GENERAL IMPLICIT VARIATIONAL INCLUSION PROBLEMS INVOLVING A-MAXIMAL RELAXED ACCRETEIVE MAPPINGS IN BANACH SPACES

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Abstract. A class of existence theorems in the context of solving a general class of nonlinear implicit inclusion problems are examined based on A-maximal relaxed accretive mappings in a real Banach space setting.

1. Introduction

We consider a real Banach space $X$ with $X^*$, its dual space. Let $\|\cdot\|$ denote the norm on $X$ and $X^*$, and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $X$ and $X^*$. We consider the implicit inclusion problem: determine a solution $u \in X$ such that

$$0 \in A(u) + M(g(u)),$$

where $A, g : X \to X$ are single-valued mappings, and $M : X \to 2^X$ is a set-valued mapping on $X$ such that $\text{range}(g) \cap \text{dom}(M) \neq \emptyset$.

Recently, Huang, Fang and Cho [4] applied a three-step algorithmic process to approximating the solution of a class of implicit variational inclusion problems of the form (1) in a Hilbert space. In their investigation, they used the resolvent operator of the form $J^M_{\rho} = (I + \rho M)^{-1}$ for $\rho > 0$, in a Hilbert space setting. Here we generalize the existence results to the case of A-maximal relaxed accretive mappings in a real uniformly smooth Banach space setting. As matter of fact, the obtained results generalize their investigation to the case of $H$-maximal accretive mappings as well. For more literature, we refer the reader to [2]–[20].

2. A-maximal relaxed accretiveness

In this section we discuss some basic properties and auxiliary results on A-maximal relaxed accretiveness. Let $X$ be a real Banach space and $X^*$ be the dual space of $X$. Let $\|\cdot\|$ denote the norm on $X$ and $X^*$ and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $X$ and $X^*$. Let $M : X \to 2^X$ be a multivalued mapping on $X$. We shall denote both the map $M$ and its graph by $M$, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset $M$ of $X \times X$, and

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we define the modulus of smoothness $q > 1$.

The inverse $M^{-1}$ of $M$ is defined (as its projection onto the first argument) by

$$D(M) = \{ x \in X : \exists y \in X : (x, y) \in M \} = \{ x \in X : M(x) \neq \emptyset \}.$$  

$D(M) = X$, shall denote the full domain of $M$, and the range of $M$ is defined by

$$R(M) = \{ y \in X : \exists x \in X : (x, y) \in M \}.$$  

The inverse $M^{-1}$ of $M$ is $\{(y, x) : (x, y) \in M \}$. For a real number $\rho$ and a mapping $M$, let $\rho M = \{x, \rho y) : (x, y) \in M \}$. If $L$ and $M$ are any mappings, we define $L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M \}$.

As we prepare for basic notions, we start with the generalized duality mapping $J_q : X \to 2^X^*$, that is defined by

$$J_q(x) = \{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \} \quad \forall x \in X ,$$

where $q > 1$. As a special case, $J_2$ is the normalized duality mapping, and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. Next, as we are heading to uniformly smooth Banach spaces, we define the modulus of smoothness $\rho_X : [0, \infty) \to [0, \infty)$ by

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$  

A Banach space $X$ is uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0 ,$$

and $X$ is $q-$uniformly smooth if there is a positive constant $c$ such that

$$\rho_X(t) \leq ct^q , \quad q > 1 .$$

Note that $J_q$ is single-valued if $X$ is uniformly smooth. In this context, we state the following Lemma from Xu [17].

**Lemma 2.1** ([17]). Let $X$ be a uniformly smooth Banach space. Then $X$ is $q$-uniformly smooth if there exists a positive constant $c_q$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q\|y\|^q .$$

**Lemma 2.2.** For any two nonnegative real numbers $a$ and $b$, we have

$$(a + b)^q \leq 2^q(a^q + b^q) .$$

**Definition 2.1.** Let $M : X \to 2^X$ be a multivalued mapping on $X$. The map $M$ is said to be:

(i) $(r)$—strongly accretive if there exists a positive constant $r$ such that

$$\langle u^* - v^*, J_q(u - v) \rangle \geq r\|u - v\|^q \quad \forall (u, u^*), (v, v^*) \in \text{graph}(M) .$$

(ii) $(m)$—relaxed accretive if there exists a positive constant $m$ such that

$$\langle u^* - v^*, J_q(u - v) \rangle \geq (-m)\|u - v\|^q \quad \forall (u, u^*), (v, v^*) \in \text{graph}(M) .$$
Definition 2.2 ([5]). Let $A: X \to X$ be a single-valued mapping. The map $M: X \to 2^X$ is said to be $A$-maximal $(m)$-relaxed accretive if:

(i) $M$ is $(m)$-relaxed accretive for $m > 0$.

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.3 ([5]). Let $A: X \to X$ be an $(r)$-strongly accretive mapping and let $M: X \to 2^X$ be an $A$-maximal accretive mapping. Then the generalized resolvent operator $J_{\rho,A}^M: X \to X$ is defined by

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u).$$

Definition 2.4 ([2]). Let $H: X \to X$ be an $(r)$-strongly accretive mapping. The map $M: X \to 2^X$ is said to be $H$-maximal accretive if

(i) $M$ is accretive,

(ii) $R(H + \rho M) = X$ for $\rho > 0$.

Definition 2.5. Let $H: X \to X$ be an $(r)$-strongly accretive mapping and let $M: X \to 2^X$ be an $H$-maximal accretive mapping. Then the generalized resolvent operator $J_{\rho,H}^M: X \to X$ is defined by

$$J_{\rho,H}^M(u) = (H + \rho M)^{-1}(u).$$

Proposition 2.1 ([5]). Let $A: X \to X$ be an $(r)$-strongly accretive single-valued mapping and let $M: X \to 2^X$ be an $A$-maximal $(m)$-relaxed accretive mapping. Then $(A + \rho M)$ is maximal accretive for $\rho > 0$.

Proposition 2.2 ([5]). Let $A: X \to X$ be an $(r)$-strongly accretive mapping and let $M: X \to 2^X$ be an $A$-maximal relaxed accretive mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Proposition 2.3 ([2]). Let $H: X \to X$ be an $(r)$-strongly accretive single-valued mapping and let $M: X \to 2^X$ be an $H$-maximal accretive mapping. Then $(H + \rho M)$ is maximal accretive for $\rho > 0$.

Proposition 2.4 ([2]). Let $H: X \to X$ be an $(r)$-strongly accretive mapping and let $M: X \to 2^X$ be an $H$-maximal accretive mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued.

3. Existence theorems

This section deals with the existence theorems on solving the implicit inclusion problem (1) based on the $A$-maximal relaxed accretiveness.

Lemma 3.1 ([5]). Let $X$ be a real Banach space, let $A: X \to X$ be $(r)$-strongly accretive, and let $M: X \to 2^X$ be $A$-maximal relaxed accretive. Then the generalized resolvent operator associated with $M$ and defined by

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u) \forall u \in X,$$

is $(\frac{1}{r-\rho m})$-Lipschitz continuous for $r - \rho m > 0$. 
Lemma 3.2. Let $X$ be a real Banach space, let $A: X \to X$ be $(r)$-strongly accretive, and let $M: X \to 2^X$ be $A$-maximal $(m)$-relaxed accretive. In addition, let $g: X \to X$ be a $(\beta)$-Lipschitz continuous mapping on $X$. Then the generalized resolvent operator associated with $M$ and defined by

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u) \forall u \in X,$$

satisfies

$$\|J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))\| \leq \frac{\beta}{r - \rho m}\|u - v\|,$$

where $r - \rho m > 0$.

Furthermore, we have

$$\langle J_q(J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))), g(u) - g(v) \rangle \geq (r - \rho m)\|J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))\|^q,$$

where $r - \rho m > 0$.

**Proof.** For any elements $u, v \in X$ (and hence $g(u), g(v) \in X$), we have from the definition of the resolvent operator $J_{\rho,A}^M$ that

$$\frac{1}{\rho}[g(u) - A(J_{\rho,A}^M(g(u)))] \in M(J_{\rho,A}^M(g(u))),$$

and

$$\frac{1}{\rho}[g(v) - A(J_{\rho,A}^M(g(v)))] \in M(J_{\rho,A}^M(g(v))).$$

Since $M$ is $A$-maximal $(m)$-relaxed accretive, it implies that

$$\langle g(u) - g(v) - [A(J_{\rho,A}^M(g(u)))] - A(J_{\rho,A}^M(g(v))), J_q(J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))) \rangle \geq (-\rho m)\|J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))\|^q.$$

Based on (2), using the $(r)$-strong accretiveness of $A$, we get

$$\langle g(u) - g(v), J_q(J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))) \rangle \geq \langle A(J_{\rho,A}^M(g(u)) - A(J_{\rho,A}^M(g(v))), J_q(J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))) \rangle - \rho m\|J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))\|^q \geq (r - \rho m)\|J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))\|^q.$$

Therefore, we have

$$\langle g(u) - g(v), J_q(J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))) \rangle \geq (r - \rho m)\|J_{\rho,A}^M(g(u)) - J_{\rho,A}^M(g(v))\|^q.$$

This completes the proof. \(\square\)

**Theorem 3.1.** Let $X$ be a real Banach space, let $A: X \to X$ be $(r)$-strongly accretive, and let $M: X \to 2^X$ be $A$-maximal $(m)$-relaxed accretive. Let $g: X \to X$ be a map on $X$. Then the following statements are equivalent:

(i) An element $u \in X$ is a solution to (1).

(ii) For an $u \in X$, we have

$$g(u) = J_{\rho,A}^M(A(g(u)) - \rho A(u)),$$
Theorem 3.2. Let \( X \) be a real Banach space, let \( H : X \to X \) be \((r)\)-strongly accretive, and let \( M : X \to 2^X \) be \(H\)-maximal accretive. Let \( g : X \to X \) be a map on \( X \). Then the following statements are equivalent:

(i) An element \( u \in X \) is a solution to (1).

(ii) For an \( u \in X \), we have

\[
g(u) = J_{\rho,H}^M(H(g(u)) - \rho H(u)) ,
\]

where

\[
J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u) .
\]

Proof. It follows from the definition of the resolvent operator \( J_{\rho,A}^M \). \( \square \)

Theorem 3.3. Let \( X \) be a real \( q \)-uniformly smooth Banach space, let \( A : X \to X \) be \((r)\)-strongly accretive and \((s)\)-Lipschitz continuous, and let \( M : X \to 2^X \) be \(A\)-maximal \((m)\)-relaxed accretive. Let \( g : X \to X \) be \((t)\)-strongly accretive and \((\beta)\)-Lipschitz continuous. Then there exists a unique solution \( x^* \in X \) to (1) for

\[
\theta = \left(1 + \frac{1}{r - \rho m} \right) \sqrt{1 - qt + c_q \beta^q} + \frac{1}{r - \rho m} \sqrt{\beta^q - q r t^q + c_q \beta^q} \]

\[
+ \frac{1}{r - \rho m} \sqrt{1 - q r \rho + c_q \rho^q s^q} < 1 ,
\]

for \( r - \rho m > 1 \) and \( c_q > 0 \).

Proof. First we define a function \( F : X \to X \) by

\[
F(u) = u - g(u) + J_{\rho,A}^M(A(g(u)) - \rho A(u)) ,
\]

and then prove that \( F \) is contractive. Applying Lemma 3.1, we estimate

\[
\|F(u) - F(v)\| = \|u - v - (g(u) - g(v)) + J_{\rho,A}^M(A(g(u)) - \rho A(u)) - J_{\rho,A}^M(A(g(v)) - \rho A(v))\| \\
\leq \|u - v - (g(u) - g(v))\| + \frac{1}{r - \rho m} \|A(g(u)) - A(g(v)) - \rho(A(u) - A(v))\| \\
\leq \left(1 + \frac{1}{r - \rho m} \right) \|u - v - (g(u) - g(v))\| \\
+ \frac{1}{r - \rho m} \|A(g(u)) - A(g(v)) - (g(u) - g(v))\| \\
+ \frac{1}{r - \rho m} \|u - v - \rho(A(u) - A(v))\| .
\]

Since \( g \) is \((t)\)-strongly accretive and \((\beta)\)-Lipschitz continuous, we have

\[
\|u - v - (g(u) - g(v))\|^q = \|u - v\|^q - q\langle g(u) - g(v), J_q(u - v) \rangle + c_q \|g(u) - g(v)\|^q \\
\leq \|u - v\|^q - qt\|u - v\|^q + c_q \beta^q \|u - v\|^q \\
= (1 - qt + c_q \beta^q) \|u - v\|^q .
\]
Therefore, we have
\begin{equation}
\|u - v - (g(u) - g(v))\| \leq \sqrt{1 - qt + c_q \beta^q}.
\end{equation}

Similarly, based on the strong accretiveness and Lipschitz continuity of $A$ and $g$, we get
\begin{equation}
\|A(g(u)) - A(g(v)) - (g(u) - g(v))\| \leq \sqrt{\beta^q - qrt^q + c_q s^q \beta^q},
\end{equation}
and
\begin{equation}
\|u - v - \rho(A(u) - A(v))\| \leq \sqrt{1 - q \rho + c_q \rho^q s^q}.
\end{equation}

In light of above arguments, we have
\begin{equation}
\|F(u) - F(v)\| \leq \theta \|u - v\|,
\end{equation}
where
\begin{equation}
\theta = \left(1 + \frac{1}{r - \rho m}\right) \sqrt{1 - qt + c_q \beta^q} + \frac{1}{r - \rho m} \sqrt{\beta^q - qrt^q + c_q s^q \beta^q}
+ \frac{1}{r - \rho m} \sqrt{1 - q \rho + c_q \rho^q s^q} < 1,
\end{equation}
for $r - \rho m > 1$.

**Corollary 3.1.** Let $X$ be a real $q$– uniformly smooth Banach space, let $H : X \to X$ be $(r)$- strongly accretive and $(s)$-Lipschitz continuous, and let $M : X \to 2^X$ be $H$-maximal accretive. Let $g : X \to X$ be $(t)$-strongly accretive and $(\beta)$-Lipschitz continuous. Then there exists a unique solution $x^* \in X$ to (1) for
\begin{equation}
\theta = \left(1 + \frac{1}{r}\right) \sqrt{1 - qt + c_q \beta^q} + \frac{1}{r} \sqrt{\beta^q - qrt^q + c_q s^q \beta^q}
+ \frac{1}{r} \sqrt{1 - q \rho + c_q \rho^q s^q} < 1,
\end{equation}
for $r > 1$.

**References**


