G_n \neq E \), then there exists \( 0 \neq f = \{f_i\} \in E^* \) such that \( f(y) = 0 \)
for all \( y \in G_n \), \( n \in \mathbb{N} \). This would imply \( f_n = 0 \) for all \( n \in \mathbb{N} \) and hence \( f = 0 \).
Therefore, by Lemma [1.2] again, there exist a Banach space \( A_1 \) associated to \( E^* \) and an operator \( T: A_1 \to E^* \) such that \( (\{v_n^*(E^*), v_n^*\}, T) \) is a fusion Banach frame for \( E^* \) with respect to \( A_1 \).

In view of the above discussion, we define the following

**Definition 2.3.** Let \( E \) be a Banach space. Let \( \{G_n\} \) be a sequence of non-trivial subspaces of \( E \) and \( \{v_n\} \) be a sequence of bounded linear projections such that \( v_n(E) = G_n \), \( n \in \mathbb{N} \). If there exist Banach spaces \( A \) and \( A_1 \) associated with \( E \) and \( E^* \) respectively and operators \( S: A \to E \) and \( T: A_1 \to E^* \) such that \( (\{G_n, v_n\}, S) \) is a fusion Banach frame for \( E \) with respect to \( A \) and \( (\{v_n^*(E^*), v_n^*\}, T) \) is a fusion Banach frame for \( E^* \) with respect to \( A_1 \), then we call the system \( (\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T) \) a **fusion bi-Banach frame** for \( E \).

In view of Remark 3.2.1 in [8], we have

Every reflexive Banach space has a fusion bi-Banach frame.

Recall that if \( E \) is a Banach space and \( E_d \) is an associated Banach space of scalar-valued sequences, indexed by \( \mathbb{N} \), \( \{x_n\} \) is a sequence in \( E \) and \( \{f_n\} \) is a sequence in \( E^* \), then the pair \( (\{f_n\}, \{x_n\}) \) is called an **atomic decomposition** for \( E \) with respect to \( E_d \) if

(i) \( \{f_n(x)\} \in E_d, x \in E \);

(ii) there exist constants \( A, B \) with \( 0 < A \leq B < \infty \) such that
\[
A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E;
\]

(iii) \( x = \sum_{n=1}^{\infty} f_n(x)x_n, x \in E. \)

The next result is regarding the existence of fusion bi-Banach frames for a Banach space having an atomic decomposition.

**Theorem 2.4.** Let \( E \) be a Banach space. If \( E \) has an atomic decomposition, then it also has a fusion bi-Banach frame.
Theorem 2.5. Let \( \{f_n, x_n\} \) be an atomic decomposition for \( E \) with respect to \( E_d \).

Define \( G_n = [x_n], n \in \mathbb{N} \) and \( v_n(x) = f_n(x)x_n, n \in \mathbb{N} \). Then there exist an associated Banach space \( A = \{(v_n(x)) : x \in E\} \) together with an operator \( S : A \to E \) such that \( \{G_n, v_n, S\} \) is a fusion Banach frame for \( E \) with respect to \( A \). Further \( \bigcup_{n=1}^{\infty} G_n = E \) (as \([x_n] = E\)). So, \( v_n^*(f) = 0 \) for all \( n \in \mathbb{N} \) imply \( f = 0 \), where \( f \in E^* \). Thus, \( \{v_n^*\} \) is total over \( E^* \) and so by Lemma 1.2 there exist an associated Banach space \( A_1 \) and an operator \( T : A_1 \to E^* \) such that \( \{v_n^*(E^*), v_n^*\}, T \) is a fusion Banach frame for \( E^* \) with respect to \( A_1 \). Hence, \( \{G_n, v_n, S; \{v_n^*(E^*), v_n^*\}, T \} \) is a fusion bi-Banach frame for \( E \). \( \square \)

Next, we observe that if \( E \) be a Banach space and \( \{G_n\} \) be a sequence of non-trivial subspaces of \( E \) with associated sequence of projections \( \{v_n\} \) with \( v_n(E) = G_n, n \in \mathbb{N} \), then it is possible that there exist a Banach space \( A_1 \) associated with \( E^* \) together with a bounded linear operator \( T : A_1 \to E^* \) such that \( \{v_n^*(E^*), v_n^*\}, T \) is a fusion Banach frame for \( E^* \) with respect to \( A_1 \) and there may not exist any Banach space \( A \) associated with \( E \) together with an operator \( S : A \to E \) such that \( \{G_n, v_n, S; \{v_n^*(E^*), v_n^*\}, T \} \) is a fusion Banach frame for \( E \) with respect to \( A \). Indeed, let

\[
E = \ell^2(X) = \left\{ \{x_n\} : x_n \in X; \sum_{n=1}^{\infty} \|x_n\|_X^2 < \infty \right\},
\]

where \((X, \| \cdot \|)\) is a Banach space, equipped with the norm given by

\[
\|\{x_n\}\|_E = \left( \sum_{n=1}^{\infty} \|x_n\|_X^2 \right)^{1/2}.
\]

Define for \( n \in \mathbb{N} \), \( G_n = \{\delta^n_1 + \delta^n_{n+1} : x \in X\} \) and \( v_n(x) = \delta^n_{n+1} \) \( x = \{x_n\} \in E \), where \( \delta^n_1 = (0, 0, \ldots, 0, x_1, 0, \ldots) \), \( x \in X \).

Then \( \bigcup_{n=1}^{\infty} G_n = E \) and \( v_i v_j = 0 \) for all \( i \neq j \).

But, since for any \( 0 \neq x \in X \), \( \delta^n_1 = (x, 0, 0, \ldots) \in E \) is such that \( v_n(\delta^n_1) = 0 \), for all \( n \in \mathbb{N} \), there exist no associated Banach space \( A \) such that \( \{G_n, v_n, S\} \) is a fusion Banach frame for \( E \) with respect to \( A \). However, there exist a Banach space \( A_0 \) and an operator \( T : A_0 \to E^* \) such that \( \{v_n^*(E^*), v_n^*\}, T \) is a fusion Banach frame for \( E^* \) with respect to \( A_0 \).

In view of the above discussion, we prove the following result

Theorem 2.5. Let \( E \) be a Banach space and \( \{G_n\} \) be a sequence of subspaces of \( E \) with \( \bigcup_{n=1}^{\infty} G_n = E \). Let \( \{v_n\} \) be a sequence of projections on \( E \) satisfying \( v_n(E) = G_n, n \in \mathbb{N} \) and \( v_i v_j = 0 \) for all \( i \neq j \). Then there exist Banach spaces \( A \) and \( A_1 \) associated with \( E \) and \( E^* \), respectively, and operators \( S : A \to E \) and \( T : A_1 \to E^* \) such that \( \{G_n, v_n, S; \{v_n^*(E^*), v_n^*\}, T \} \) is a fusion bi-Banach frame.
for $E$ if every sequence $\{x_n\} \subset E$ such that $x_n \in G_n$ and $x_n \neq 0$, $n \in \mathbb{N}$ satisfies
$$\bigcap_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \ldots] = \{0\}.$$

**Proof.** Since $[\bigcup_{n=1}^{\infty} G_n] = E$, there exist an associated Banach space $A_1$ and a bounded linear operator $T: A_1 \to E^*$ such that $\{v_n^*(E^*), v_n^*\} = T$ is a fusion Banach frame for $E^*$ with respect to $A_1$. Let, if possible, there exist no Banach space $A$ associated with $E$ such that $\{(G_n, v_n), S\}$ is a fusion Banach frame for $E$ with respect to $A$ where $S: A \to E$ is a bounded linear operator. Now, since $[\bigcup_{n=1}^{\infty} G_n] = E$ and $v_iv_j = 0$ for all $i \neq j$, $u_n = \sum_{i=1}^{n} v_i$ is a bounded linear projection of $E$ onto $[\bigcup_{i=1}^{n} G_i]$ along $[\bigcup_{i=n+1}^{\infty} G_i]$, $n \in \mathbb{N}$. Write $E = [\bigcup_{i=1}^{n} G_i] \oplus [\bigcup_{i=n+1}^{\infty} G_i]$, $n \in \mathbb{N}$. Then
$$\{x \in E : v_i(x) = 0, i = 1, 2, \ldots, n\} = [\bigcup_{i=n+1}^{\infty} G_i], \quad n \in \mathbb{N}.$$

Since $\{(G_n, v_n), S\}$ is not a fusion Banach frame for $E$ with respect to any associated Banach space, there exists $0 \neq x \in \bigcap_{n=1}^{\infty} [\bigcup_{i=n+1}^{\infty} G_i]$. So, there exists

$$y_1 = \sum_{i=1}^{m_1} z_i$$

where $z_i \in G_i$ $(1 \leq i \leq m_1)$ such that $\text{dist}(x, y_1) < 1$, that is, $\text{dist}(x, [\bigcup_{i=1}^{m_1} G_i]) < 1$. Also, $x \in [\bigcup_{i=m_1+1}^{\infty} G_i]$. So, we can choose $m_2 > m_1$ and

$$y_2 = \sum_{i=m_1+1}^{m_2} z_i$$

where $z_i \in G_i$ $(m_1 + 1 \leq i \leq m_2)$ such that $\text{dist}(x, [\bigcup_{i=m_1+1}^{m_2} G_i]) < \frac{1}{2}$. Proceeding like this, for each $n \in \mathbb{N}$, we get a sequence $\{z_n\} \subset E$ and an increasing sequence $\{m_n\}$ of positive integers such that $z_n \in G_n$, $n \in \mathbb{N}$ and

$$\text{dist}(x, [\bigcup_{i=m_n+1}^{m_n} G_i]) < \frac{1}{n}.$$ 

Thus $x \in [z_{n+1}, z_{n+2}, \ldots], n \in \mathbb{N}$. Consider a sequence $\{x_n\} \subset E$ with $0 \neq x_n \in G_n$, $n \in \mathbb{N}$ such that $x_n = z_n$ whenever $z_n \neq 0$. Then $x \in [x_{n+1}, x_{n+2}, \ldots], n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \ldots] \neq \{0\}$. \hfill $\square$

Finally, we give a characterization of fusion bi-Banach frames in terms of a sequence in $bv_0$, where $bv_0$ is the linear space of all sequences $\{\alpha_n\}$ of scalars with

$$\lim_{n \to \infty} \alpha_n = 0$$

and for which the norm $\|\{\alpha_n\}\| = \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$ is finite.

**Theorem 2.6.** Let $E$ be a Banach space and $\{(G_n, v_n), S\}$ be a fusion Banach frame for $E$, where the projections $\{v_n\}$ on $E$ are such that $v_i v_j = 0$ for all $i \neq j$. Then $\{(G_n, v_n), S; \{v_n^*(E^*), v_n^*\}, T\}$ is a fusion bi-Banach frame for $E$ if and only
if for every $x \in E$, there exist $\{\alpha_j\} \in bv_0$ and $z \in E$ such that $v_n(x) = \alpha_n v_n(z)$, $n \in \mathbb{N}$ and $\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} v_i(z) \right\| < \infty$.

**Proof.** Let $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ be a fusion bi-Banach frame for $E$. For each $k \in \mathbb{N}$, write $u_k = \sum_{i=1}^{k} v_i$. Then $\lim_{k \to \infty} u_k(x) = x$, $x \in E$. Therefore, there exists a sequence $\{m_n\}$ of positive integers such that

$$\|x - u_k(x)\| < \frac{1}{4n+1}, \quad k \geq m_n, \quad n \in \mathbb{N}.$$ 

Take $y_n = \sum_{i=m_{n-1}+1}^{m_n} v_i(x)$, $n \in \mathbb{N}$. Then $\|y_n\| \leq \frac{2}{4^n}$, $n \in \mathbb{N}$.

So, $\sum_{n=1}^{\infty} 2^{n-1}\|y_n\| \leq \sum_{n=1}^{\infty} 2^{-n}$. Thus, the series $\sum_{n=1}^{\infty} 2^{n-1}y_n$ converges.

Put $z = \sum_{n=1}^{\infty} 2^{n-1}y_n$ and $\alpha_j = 2^{1-n}, m_{n-1} + 1 \leq j \leq m_n, n \in \mathbb{N}$.

Therefore, $\{\alpha_j\} \in bv_0$. Also, we have

$$v_j(z) = 2^{n-1}v_j(x), \quad m_{n-1} + 1 \leq j \leq m_n, \quad n \in \mathbb{N}.$$ 

Hence, $v_j(x) = \alpha_j v_j(z)$, $j \in \mathbb{N}$.

Conversely, for integers $p < q$, we have

$$\left\| \sum_{i=p}^{q} v_i(x) \right\| = \left\| \sum_{i=p}^{q} \alpha_i \left( \sum_{j=1}^{i} v_j(z) - \sum_{j=1}^{i-1} v_j(z) \right) \right\|$$

$$\leq \left( |\alpha_p| + \sum_{i=p+1}^{q-1} |\alpha_i - \alpha_{i+1}| + |\alpha_q| \right) \sup_{1 \leq n < \infty} \left\| \sum_{j=1}^{n} v_j(z) \right\|$$

Since, $\{\alpha_j\} \in bv_0$, $\{ \sum_{i=1}^{n} v_i(x) \}$ is a Cauchy sequence and hence converges.

Also, since $\{v_n\}$ is total on $E$ and

$$v_j \left( x - \lim_{n \to \infty} \sum_{i=1}^{n} v_i(x) \right) = 0, \quad \text{for all} \quad j \in \mathbb{N},$$

it follows that $x = \lim_{n \to \infty} \sum_{i=1}^{n} v_i(x)$. Therefore, $\left( \bigcup_{n=1}^{\infty} G_n \right) = E$. Thus, $\{v_n^*\}$ is total over $E^*$ and so by Lemma 1.2 there exist a Banach space $A_2$ associated with $E^*$ and an operator $T: A_2 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $A_2$. Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for $E$. \[ \square \]

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