LOWER BOUNDS FOR EXPRESSIONS OF LARGE SIEVE TYPE

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ABSTRACT. We show that the large sieve is optimal for almost all exponential sums.

Let \( a_n, 1 \leq n \leq N \) be complex numbers, and set \( S(\alpha) = \sum_{n \leq N} a_n e(n \alpha) \), where \( e(\alpha) = \exp(2\pi i \alpha) \). Large Sieve inequalities aim at bounding the number of places where this sum can be extraordinarily large, the basic one being the bound

\[
\sum_{q \leq Q} \sum_{1 \leq a \leq q \atop (a, q) = 1} \left| S\left(\frac{a}{q}\right)\right|^2 \leq (N + Q^2) \sum_{n \leq N} |a_n|^2
\]

(see e.g. [3] for variations and applications). P. Erdős and A. Rényi [1] considered lower bounds of the same type, in particular they showed that the bound

\[
\sum_{q \leq Q} \sum_{1 \leq a \leq q \atop (a, q) = 1} \left| S\left(\frac{a}{q}\right)\right|^2 \ll N \sum_{n \leq N} |a_n|^2,
\]

(1)

valid for \( Q \ll \sqrt{N} \), is wrong for almost all choices of coefficients \( a_n \in \{1, -1\} \), provided that \( Q > C \sqrt{N} \log N \), and that the standard probabilistic argument fails to decide whether (1) is true in the range \( \sqrt{N} < Q < \sqrt{N} \log N \). In this note, we show that (1) indeed fails throughout this range.

Theorem 1. Let \( S(\alpha) \) be as above. Then

\[
\sum_{q \leq Q} \sum_{1 \leq a \leq q \atop (a, q) = 1} \left| S\left(\frac{a}{q}\right)\right|^2 \geq \varepsilon Q^2 \sum_{n \leq N} |a_n|^2
\]

holds true with probability tending to 1 provided \( \varepsilon \) tends to 0, and \( Q^2/N \) tends to infinity.

Our approach differs from [1] in so far as we first prove an unconditional lower bound, which involves an awkward expression, and show then that almost always this expression is small. We show the following.

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Lemma 1. Let $S(\alpha)$ be as above, and define

$$M(x) = \sup_m \frac{\int |S(u)|^2 \, du}{\int_0^x |S(u)|^2 \, du},$$

where $m$ ranges over all measurable subsets of $[0,1]$ of measure $x$. Then for any real parameter $A > 1$ we have the estimate

$$(3) \quad \sum_{q \leq Q} \sum_{(a,q) = 1} |S\left(\frac{a}{q}\right)|^2 \geq \left(\frac{Q^2}{A} \left(1 - M\left(\frac{1}{A}\right)\right) - 6\pi N A\right) \sum_{n \leq N} |a_n|^2.$$

Proof. Our proof adapts Gallagher’s proof of an upper bound large sieve [2]. For every $f \in C^1([0,1])$, we have

$$f(1/2) = \int_0^1 f(u) \, du + \int_0^{1/2} uf'(u) \, du - \int_{1/2}^1 (1-u)f'(u) \, du.$$

Putting $f(u) = |S(u)|^2$, and using the linear substitution $u \mapsto (\alpha - \delta/2) + \delta u$, we obtain for every $\delta > 0$ and any $\alpha \in [0,1]$

$$|S(\alpha)|^2 = \frac{1}{\delta} \int_{\alpha - \delta/2}^{\alpha + \delta/2} |S(u)|^2 \, du + \frac{1}{\delta} \int_{\alpha - \delta/2}^{\alpha} \left(\frac{\delta}{2} - |u - \alpha|\right) \left(S'(u)S(-u) - S(u)S'(-u)\right) \, du$$

$$- \frac{1}{\delta} \int_{\alpha}^{\alpha + \delta/2} \left(\frac{\delta}{2} - |u - \alpha|\right) \left(S'(u)S(-u) - S(u)S'(-u)\right) \, du.$$

We have $|S(u)| = |S(-u)|$ and $|S'(-u)| = |S'(u)|$, thus $|S'(u)S(-u) - S(u)S'(-u)| \leq 2|S(u)S'(u)|$, and we obtain

$$|S(\alpha)|^2 \geq \frac{1}{\delta} \int_{\alpha - \delta/2}^{\alpha + \delta/2} |S(u)|^2 \, du - \frac{1}{\delta} \int_{\alpha - \delta/2}^{\alpha} 2\left(\frac{1}{2} - \frac{|u - \alpha|}{\delta}\right) |S(u)S'(u)| \, du$$

$$\geq \frac{1}{\delta} \int_{\alpha - \delta/2}^{\alpha + \delta/2} |S(u)|^2 \, du - \int_{\alpha - \delta/2}^{\alpha} |S(u)S'(u)| \, du.$$
We now set $\delta = A/Q^2$. We can safely assume that $\delta < \frac{1}{2}$, since our claim would be trivial otherwise. Summing over all fractions $\alpha = \frac{a}{q}$ with $q \leq Q$, $(a, q) = 1$, we get

$$\sum_{q \leq Q} \sum_{(a, q) = 1} \left| S\left( \frac{a}{q} \right) \right|^2 \geq \frac{Q^2}{A} \int_0^1 |S(u)|^2 du - \frac{Q^2}{A} \int_{m(Q, A)} \| S(u) \| du,$$

where

$$R(u) = \# \left\{ a, q : (a, q) = 1, q \leq Q, \left| u - \frac{a}{q} \right| \leq \frac{A}{Q^2} \right\},$$

and

$$m(Q, A) = \{ u \in [0, 1] : R(u) = 0 \}.$$
Hence, the last term in (4) is bounded above by $3A(2\pi N)\sum_{n\leq N} |a_n|^2$, and inserting our bounds into (4) yields the claim of our lemma.

Now we deduce Theorem 1. Let $S(\alpha)$ be a random sum in the sense that the coefficients $a_n \in \{1, -1\}$ are chosen at random. We compute the expectation of the fourth moment of $S(\alpha)$.

$$E \int_0^1 |S(u)|^4 \, du = E \sum_{\mu_1 + \mu_2 = \nu_1 + \nu_2} a_{\mu_1}a_{\nu_1}a_{\mu_2}a_{\nu_2}$$

$$= \# \{\mu_1, \mu_2, \nu_1, \nu_2 \leq N : \{\mu_1, \mu_2\} = \{\nu_1, \nu_2\} \}$$

$$= 2N^2 - N.$$

If $m \subseteq [0, 1]$ is of measure $x$, then $\int |S(u)|^2 \, du \leq \sqrt{x}(\int |S(u)|^4 \, du)^{1/2}$, thus $EM(x) \leq \sqrt{2x}$. In particular, we have $M(x) \leq 1/2$ with probability $\geq 1 - \sqrt{8x}$. Let $\delta > 0$ be given, and set $A = 8\delta^{-2}$. Then with probability $\geq 1 - \delta$ we have $M(1/A) \leq 1/2$, and (3) becomes

$$\sum_{q \leq Q} \sum_{(a,q)=1} \left| S\left(\frac{a}{q}\right) \right|^2 \geq \left( \frac{Q^2\delta^2}{16} - 48\delta^{-2}\pi N \right) \sum_{n \leq N} |a_n|^2$$

$$\geq \frac{Q^2\delta^2}{32} \sum_{n \leq N} |a_n|^2,$$

provided that $Q^2 > 1536\delta^4 N$. Hence, for fixed $\epsilon$, the relation (2) becomes true with probability $1 - \sqrt{1024\epsilon}$, provided that $Q^2/N$ is sufficiently large. Hence, our claim follows.

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References

