ON RINGS ALL OF WHOSE MODULES ARE RETRACTABLE

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Abstract. Let $R$ be a ring. A right $R$-module $M$ is said to be retractable if $\text{Hom}_R(M,N) \neq 0$ whenever $N$ is a non-zero submodule of $M$. The goal of this article is to investigate a ring $R$ for which every right $R$-module is retractable. Such a ring will be called right mod-retractable. We proved that

1. The ring $\prod_{i \in I} R_i$ is right mod-retractable if and only if each $R_i$ is a right mod-retractable ring for each $i \in I$, where $I$ is an arbitrary finite set.
2. If $R[x]$ is a mod-retractable ring then $R$ is a mod-retractable ring.

Throughout this paper, $R$ is an associative ring with unity and all modules are unital right $R$-modules.

Khuri [1] introduced the notion of retractable modules and gave some results for non-singular retractable modules when the endomorphism ring is (quasi-)continuous. For retractable modules, we direct the reader to the excellent papers [1], [2], [3] and [4] for nice introduction to this topic in the literature.

Let $M$ be an $R$-module. $M$ is said to be a retractable module if $\text{Hom}_R(M,N) \neq 0$ whenever $N$ is a non-zero submodule of $M$ ([1]). We give some examples.

(i) Free modules and semisimple modules are retractable.
(ii) Any direct sum of $\mathbb{Z}_{p^i}$ is retractable, where $p$ is a prime number.
(iii) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not retractable.
(iv) Let $R$ be an integral domain with quotient ring $F$ and $F \neq R$. Then $R \oplus F$ is a retractable $R$-module, because $\text{End}_R(M) = \begin{pmatrix} F & F \\ 0 & R \end{pmatrix}$.
(v) Assume that $M_R$ is a finitely generated semisimple right $R$-module. Then the module $M_R$ is retractable and $\text{End}_R(M)$ is semisimple artinian By [3] Corollary 2.2
(vi) Take an $R$-module $M$. Let $0 \neq N \leq R \oplus M$; take $0 \neq n \in N$ and construct the map $\varphi: R \oplus M \to N$ by $\varphi(1) = n$ and $\varphi(m) = 0$ for all $m \in M$. Since $0 \neq \varphi \in \text{Hom}_R(R \oplus M, N)$, we have $\text{Hom}_R(R \oplus M, N) \neq 0$, thus $R \oplus M$ is retractable.

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In this note, we deal with some ring extensions of a ring $R$ for which every (right) $R$-module is retractable. Hence, such a ring will be called right *mod-retractable*. This will avoid a conflict of nomenclature with the existing concept of retractability. The following examples show that this definition is not meaningless.

We take $\mathbb{Z}$-modules $M = \mathbb{Q}$ and $N = \mathbb{Z}$. Note that $\mathbb{Q}$ is a divisible group, so every its homomorphic image is a divisible group as well. Since the only divisible subgroup of $\mathbb{Z}$ is 0, the only homomorphism of $\mathbb{Q}$ into $\mathbb{Z}$ is the zero homomorphism.

Let $R$, $S$ be two rings and $M$ be an $R$-$S$-bimodule. Then we consider the ring $R' = (R \times M / 0 \times S)$. Let $I = (0 \times M / 0 \times 0)$ and $K = eR'$, where $e = (1 \times 0 / 0 \times 0)$. We claim that $\text{Hom}_{R'}(K, I) = 0$. Note that $I \nsubseteq K$. Let $f \in \text{Hom}_{R'}(K, I)$. Then $f(K) = f(eR) = f(eeR) = f(e)eR = f(e)K \subseteq IK = 0$, i.e., $R'$ is retractable.

A ring $R$ is called (finitely) *mod-retractable* if all (finitely generated) right $R$-modules are retractable.

**Example 1.** (i) Any semisimple artinian ring is mod-retractable.

(ii) $\mathbb{Z}$ is a finitely mod-retractable ring but is not mod-retractable ring.

We start the Morita invariant property for (finitely) mod-retractable rings.

**Theorem 2.** (Finite) *mod-retractability is Morita invariant.*

**Proof.** Let $R$ and $S$ be two Morita equivalent rings. Assume that $f : \text{Mod}-R \to \text{Mod}-S$ and $g : \text{Mod}-S \to \text{Mod}-R$ are two category equivalences. Let $M$ be a retractable $R$-module. Then $M$ is a retractable object in $\text{Mod}-R$. Let $0 \neq N \leq f(M)$. Then $\text{Hom}_R(M, g(N)) \neq 0$ since $g(N)$ is isomorphic to a submodule of $M$. Thus, we have $0 \neq \text{Hom}_S(f(M), fg(N)) \cong \text{Hom}_S(f(M), N)$. This follows that $f(M)$ is a retractable object in $\text{Mod}-S$. □

Let $R$ be a ring, $n$ a positive integer and the ring $M_n(R)$ of all $n \times n$ matrices with entries in $R$.

**Corollary 3.** If $R$ is (finitely) *mod-retractable*, then $M_n(R)$ is (finitely) mod-retractable.

**Proof.** By Theorem 2. □

**Theorem 4.** The class of (finite) mod-retractable rings is closed under taking homomorphic images.

**Proof.** Suppose $R$ is a (finite) mod-retractable ring. It is well-known that

$$\text{Hom}_R(M, N) = \text{Hom}_{R/I}(M, N)$$

for each ideal $I$ of $R$ and $M, N \in \text{Mod}-R/I$. Now the proof is clear. □

Recall that a module $M$ is said to be *e-retractable* if, for all every essential submodule $N$ of $M$, $\text{Hom}_R(M, N) \neq 0$ (see [1]).

**Lemma 5.** The following statements are equivalent for a ring $R$.

1. $R$ is (finitely) mod-retractable.
2. Every (finitely generated) $R$-module $M$ is e-retractable.
(3) For every (finitely generated) \( R \)-module \( M \) and \( N \) \( \leq M \), \( \text{Hom}_R(M,N) = 0 \) if and only if \( \text{Hom}_R(M,E(N)) = 0 \), where \( E(N) \) is an injective hull of \( N \).

**Proof.** (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are clear.

(2) \( \Rightarrow \) (1) Let \( M \) be a (finitely generated) right \( R \)-module and \( N \) be a submodule of \( M \). Since \( E(N) \) is an injective module, we extend the inclusion \( N \subseteq E(N) \) to the map \( \alpha : M \to E(N) \). This implies that \( \alpha(N) = N \). Thus \( \alpha(M) \cap N = N \). Since \( N \leq e \), we have \( N \leq e \alpha(M) \). This implies that \( \text{Hom}_R(\alpha(M),N) \neq 0 \). Moreover, for \( K = \text{Ker}(\alpha) \),

\[
\text{Hom}_R(\alpha(M),N) = \text{Hom}_R(M/K,N) \subseteq \text{Hom}_R(M,N).
\]

As such, \( \text{Hom}_R(M,N) \neq 0 \).

(3) \( \Rightarrow \) (2) Let \( N \) be an essential submodule of a (finitely generated) right \( R \)-module \( M \). Then \( E(N) \cong E(M) \). By (3), we can obtain that \( \text{Hom}_R(M,N) = 0 \), and so \( \text{Hom}_R(M,E(N)) = 0 \). Hence \( \text{Hom}_R(M,E(M)) = 0 \).

By Example 1 a commutative ring need not be retractable.

**Theorem 6.** Any ring that is Morita equivalent to a commutative ring is finitely mod-retractable.

**Proof.** By Theorem 2 it suffices to prove the claim for a commutative ring \( R \). Let \( M \) be a finitely generated \( R \)-module and \( N \leq M \). Assume that \( \text{Hom}_R(M,E(N)) \neq 0 \), and take \( 0 \neq \alpha \in \text{Hom}_R(M,E(N)) \). Since \( M \) is a finitely generated \( R \)-module, we can write \( \alpha(M) \) as follows (where the sum is not necessarily direct): \( \alpha(M) = Rm_1 + Rm_2 + \ldots Rm_n \) with \( m_i \in E(N), 1 \leq i \leq n \). Since \( N \) is essential in \( E(N) \), thus there exists \( r \in R \) such that \( rm_i \in N \) for all \( i \) and \( r\alpha(M) \neq 0 \). Now we can define \( 0 \neq \beta : \alpha(M) \to N \) such that \( \beta(m_i) = rm_i \) for all \( 1 \leq i \leq n \). Thus \( 0 \neq \beta \alpha \in \text{Hom}_R(M,N) \). This implies that \( \text{Hom}_R(M,N) \neq 0 \). By Lemma 5 the \( R \)-module \( M \) is retractable.

**Example 7.** Let \( R \) be a commutative artin ring. Assume that a ring \( S \) is Morita equivalent to \( R \). First, note that every \( S \)-module is retractable and has a maximal submodule. We consider an \( S \)-module \( M \). Let \( N \) be a maximal submodule of \( M \). Hence we have a simple submodule \( K \) of \( N \). Then there exits an \( S \)-homomorphism \( f : M \to E(K) \), where \( E(K) \) is the injective hull of \( K \). Clearly, \( f(M) \) is a finitely generated \( S \)-module. By Theorem 6, \( f(M) \) is a retractable \( S \)-module and so \( M \) is a retractable \( S \)-module.

Example 7 shows that the class of right mod-retractable rings is not closed under direct sums.

**Theorem 8.** The ring \( \prod_{i \in \mathcal{I}} R_i \) is right mod-retractable if and only if each \( R_i \) is a right mod-retractable ring for each \( i \in \mathcal{I} \), where \( \mathcal{I} \) is an arbitrary finite set.

**Proof.** \( \Rightarrow \) Indeed, \( R_i \) is a homomorphic image of \( \prod_{i \in \mathcal{I}} R_i \). So the result follows from Theorem 4.

\( \Leftarrow \): Let each \( e_i \) denote the unit element of \( R_i \) for all \( i \in \mathcal{I} \). A module \( M \) over \( \prod_{i \in \mathcal{I}} R_i \) can be written as set direct product \( \prod_{i \in \mathcal{I}} M_i \), where \( M_i R_i = Me_i \) and external operation defined as \( (r_i)_{i \in \mathcal{I}} (m_i)_{i \in \mathcal{I}} = (r_i m_i)_{i \in \mathcal{I}} \). Thus retractability of \( M \)
is given by retractability of each \( M_{ii} \in T \). But, since each \( R_i \) is mod-retractable, this condition is satisfied.

**Corollary 9.** The class of all right mod-retractable rings is closed under taking finite direct products.

**Proof.** By Theorem 8.

Giving a ring \( R \), \( R[X] \) denotes the polynomial ring with \( X \) a set of commuting indeterminate over \( R \). If \( X = \{x\} \), then we use \( R[x] \) in place of \( R[\{x\}] \).

**Theorem 10.** If \( R[x] \) is a mod-retractable ring then \( R \) is a mod-retractable ring.

**Proof.** Since \( R \cong R[x]/R[x]x \), the result is clear from Theorem 4.

**References**


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