ESTIMATIONS OF NONCONTINUABLE SOLUTIONS
OF SECOND ORDER DIFFERENTIAL EQUATIONS
WITH \( p \)-LAPLACIAN

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ABSTRACT. We study asymptotic properties of solutions for a system of second
differential equations with \( p \)-Laplacian. The main purpose is to investigate
lower estimates of singular solutions of second order differential equations
with \( p \)-Laplacian
\[
(A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t),
\]
Furthermore, we obtain results for a scalar equation.

1. INTRODUCTION

Consider the differential equation
\[
(1) \quad (A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t),
\]
where \( p > 0 \), \( A(t) \), \( B(t) \), \( R(t) \) are continuous, matrix-valued function on
\( \mathbb{R}_+ := [0, \infty) \), \( A(t) \) is regular for all \( t \in \mathbb{R}_+ \), \( e: \mathbb{R}_+ \to \mathbb{R}^n \) and \( f, g: \mathbb{R}^n \to \mathbb{R}^n \) are continuous mappings and \( \Phi_p(u) = (|u_1|^{p-1}u_1, \ldots, |u_n|^{p-1}u_n) \) for \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n \). We shall use the norm \( \|u\| = \max_{1 \leq i \leq n} |u_i| \) where \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n \).

Definition 1. A solution \( y \) of (1) defined on \( t \in [0, T) \) is called noncontinuable
or nonextendable if \( T < \infty \) and \( \limsup_{t \to T^-} \|y'(t)\| = \infty \). The solution \( y \) is called
continuable if \( T = \infty \).

Note, that noncontinuable solutions are also called singular of the second kind,
see e.g. [3], [8], [13].

Definition 2. A noncontinuable solution \( y: [0, T] \to \mathbb{R}^n \) is called oscillatory if
there exists an increasing sequence \( \{t_k\}_{k=1}^\infty \) of zeros of \( y \) such that \( \lim_{k \to \infty} t_k = T \);
otherwise \( y \) is called nonoscillatory.

In the last two decades the existence and properties of noncontinuable solutions
of special types of (1) are investigated. For the scalar case, see e.g. [3], [4], [5].
and references therein. In particular, noncontinuable solutions do not exist if $f$ and $g$ satisfy the following conditions
\begin{equation}
|g(x)| \leq |x|^p \quad \text{and} \quad |f(x)| \leq |x|^p \quad \text{for } |x| \text{ large}
\end{equation}
and $R$ is positive. Hence, noncontinuable solutions may exist mainly in the case $|f(x)| \geq |x|^m$ with $m > p$.

As concern the system (1), see papers [7], [14], where sufficient conditions are given for (1) to have continuable solutions.

The scalar equation (1) can be applied in problems of radially symmetric solutions of the $p$-Laplace differential equation, see e.g. [14]; noncontinuable solutions appear e.g. in water flow problems (flood waves, a flow in sewerage systems), see e.g. [10].

The present paper deals with the estimations from below of norms of a noncontinuable solution of (1) and its derivative. Estimations of solutions are important e.g. in proofs of the existence of such solutions, see e.g. [4], [8] for
\begin{equation}
y^{(n)} = f(t, y, \ldots, y^{(n-1)})
\end{equation}
with $n \geq 2$ and $f \in C^0(\mathbb{R}_+, \mathbb{R}^n)$. For generalized Emden-Fowler equation of the form (3), some estimation are proved in [1].

In the paper [14] the differential equation (1) is studied with the initial conditions
\begin{equation}
y(0) = y_0, \quad y'(0) = y_1
\end{equation}
where $y_0, y_1 \in \mathbb{R}^n$.

We will use results from [7, Theorem 1.2].

**Theorem A.** Let $m > p$ and there exist positive constants $K_1$, $K_2$ such that
\begin{equation}
\|g(u)\| \leq K_1\|u\|^m, \quad \|f(v)\| \leq K_2\|v\|^m, \quad u, v \in \mathbb{R}^n.
\end{equation}
and $\int_0^\infty \|R(s)\| s^m ds < \infty$. Denote
\begin{align*}
A_\infty & := \sup_{0 \leq t < \infty} \|A(t)^{-1}\| < \infty, \quad E_\infty := \sup_{0 \leq t < \infty} \int_0^t \|e(s)\| ds < \infty, \\
R_\infty & := \int_0^\infty \|R(s)\| ds, \quad B_\infty := \int_0^\infty \|B(t)\| dt.
\end{align*}

Let the following conditions be satisfied:
\begin{enumerate}
\item[(i)] Let $m > 1$ and \( \frac{m-p}{p} A_\infty D_1^{\frac{m-p}{p}} \int_0^\infty (K_1\|B(s)\| + 2^{m-1}K_2 s^m\|R(s)\|) ds < 1 \)
for all $t \in \mathbb{R}_+$, where
\( D_1 = A_\infty \\{\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_\infty + E_\infty \} \).
\item[(ii)] Let $m \leq 1$ and \( \frac{2m+1-m-p}{p} A_\infty D_2^{\frac{m-p}{p}} \int_0^\infty (K_1\|B(s)\| + K_2 s^m\|R(s)\|) ds < 1 \)
for all $t \in \mathbb{R}_+$, where
\( D_2 = A_\infty \\{\|A(0)\Phi_p(y_1)\| + 2^m K_1\|y_1\|^m B_\infty + 2^{2m+1}K_2 R_\infty\|y_0\|^m + E_\infty \} \).
\end{enumerate}
Then any solution $y(t)$ of the initial value problem \((1)\), \((4)\) is continuable.

**Proof.** First let us prove the assertion (i). We will use \([7, \text{Theorem 1.2}]\). From \((5)\) and its proof, it follows that equation (2.3) in \([7]\) may have form

\[
\| \Phi_p(u(t)) \| \leq \| A(t)^{-1} \| \left\{ \| A(0) \Phi_p(y_1) \| + K_1 \int_0^t \| B(s) \| \| u(s) \|^m \, ds \right\} + K_2 \int_0^t \| R(s) \| \| y_0 \| + \int_0^s u(\tau) d\tau \|^m \, ds
\]

where

\[
c = A_\infty \{ \| A(0) \Phi_p(y_1) \| + 2^{m-1} K_2 \| y_0 \| R_\infty \}
\]

and

\[
F(t) = 2^{m-1} K_2 A_\infty \int_t^\infty \| R(s) \| s^{m-1} \, ds + K_1 A_\infty \| B(t) \|.
\]

Now, the results follows from \([7, \text{Theorem 1.2}]\).

The assertion (ii) follows from \([7, \text{Theorem 1.2}]\). □

2. **Main results**

In this chapter we will derive estimates for a noncontinuable solution $y$ on the fixed definition interval $[T, \tau) \subset \mathbb{R}_+$, $\tau < \infty$.

**Theorem 1.** Let $y$ be a noncontinuable solution of the system \((1)\) on the interval $[T, \tau) \subset \mathbb{R}_+$, $\tau - T \leq 1$,

\[
A_0 := \max_{T \leq t \leq \tau} \| A(t)^{-1} \|, \quad B_0 := \max_{T \leq t \leq \tau} \| B(t) \|, \quad E_0 := \max_{T \leq t \leq \tau} \| e(t) \|, \quad R_0 := \max_{T \leq t \leq \tau} \| R(t) \|, \quad \int_0^\infty \| R(s) \| s^m \, ds < \infty
\]

and let there exist positive constants $K_1, K_2$ and $m > p$ such that

\[
\| g(u) \| \leq K_1 \| u \|^m, \quad \| f(v) \| \leq K_2 \| v \|^m, \quad u, v \in \mathbb{R}^n.
\]

Then the following assertions hold:

(i) If $p > 1$ and $M = \frac{2^{m+1}(2m+3)}{(m+1)(m+2)}$, then

\[
\| A(t) \Phi_p(y(t)) \| + 2^{m-1} K_2 \| y(t) \|^m R_0 + 2E_0(\tau - t) \geq C_1 (\tau - t)^{-\frac{m}{p-1}}
\]

for $t \in [T, \tau)$, where

\[
C_1 = A_0^{-\frac{m}{m-p}} \left( \frac{m-p}{p} \right)^{-\frac{m}{m-p}} \left[ \frac{3}{2} K_1 B_0 + MK_2 R_0 \right]^{-\frac{m}{m-p}}.
\]

(ii) If $p \leq 1$, then

\[
\| A(t) \Phi_p(y(t)) \| + 2^m K_1 B_0 \| y(t) \|^m + 2^{2m+1} K_2 R_0 \| y(t) \|^m
\]

\[
+ 2E_0(\tau - t) \geq C_2 (\tau - t)^{-\frac{m}{p-1}}
\]

for $t \in [T, \tau)$. The constants $C_1$ and $C_2$ are determined by $p$, $m$, $K_1$, $K_2$, and the initial data $y_0$. □
for \( t \in [T, \tau) \) where
\[
C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \left( \frac{m-p}{p} \right)^{-\frac{n}{m-p}} \left[ \frac{3}{2} K_1 B_0 + MK_2 R_0 \right]^{-\frac{n}{m-p}}.
\]

**Proof.** First let us prove the assertion (i). Let \( y \) be a singular solution of system (1) on the interval \([T, \tau)\). We take \( t \) to be fixed in the interval \([T, \tau)\) and for the simplicity denote
\[
D = A_0^{-\frac{m}{m-p}} \left( \frac{m-p}{p} \right)^{-\frac{n}{m-p}}.
\]
Assume, by contradiction, that
\[
\|A(t)\Phi_p(y(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0 + 2E_0(\tau - t) < D \left[ \frac{3}{2} K_1 B_0 + MK_2 R_0 \right]^{-\frac{n}{m-p}} (\tau - t)^{-\frac{n}{m-p}}.
\]
Together with the Cauchy problem
\[
(A(x)\Phi_p(y'))' + B(x)g(y') + R(x)f(y) = e(x), \quad x \in [t, \tau)
\]
and
\[
y(t) = y_0, \quad y'(t) = y_1
\]
we construct an auxiliary system
\[
(\tilde{A}(s)\Phi_p(z'))' + \tilde{B}(s)g(z') + \tilde{R}(s)f(z) = \tilde{e}(s),
\]
where \( s \in \mathbb{R}_+, z_0, z_1 \in \mathbb{R}^n \), \( \tilde{A}(s), \tilde{B}(s), \tilde{R}(s) \) are continuous, matrix-valued function on \( \mathbb{R}_+ \) given by
\[
\tilde{A}(s) = \begin{cases} A(s + t) & \text{if } 0 \leq s < \tau - t, \\ A(\tau) & \text{if } \tau - t \leq s < \infty, \end{cases}
\]
\[
\tilde{B}(s) = \begin{cases} B(s + t) & \text{if } 0 \leq s < \tau - t, \\ -\frac{B(\tau - t)}{\tau - t} s + 2B(\tau - t) & \text{if } \tau - t \leq s < 2(\tau - t), \\ 0 & \text{if } 2(\tau - t) \leq s < \infty, \end{cases}
\]
\[
\tilde{R}(s) = \begin{cases} R(s + t) & \text{if } 0 \leq s < \tau - t, \\ -\frac{R(\tau - t)}{\tau - t} s + 2R(\tau - t) & \text{if } \tau - t \leq s < 2(\tau - t), \\ 0 & \text{if } 2(\tau - t) \leq s < \infty, \end{cases}
\]
\[
\tilde{e}(s) = \begin{cases} e(s) & \text{if } 0 \leq s < \tau - t, \\ -\frac{e(\tau - t)}{\tau - t} s + 2e(\tau - t) & \text{if } \tau - t \leq s < 2(\tau - t), \\ 0 & \text{if } 2(\tau - t) \leq s < \infty. \end{cases}
\]
We can see that \( \tilde{A}(s) \) is regular for all \( s \in \mathbb{R}_+ \).
Hence, the systems (12) on \([t, \tau]\) and (14) on \([0, \tau - t]\) are equivalent with the change of independent variable \(x - t \rightarrow s\). Let \(z_0 = y(t)\) and \(z_1 = y'(t)\). Then the definitions of the functions \(A, B, \tilde{R}, \tilde{e}\) give that

\[
(20) \quad z(s) = y(s + t), \quad s \in [0, \tau - t) \quad \text{is a noncontinuable solution}
\]
of the system (14), (15) on \([0, \tau - t)\). By the application of Theorem A (i) to the system (14), (15) we will see that every solution \(z\) of the system (14), (15) satisfying

\[
\|A(0)\Phi_p(z_1)\| + 2^{m-1}K_2\|z_0\|^mR_0 + \int_0^\infty \|\tilde{e}(s)\| \, ds
\]

\[
< D\left[\int_0^\infty (K_1\|\tilde{B}(w)\| + 2^{m-1}K_2\|\tilde{R}(w)\|w^m) \, dw\right]^{-\frac{p}{m-p}}
\]

is continuable. Note, that according to (16)-(21) all assumptions of Theorem A are valid. Furthermore, we will show that (11) yields (21).

We estimate the right-hand side of inequality (21):

\[
G := D\left[\int_0^\infty (K_1\|\tilde{B}(w)\| + 2^{m-1}K_2\|\tilde{R}(w)\|w^m) \, dw\right]^{-\frac{p}{m-p}}
\]

\[
\geq D\left[\int_0^{2(\tau-t)} (K_1\|\tilde{B}(w)\| + 2^{m-1}K_2\|\tilde{R}(w)\|w^m) \, dw\right]^{-\frac{p}{m-p}}
\]

\[
\geq D\left\{K_1 \max_{0 \leq s \leq \tau-t} \|B(s + t)\|(\tau - t) + K_2 \int_{\tau-t}^{2(\tau-t)} \left\| - \frac{B(\tau - t)}{\tau - t} w + 2B(\tau - t) \right\| dw
\]

\[
+ 2^{m-1}K_2 \max_{0 \leq s \leq (\tau-t)} \|R(s + t)\| \frac{(\tau - t)^{m+1}}{m + 1} \, dw
\]

\[
+ 2^{m-1}K_2 \int_{\tau-t}^{2(\tau-t)} \left\| - \frac{R(\tau - t)}{\tau - t} w + 2R(\tau - t) \right\| w^m \, dw\right\]^{-\frac{p}{m-p}},
\]

\[
G \geq D\left\{K_1 \max_{T \leq t \leq \tau} \|B(t)\|(\tau - t) + \frac{1}{2}K_1\|B(\tau - t)\|(\tau - t) + M_1K_2 \max_{T \leq t \leq \tau} \|R(t)\|(\tau - t)^{m+1} + M_2K_2\|R(\tau - t)\|(\tau - t)^{m+1}\right\]^{-\frac{p}{m-p}},
\]

where

\[
M_1 = \frac{2^{m-1}}{m + 1} \quad \text{and} \quad M_2 = 2^{m-1}\frac{2^{m+2}(2m + 3) - 3m - 5}{(m + 1)(m + 2)}.
\]

Hence,

\[
(22) \quad G > D\left[\frac{3}{2}K_1B_0(\tau - t) + MK_2R_0(\tau - t)^{m+1}\right]^{-\frac{p}{m-p}}
\]
as \(M > M_1 + M_2\).
As we assume that $\tau - t \leq 1$, inequalities (11) and (22) imply
\[
G > D \left[ \frac{3}{2} K_1 B_0 + MK_2 R_0 \right]^{-\frac{p}{m-p}} (\tau - t)^{-\frac{p}{m-p}} = C_1(\tau - t)^{-\frac{p}{m-p}}
\]
\[
\geq ||A(t)\Phi_p(y'(t))|| + 2^{m-1} K_2 ||y(t)||^m R_0 + 2 E_0 (\tau - t)
\]
(23)
\[
\geq ||\tilde{A}(0)\Phi_p(z_1)|| + 2^{m-1} K_2 ||z_0||^m R_0 + \int_0^\infty ||\tilde{e}(s)|| \, ds,
\]
where $C_1 = D \left[ \frac{3}{2} K_1 B_0 + MK_2 R_0 \right]^{-\frac{p}{m-p}}$. Hence (21) holds and the solution $z$ of (14) satisfying the initial condition $z(0) = y_0$ and $z'(0) = y_1$ is continuable. This contradiction with (20) proves the statement.

Now we shall prove the assertion (ii). If $p \leq 1$ then the proof is similar, we have to use only Theorem A (ii) instead of Theorem A (i).

Now consider the following special case of equation (1):
(24) \[
(A(t)\Phi_p(y'))' + R(t)f(y) = 0
\]
for all $t \in \mathbb{R}_+$. In this case a better estimation than before can be proved.

**Theorem 2.** Let $m > p$ and $y$ be a noncontinuable solution of system (24) on interval $[T, \tau) \subset \mathbb{R}_+$. Let there exists a constant $K_2 > 0$ such that
(25) \[
||f(v)|| \leq K_2 ||v||^m, \quad v \in \mathbb{R}^n.
\]
Let $R_0$ and $M$ to be given by Theorem 1. Then
(26) \[
||A(t)\Phi_p(y'(t))|| + 2^{m+2} K_2 ||y(t)||^m R_0 \geq C_1(\tau - t)^{-\frac{p(m+1)}{m-p}}
\]
where
\[
C_1 = A_0^{-\frac{m}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} [MK_2 R_0]^{-\frac{p}{m-p}} \quad \text{in case } p > 1
\]
and
\[
||A(t)\Phi_p(y')|| + 2^{2m+1} K_2 ||y(t)||^m R_0 \geq C_2(\tau - t)^{-\frac{p(m+1)}{m-p}}
\]
with
\[
C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \left(\frac{m-p}{p}\right)^{-\frac{p}{m-p}} [MK_2 R_0]^{-\frac{p}{m-p}} \quad \text{in case } p \leq 1.
\]
**Proof.** Proof is similar the one of the Theorem 1 for $B(t) \equiv 0$ and $e(t) \equiv 0$. Let $p > 1$. We do not use assumption $\tau - t \leq 1$ and we are able to improve an exponent of the estimation (8). The inequality (23) has changed to
(27) \[
G \geq C_1(\tau - t)^{-\frac{p(m+1)}{m-p}}
\]
\[
\geq ||A(t)\Phi_p(y'(t))|| + 2^{m-1} K_2 ||y(t)||^m R_0
\]
\[
\geq ||\tilde{A}(0)\Phi_p(z'(0))|| + 2^{m-1} K_2 ||z(0)||^m R_0,
\]
where $C_1 = D[MK_2 R_0]^{-\frac{p}{(m-p)}}$. If $p \leq 1$, the proof is similar.
3. Applications

In this case we study the scalar differential equation
\[(a(t)\Phi_p(y'))' + r(t)f(y) = 0,\]
where \(p > 0, a(t), r(t)\) are continuous functions on \(\mathbb{R}_+\), \(a(t) > 0\) for \(t \in \mathbb{R}_+,\)
\(f: \mathbb{R} \to \mathbb{R}\) is a continuous mapping and \(\Phi_p(u) = |u|^{p-1}u.\)

Corollary 3. Let \(y\) be a noncontinuable oscillatory solution of equation (28) defined on \([T, \tau]\). Let there exist constants \(K_2 > 0\) and \(m > 0\) such that
\[|f(v)| \leq K_2|v|^m, \quad v \in \mathbb{R}\]
and let \(\{t_k\}_1^\infty\) and \(\{\tau_k\}_1^\infty\) be increasing sequences of all local extrema of the solution \(y\) and of \(y^{[1]} = a(t)\Phi_p(y')\) on \([T, \tau]\), respectively. Then there exist constants \(C_1\) and \(C_2\) such that
\[|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}},\]
and, in the case \(r \neq 0\) on \(\mathbb{R}_+\),
\[|y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}}\]
for \(k \geq 1, 2, \ldots\).

Proof. Let \(m > p\) and \(y\) be an oscillatory noncontinuable solution of equation (28) defined on \([T, \tau]\). An application of Theorem 2 to (28) gives
\[|y^{[1]}(t)| + 2^{2m+1}K_2|y(t)|^m r_0 \geq C(\tau - t)^{-\frac{p(m+1)}{m-p}},\]
where \(C\) is a suitable constant and \(r_0 = \max_{T \leq t \leq \tau} |r(t)|.\) Note that according to (30), \(x(x^{[1]})\) has a local extremum at \(t_0 \in (T, \tau)\) if and only if \(x^{[1]}(t_0) = 0\) \((x(t_0) = 0).\) From this it follows that an accumulation point of zeros of \(x(x^{[1]})\) does not exist in \([T, \tau].\) Otherwise, it holds \(y(\tau) = 0\) and \(y'(\tau) = 0.\) That is in contradiction with (32). If \(\{t_k\}_1^\infty\) is the sequence of all extrema of a solution \(y,\) then \(y'(t_k) = 0,\) i.e. \(y^{[1]}(t_k) = 0.\) We obtain the following estimate for \(y(t_k)\) from (32)
\[|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}},\]
where \(C_1 = C \frac{1}{m} (2^{2m+1}K_2 r_0)^{-\frac{1}{m}}\) and (30) is valid. If \(\{\tau_k\}_1^\infty\) is the sequence of all extrema of \(y^{[1]}(\tau_k),\) then \(y(\tau_k) = 0.\) We obtain the following estimate for \(y^{[1]}(\tau_k)\)
\[|y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}},\]
where \(C_2 = C.\) \(\square\)

Example 1. Consider (28) and (29) with \(m = 2, p = 1.\) Then from Corollary 3 we obtain the following estimates
\[|y(t_k)| \geq C_1(\tau - t_k)^{-\frac{3}{2}}, \quad |y^{[1]}(\tau_k)| \geq C_2(\tau - \tau_k)^{-3},\]
where \(M = \frac{\sqrt{72}}{3}, C_1 = \frac{\sqrt{72}}{448K_2 a_0 r_0}\) and \(C_2 = \frac{3}{448K_2 a_0^2 r_0}.\)
Example 2. Consider (28) and (29) with $m = 3, p = 2$. Then from Corollary 3 we obtain the following estimates

$$|y(t_k)| \geq C_1 (\tau - t_k)^{- \frac{8}{3}}, \quad |y^{[1]}(\tau_k)| \geq C_2 (\tau - \tau_k)^{- 8},$$

where $M = \frac{288}{5}, C_1 = \frac{1}{32K_2r_0} (\frac{10a_0}{9})^\frac{2}{3}$ and $C_2 = (\frac{5a_0}{144K_2r_0})^2$.

The following lemma is a special case of [13 Lemma 11.2].

Lemma 1. Let $y \in C^2[a, b], \; \delta \in (0, \frac{1}{2})$ and $y'(t)y(t) > 0, \; y''(t)y(t) \geq 0$ on $[a, b)$. Then

$$\left( y'(t)y(t) \right)^{\frac{1}{1 - 2\delta}} \geq \omega \int_t^b |y''(s)|^{\delta} |y(s)|^{3\delta - 2} \; ds, \quad t \in [a, b),$$

where $\omega = [(1 - 2\delta) \delta^\delta (1 - \delta)^{1 - \delta}]^{-1}$.

Now, let us turn our attention to nonoscillatory solutions of (28).

Theorem 4. Let $m > p$ and $M \geq 0$ be such that

$$|f(x)| \leq |x|^m \quad \text{for} \quad |x| \geq M.$$ 

If $y$ is a nonoscillatory noncontinuable solution of (28) defined on $[T, \tau)$, then constants $C, C_0$ and a left neighborhood $J$ of $\tau$ exist such that

$$|y'(t)| \geq C (\tau - t)^{- \frac{p(m+1)}{m(m-p)}}, \quad t \in J.$$

Let, moreover, $m < p + \sqrt{p^2 + p}$. Then

$$|y(t)| \geq C_0 (\tau - t)^{m_1} \quad \text{with} \quad m_1 = \frac{m^2 - 2mp - p}{m(m-p)} < 0.$$

Proof. Let $y$ be a nonoscillatory noncontinuable solutions of (28) defined on $[T, \tau)$. Then there exists $t_0 \in [T, \tau)$ such that $y(t)y^{[1]}(t) > 0$ for $t \in [t_0, \tau)$. Let

$$y(t) > 0 \quad \text{and} \quad y'(t) > 0 \quad \text{for} \quad t \in J := [t_0, \tau);$$

the opposite case $y(t) < 0$ and $y'(t) < 0$ can be studied similarly. As $y$ is noncontinuable, $\lim_{t \to \tau^-} y'(t) = \infty$. Moreover, $\lim_{t \to \infty} y(t) = \infty$ as, otherwise, $y^{[1]}$ and $y$ are bounded on the finite interval $J$. Hence, there exists $t_1 \in J$ such that $y'(t) \geq 1$ for $[t_1, \tau), y(t) \geq M$ for $t \geq t_1$ and

$$y(t) = y(t_0) + \int_{t_0}^t y'(s) \; ds \leq y(t_0) + \tau y'(t) \leq 2\tau y'(t), \quad t \in [t_1, \tau).$$

Note, that due to $y \geq M$ it is sufficient to suppose (36) instead of (25) for an application of Theorem 2. Hence, Theorem 2 applied to (28), (39) and $y' \geq 1$ imply

$$C_1 (\tau - t)^{- \frac{p(m+1)}{m(m-p)}} \leq a(t)(y'(t))^p + C_2 y^m(t) \leq a(t)(y'(t))^p + C_2 (2\tau)^m (y'(t))^m \leq C_3 (y'(t))^m$$

for $t \geq t_1$. Therefore, $y(t) \geq M$ for $t \geq t_1$ and $|y(t)| \geq C_0 (\tau - t)^{m_1}$ for $t \in J$.
or
\[ y'(t) \geq C_4(\tau - t)^{-\frac{p(m+1)}{m(m-p)}} \quad \text{on} \quad [t_1, \tau), \]
where \( C_1, C_2, C_3 \) and \( C_4 \) are positive constants which do not depend on \( y \). Moreover, the integration of (37) yields
\[ y(t) = y(t_0) + \int_{t_0}^{t} y'(s) \, ds \geq C \int_{t_0}^{t} (\tau - s)^{-\frac{p(m+1)}{m(m-p)}} \, ds \]
\[ \geq \frac{C}{|m_1|} [(\tau - t)^{m_1} - (\tau - t_0)^{m_1}] \geq \frac{C}{2|m_1|} (\tau - t)^{m_1} \]
for \( t \) lying in a left neighbourhood \( I_1 \) of \( \tau \). Hence, (37) and (38) are valid. \( \square \)

Our last application is devoted to the equation
\[ y'' = r(t)|y|^m \, \text{sgn} \, y, \tag{40} \]
where \( r \in C^0(\mathbb{R}_+) \), \( m > 1 \).

**Theorem 5.** Let \( \tau \in (0, \infty), T \in [0, \tau) \) and \( r(t) > 0 \) on \([t, \tau]\).

(i) Then (40) has a nonoscillatory noncontinuable solution which is defined in a left neighbourhood of \( \tau \).

(ii) Let \( y \) be a nonoscillatory noncontinuable solution of (40) defined on \([T, \tau]\). Then constants \( C, C_1, C_2 \) and a left neighbourhood \( I \) of \( \tau \) exist such that
\[ |y(t)| \leq C(\tau - t)^{-\frac{2(m+3)}{m+1}} \quad \text{and} \quad |y'(t)| \geq C_1(\tau - t)^{-\frac{m+1}{m(m-1)}}, \quad t \in I. \]
If, moreover, \( m < 1 + \sqrt{2} \), then
\[ |y(t)| \leq C_2(\tau - t)^{-m_1} \quad \text{with} \quad m_1 = \frac{m^2 - 2m - 1}{m(m-1)} < 0. \]

**Proof.** The assertion (i) follows from [2] Theorem 2.

Let us prove the assertion (ii). Let \( y \) be a noncontinuable solution of (40) defined on \([T, \tau]\). According to Theorem 4 and its proof we have \( \lim_{t \to \tau^-} |y(t)| = \infty \) and (37) holds. Hence, suppose that \( t_0 \in [T, \tau) \) is such that
\[ y(t) \geq 1 \quad \text{and} \quad y'(t) > 0 \quad \text{on} \quad [t_0, \tau). \]

Furthermore, there exists \( t_1 \in [t_0, \tau) \) such that
\[ y(t) = y(t_0) + \int_{t_0}^{t} y'(s) \, ds \leq y(t_0) + y'(t)(\tau - t_0) \leq C_3 y'(t) \tag{41} \]
for \( t \in [t_1, \tau) \) with \( C_3 = 2(\tau - t_0) \). Now, we estimate \( y \) from below. By applying Lemma 1 with \([a, b] = [t_1, \tau)\) and \( \delta = \frac{2}{m+3} \in (0, \frac{1}{2}) \). We have \( \delta m + 3\delta - 2 = 0 \) and
\[ C_3 y^{-\frac{2(m+3)}{m+1}} m \geq (y'(t)y(t))^{-\frac{1}{1-2\delta}} \geq \omega \int_{t}^{\tau} (y''(s))^{\delta}(y(s))^{3\delta-2} \, ds \]
\[ \geq C_4 \int_{t}^{\tau} y^{\delta m+3\delta-2} \, ds = C_4(\tau - t) \quad \text{on} \quad [t_1, \tau), \tag{42} \]
where $C_4 = \omega \min_{t_0 \leq \sigma \leq \tau} |r(\sigma)|$. From this we have

$$y(t) \leq C(\tau - t)^{-\frac{m-1}{2(m+3)}} \text{ on } [t_1, \tau]$$

with a suitable positive $C$. The rest of the statement follows from Theorem 4. □

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**References**


