

A GENERALIZATION OF THOM'S TRANSVERSALITY THEOREM

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ABSTRACT. We prove a generalization of Thom's transversality theorem. It gives conditions under which the jet map $f_*|_Y: Y \subseteq J^r(D, M) \rightarrow J^r(D, N)$ is generically (for $f: M \rightarrow N$) transverse to a submanifold $Z \subseteq J^r(D, N)$. We apply this to study transversality properties of a restriction of a fixed map $g: M \rightarrow P$ to the preimage $(j^s f)^{-1}(A)$ of a submanifold $A \subseteq J^s(M, N)$ in terms of transversality properties of the original map f . Our main result is that for a reasonable class of submanifolds A and a generic map f the restriction $g|_{(j^s f)^{-1}(A)}$ is also generic. We also present an example of A where the theorem fails.

0. INTRODUCTION

We start by reminding that for smooth manifolds M and N the set $C^\infty(M, N)$ of smooth maps is endowed with two topologies called weak (compact-open) and strong (Whitney) topology. They agree when M is compact. We say that a subset of a topological space is *residual* if it contains a countable intersection of open dense subsets. The Baire property for $C^\infty(M, N)$ then guarantees that it is automatically dense. This holds for both topologies but is almost exclusively used for the strong one. Clearly every residual subset of $C^\infty(M, N)$ for the strong topology is also residual for the weak topology. The following is our main theorem in which we denote by $J_{\text{imm}}^r(D, M)$ the subspace of all jets of immersions.

Theorem A. *Let D, M, N be manifolds, $Y \subseteq J_{\text{imm}}^r(D, M)$ and $Z \subseteq J^r(D, N)$ submanifolds. Let us further assume that $\sigma_Y \pitchfork \sigma_Z$, where*

$$\sigma_Y = \sigma|_Y: Y \subseteq J^r(D, M) \rightarrow D \quad \text{and} \quad \sigma_Z = \sigma|_Z: Z \subseteq J^r(D, N) \rightarrow D$$

are the restrictions of the source maps. For a smooth map $f: M \rightarrow N$ let $f_|_Y$ denote the map*

$$Y \hookrightarrow J_{\text{imm}}^r(D, M) \xrightarrow{f_*} J^r(D, N).$$

Then the subset

$$\mathfrak{X} := \{f \in C^\infty(M, N) \mid f_*|_Y \pitchfork Z\}$$

2000 *Mathematics Subject Classification:* primary 57R35; secondary 57R45.

Key words and phrases: transversality, residual, generic, restriction, fibrewise singularity.

Research was supported by the grant MSM 0021622409 of the Czech Ministry of Education and by the grant 201/05/2117 of the Grant Agency of the Czech Republic.

is residual in $C^\infty(M, N)$ with the strong topology, and open if Z is closed (as a subset) and the target map $\tau_Y : Y \rightarrow M$ is proper.

We are interested in the theorem mainly because of the following two applications, the first of which is the classical theorem of Thom.

Theorem B (Thom’s Transversality Theorem). *Let M, N be manifolds, $Z \subseteq J^r(M, N)$ a submanifold. Then the subset*

$$\mathfrak{X} := \{f \in C^\infty(M, N) \mid j^r f \pitchfork Z\}$$

is residual in $C^\infty(M, N)$. It is moreover open provided that Z is closed (as a subset).

To explain the second application we need to introduce a bit of notation. Let us consider the following diagram

$$\begin{array}{ccccc}
 & & f^{-1}(A) & \longrightarrow & A \\
 & & \downarrow j & & \downarrow \\
 g^{-1}(B) & \hookrightarrow & M & \xrightarrow{f} & N \\
 \downarrow & & \downarrow g & & \\
 B & \hookrightarrow & P & &
 \end{array}$$

of smooth manifolds and smooth maps between them where \hookrightarrow indicates embeddings. We assume that $f \pitchfork A$ and $g \pitchfork B$ for the two pullbacks to be defined (which we emphasize by saying that they are transverse pullbacks) and also for some technical reasons. Suppose that we fix g and allow ourselves to change f (but only in such a way that $f \pitchfork A$). It is not hard to describe when the composition $gj = g|_{f^{-1}(A)}$ is transverse to B . A more general condition on gj is whether it satisfies some form of jet transversality. First we have to solve the problem of not having the source of $J^r(f^{-1}(A), P)$ fixed.

Construction. Let D be a d -dimensional manifold and

$$\text{Diff}(\mathbb{R}^d, 0) = \text{inv}G_0(\mathbb{R}^d, \mathbb{R}^d)_0$$

the group¹ of germs at 0 of local diffeomorphisms $\mathbb{R}^d \rightarrow \mathbb{R}^d$ fixing 0. Define a principal $\text{Diff}(\mathbb{R}^d, 0)$ -bundle

$$\text{Charts}_D = \text{inv}G_0(\mathbb{R}^d, D) \xrightarrow{\text{ev}_0} D$$

of germs at 0 of local diffeomorphisms $\mathbb{R}^d \rightarrow D$. If F is any manifold with a (smooth in some sense) action of $\text{Diff}(\mathbb{R}^d, 0)$ then we can construct an associated bundle

$$D[F] := \text{Charts}_D \times_{\text{Diff}(\mathbb{R}^d, 0)} F \longrightarrow D.$$

¹If we wanted to give $\text{Diff}(\mathbb{R}^d, 0)$ a topology we could do so by inducing the topology via the map $\text{Diff}(\mathbb{R}^d, 0) \rightarrow J_0^\infty(\mathbb{R}^d, \mathbb{R}^d)_0$. Or one could replace $\text{Diff}(\mathbb{R}^d, 0)$ by its image - the subspace of invertible ∞ -jets.

Any bundle of this form is “local”. Observe that this construction is functorial in D on the category of d -dimensional manifolds and local diffeomorphisms. As an example the bundle $J^r(D, P)$ of r -jets of maps $D \rightarrow P$ is a local bundle as

$$D[J_0^r(\mathbb{R}^d, P)] = \text{Charts}_D \times_{\text{Diff}(\mathbb{R}^d, 0)} J_0^r(\mathbb{R}^d, P) \cong J^r(D, P)$$

where $J_0^r(\mathbb{R}^d, P)$ is the subspace of $J^r(\mathbb{R}^d, P)$ of r -jets with source 0. The bijection is provided by the map

$$[u, \alpha] \mapsto \alpha \circ j_{u(0)}^r(u^{-1}).$$

Having a $\text{Diff}(\mathbb{R}^d, 0)$ -invariant submanifold $B \subseteq J_0^r(\mathbb{R}^d, P)$ we get an associated subbundle $D[B] \subseteq J^r(D, P)$ for any d -dimensional manifold D . This allows us to talk about jet transversality conditions on a map $D \rightarrow P$ without specifying what D (and hence also $J^r(D, P)$) is.

Let $A \subseteq J^s(M, N)$ be a submanifold and $j^s f \pitchfork A$. Then $f^*A := (j^s f)^{-1}(A)$ is a submanifold of M and for any $\text{Diff}(\mathbb{R}^d, 0)$ -invariant submanifold $B \subseteq J_0^r(\mathbb{R}^d, P)$ and the associated submanifold

$$f^*A[B] \subseteq J^r(f^*A, P)$$

we might ask whether $j^r(g|_{f^*A})$ is transverse to $f^*A[B]$. To state the theorem we make the following notation: for a map $g: M \rightarrow P$ we write $g \pitchfork B$ in place of $g_* \pitchfork B$ where again

$$g_*: J_{0, \text{imm}}^r(\mathbb{R}^d, M) \longrightarrow J_0^r(\mathbb{R}^d, P).$$

This condition is satisfied e.g. by all submersions. Also consider the following map

$$(1) \quad \begin{aligned} \kappa: J^{r+s}(M, N) \times_M J_{0, \text{imm}}^r(\mathbb{R}^d, M) &\longrightarrow J_0^r(\mathbb{R}^d, J^s(M, N)), \\ (j_x^{r+s}\varphi, j_0^r\psi) &\longmapsto j_0^r(j^s(\varphi) \circ \psi). \end{aligned}$$

Theorem C. *Let*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & & \downarrow g \\ & & P \end{array}$$

be a pair of smooth maps. Let $A \subseteq J^s(M, N)$ and $B \subseteq J_0^r(\mathbb{R}^d, P)$ be smooth submanifolds with $d = \dim M - \text{codim } A$. Assuming that B is $\text{Diff}(\mathbb{R}^d, 0)$ -invariant, $\kappa \pitchfork J_0^r(\mathbb{R}^d, A)$ and $g \pitchfork B$ the subset

$$\mathfrak{X} := \{f \in C^\infty(M, N) \mid j^s f \pitchfork A, j^r(g|_{f^*A}) \pitchfork f^*A[B]\}$$

is residual in $C^\infty(M, N)$ with the strong topology. If either $r = 0$ or $s = 0$ the transversality condition $\kappa \pitchfork J_0^r(\mathbb{R}^d, A)$ is automatically satisfied.

Remark. The transversality condition $\kappa \pitchfork J_0^r(\mathbb{R}^d, A)$ cannot be removed as we illustrate by an example in the next section. An interesting question arises whether there is a reasonable sufficient condition on A for which the transversality is automatic (an example of such is e.g. $s = 0$).

Remark. It is also possible to state conditions under which \mathfrak{X} is open.

1. AN EXAMPLE

We describe here a family of examples. For $k = 1$ (and partially for $k = 2$) Theorem C can be applied whereas for $k \geq 3$ the theorem fails due to $\kappa \notin J_0^r(\mathbb{R}^d, A)$. We will not be interested in a particular choice of B since the transversality condition in question depends only on A . Let us start with the following diagram

$$\begin{array}{ccc} D^k \times M & \xrightarrow{f} & \mathbb{R} \\ \downarrow & & \\ D^k & & \end{array}$$

where we think of f as a family of functions $M \rightarrow \mathbb{R}$ parametrized by a disc D^k . To fit into our situation we should replace D^k by \mathbb{R}^k but the boundary is not the issue in our example. Let $s = 1$ and let $A \subseteq J^1(D^k \times M, \mathbb{R})$ denote the subspace of all those jets which have zero derivative in the direction of M . Clearly A has codimension $\dim M$ and thus $d = k$.

For a function $\varphi: D^k \times M \rightarrow \mathbb{R}$ we have $j_{(x,y)}^1 \varphi \in A$ iff the composition

$$T_y M \hookrightarrow T_x D^k \times T_y M \xrightarrow{d\varphi} \mathbb{R}$$

is zero. We express this by saying that the map

$$d|_{TM}: J^1(D^k \times M, \mathbb{R}) \rightarrow T^*M$$

describes A as the preimage of 0. We will now show how to describe $J_0^r(\mathbb{R}^k, A)$ (or f^*A in fact) in a similar way. Compose the defining equation with an immersion $\psi: \mathbb{R}^k \rightarrow D^k \times M$ as in (1) and differentiate to get

$$T_y M \otimes \mathbb{R}^k \xrightarrow{\text{incl} \otimes \psi_*} S^2(T_x D^k \times T_y M) \xrightarrow{d^2 \varphi} \mathbb{R}.$$

Differentiating further and putting all the maps together we get a single map

$$J^{r+1}(D^k \times M, \mathbb{R}) \times_M J_{0, \text{imm}}^r(\mathbb{R}^k, D^k \times M) \xrightarrow{\chi} T^*M \otimes (\mathbb{R} \oplus \mathbb{R}^k \oplus \dots \oplus S^r \mathbb{R}^k)^*$$

describing f^*A . Moreover χ is a submersion at f^*A iff $\kappa \in J_0^r(\mathbb{R}^k, A)$. Fixing $j_0^r \psi$ the $T^*M \otimes (S^i \mathbb{R}^k)^*$ -coordinate is a sum of various restrictions of the derivatives of φ with only a single term involving the highest derivative $d^{i+1} \varphi$, namely

$$T_y M \otimes S^i \mathbb{R}^k \xrightarrow{\text{incl} \otimes S^i \psi_*} S^{i+1}(T_x D^k \times T_y M) \xrightarrow{d^{i+1} \varphi} \mathbb{R}.$$

This implies that if the image of the derivative of ψ_* at 0 intersects TM in a subspace of dimension at most 1, the highest order term is a surjective linear map and quite easily the whole map χ is a submersion (even for a fixed ψ). The opposite implication also holds as considerations at pairs with $j^{r+1} \varphi = 0$ show.

For $k = 1$ we have shown that $\kappa \in J_0^r(\mathbb{R}, A)$ and therefore for a generic map f the fibrewise singularity set $\Sigma_f = f^*A$ is a 1-dimensional submanifold for which the projection $\Sigma_f \rightarrow D^1$ is generic. This means that only regular points and folds appear. The regular points of the projection are exactly the fibrewise Morse

singularities of f while folds correspond to the fibrewise A_3 -singularities (those of the form $x_1^3 \pm x_2^2 \pm \dots \pm x_m^2$).

For $k \geq 2$ it is *not* the case that $\kappa \pitchfork J_0^r(\mathbb{R}^k, A)$, namely at pairs $(j_x^{r+s}\varphi, j_0^r\psi)$ where the image of the tangential map ψ_* at 0 intersects the tangent space of the fibre M in a subspace of dimension at least 2.

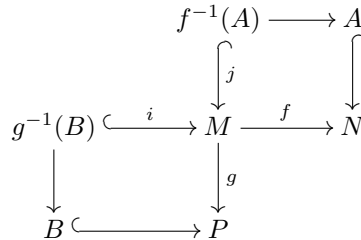
Nevertheless for $k = 2$ the conclusion of Theorem C still holds. This is because the condition $\kappa \pitchfork J_0^r(\mathbb{R}^k, A)$ is only required at those $(j_x^{r+s}\varphi, j_0^r\psi)$ for which $\psi(\mathbb{R}^d) \subseteq f^*A$ and for a generic map f such ψ cannot be tangent to the fibre M (if that was the case then the rank of the fibrewise Hessian of f at x would drop by 2 and this does not happen generically). Therefore again Σ_f is a 2-dimensional submanifold of $D^2 \times M$ partitioned into three parts: the cusps of the projection $\Sigma_f \rightarrow D^2$ (where f attains an isolated fibrewise A_4 -singularity); the folds form a 1-dimensional submanifold (where the fibrewise A_3 -singularities occur); and the fibrewise Morse singularities.

For $k = 3$ the conclusion of Theorem C fails. This is due to the fact that for a generic map between 3-manifolds the rank drops at most by 1 whereas for a generic 3-parameter family of functions $M \rightarrow \mathbb{R}$ the fibrewise Hessian can drop rank by 2.

2. PROOFS

First we will show how to translate transversality conditions for a restriction of a fixed map $g: M \rightarrow P$ to the preimage $f^{-1}(A)$ of a submanifold along a map $f: M \rightarrow N$ in terms of transversality conditions on f itself. At the end we prove that such properties are generic in the sense that maps satisfying them form a residual subset of $C^\infty(M, N)$. First version does not involve any jet conditions.

Lemma 1. *Let*



be a diagram where we assume that $f \pitchfork A$ and $g \pitchfork B$. Then the following conditions are equivalent:

- (i) $gj \pitchfork B$,
- (ii) $fi \pitchfork A$,
- (iii) $f^{-1}(A) \pitchfork g^{-1}(B)$.

Proof. Since (iii) is symmetric it is enough to show the equivalence of (i) and (iii). But (i) is equivalent to the map

$$g_* : T_x f^{-1}(A) \longrightarrow T_{g(x)}P/T_{g(x)}B$$

induced by the derivative of g being surjective for every $x \in f^{-1}(A) \cap g^{-1}(B)$. Because of the assumption $g \pitchfork B$, we have a commutative diagram

$$\begin{array}{ccc} T_x f^{-1}(A) & \longrightarrow & T_{g(x)} P / T_{g(x)} B \\ \downarrow & \nearrow \cong & \\ T_x M / T_x g^{-1}(B) & & \end{array}$$

and so (i) is equivalent to (iii). □

Now we will generalize the lemma to jet transversality conditions. Let us recall that for a $\text{Diff}(\mathbb{R}^d, 0)$ -invariant submanifold $B \subseteq J_0^r(\mathbb{R}^d, P)$ we have constructed an associated subbundle $D[B] \subseteq J^r(D, P)$ for any d -dimensional manifold D .

Lemma 2. *For $h : D \rightarrow P$ the following conditions are equivalent*

- (i) $h_* : J_{0,\text{imm}}^r(\mathbb{R}^d, D) \rightarrow J_0^r(\mathbb{R}^d, P)$ is transverse to B ,
- (ii) $j^r(h) : D \rightarrow J^r(D, P)$ is transverse to $D[B]$.

Proof. Taking associated bundles (i) is clearly equivalent to the transversality of

$$h_* : J_{\text{imm}}^r(D, D) \rightarrow J^r(D, P)$$

to $D[B]$. Let $j_x^r(k) \in J_{x,\text{imm}}^r(D, D)_y$ be an r -jet of a diffeomorphism $k : V \rightarrow W$ between open subsets of D . Then we have a diagram

$$\begin{array}{ccccc} J_{\text{imm}}^r(V, D) & \xrightarrow{h_*} & J^r(V, P) & \longleftarrow & V[B] \\ \cong \downarrow k_* & & \cong \downarrow k_* & & \cong \downarrow k_* \\ J_{\text{imm}}^r(W, D) & \xrightarrow{h_*} & J^r(W, P) & \longleftarrow & W[B] \end{array}$$

Now $j_x^r(k)$ in the top left corner is mapped by k_* down to $j_y^r(\text{id})$. Hence we see that it is enough (equivalent) to check the transversality only at $j_y^r(\text{id})$'s for all $y \in D$ for which $h_*(j_y^r(\text{id})) = j_y^r(h) \in D[B]$. For such y the same diagram shows that every $j_x^r(k)$ with target y is mapped by h_* to $D[B]$. Thus the whole fibre over y of the target map

$$J_{\text{imm}}^r(D, D) \xrightarrow{\tau} D$$

is mapped to $D[B]$. The target map τ has a section

$$j^r(\text{id}) : D \rightarrow J_{\text{imm}}^r(D, D)$$

and so (i) is finally equivalent to the composite

$$D \xrightarrow{j^r(\text{id})} J_{\text{imm}}^r(D, D) \xrightarrow{h_*} J^r(D, P)$$

being transverse to $D[B]$. This is (ii). □

We say that a map $g : M \rightarrow P$ is transverse to a $\text{Diff}(\mathbb{R}^d, 0)$ -invariant submanifold $B \subseteq J_0^r(\mathbb{R}^d, P)$, denoted $g \pitchfork B$, if

$$g_* : J_{0,\text{imm}}^r(\mathbb{R}^d, M) \rightarrow J_0^r(\mathbb{R}^d, P)$$

is transverse to B . When $r = 0$ this is equivalent to the usual transversality of a map to a submanifold. Let $f \pitchfork A$ where $f : M \rightarrow N$ and $A \subseteq N$ is a submanifold. Then we have the following diagram

$$\begin{array}{ccccc}
 J_{0,\text{imm}}^r(\mathbb{R}^d, f^{-1}(A)) & \hookrightarrow & J_0^r(\mathbb{R}^d, f^{-1}(A)) & \longrightarrow & J_0^r(\mathbb{R}^d, A) \\
 \downarrow j_* & & \downarrow j_* & & \downarrow \\
 J_{0,\text{imm}}^r(\mathbb{R}^d, M) & \hookrightarrow & J_0^r(\mathbb{R}^d, M) & \xrightarrow{f_*} & J_0^r(\mathbb{R}^d, N)
 \end{array}$$

where both squares are transverse pullbacks. This can be easily seen in local coordinates. Combining Lemma 1 with Lemma 2 we get:

Lemma 3. *Given a diagram*

$$\begin{array}{ccc}
 f^{-1}(A) & \longrightarrow & A \\
 \downarrow j & & \downarrow \\
 M & \xrightarrow{f} & N \\
 \downarrow g & & \\
 P & &
 \end{array}$$

assume that $f \pitchfork A$ and $g \pitchfork B$, where $B \subseteq J_0^r(\mathbb{R}^d, P)$ is a $\text{Diff}(\mathbb{R}^d, 0)$ -invariant submanifold with $d = \dim M + \dim A - \dim N$. Then the following conditions are equivalent:

(i) $j^r(gj) \pitchfork (f^{-1}(A))[B]$, where

$$j^r(gj) : f^{-1}(A) \rightarrow J^r(f^{-1}(A), P)$$

is the jet prolongation,

(ii) $f_*|_Y \pitchfork J_0^r(\mathbb{R}^d, A)$, where $Y = (g_*)^{-1}(B)$ is defined by a pullback diagram

$$\begin{array}{ccc}
 Y & \hookrightarrow & J_{0,\text{imm}}^r(\mathbb{R}^d, M) \\
 \downarrow & & \downarrow g_* \\
 B & \hookrightarrow & J_0^r(\mathbb{R}^d, P)
 \end{array}$$

Proof. Applying Lemma 1 to the diagram

$$\begin{array}{ccccc}
 J_{0,\text{imm}}^r(\mathbb{R}^d, f^{-1}(A)) & \longrightarrow & J_0^r(\mathbb{R}^d, A) & & \\
 \downarrow j_* & & \downarrow & & \\
 Y & \hookrightarrow & J_{0,\text{imm}}^r(\mathbb{R}^d, M) & \xrightarrow{f_*} & J_0^r(\mathbb{R}^d, N) \\
 \downarrow & & \downarrow g_* & & \\
 B & \hookrightarrow & J_0^r(\mathbb{R}^d, P) & &
 \end{array}$$

gives an equivalence of (ii) with the transversality of

$$(gj)_* : J_{0,\text{imm}}^r(\mathbb{R}^d, f^{-1}(A)) \longrightarrow J_0^r(\mathbb{R}^d, P)$$

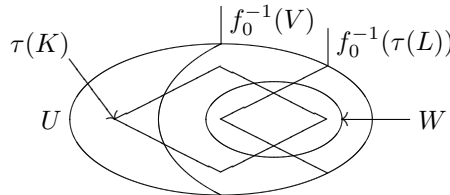
to B . By Lemma 2 this is equivalent to (i). □

Now that we know how f controls the transversality of a map defined on the preimage $f^{-1}(A)$ of some submanifold, we would like to see that this transversality condition (any of the two equivalent conditions in Lemma 3) is generic. This is indeed the case. We first prove a more general result which at the same time happens to generalize the Thom Transversality Theorem.

Proof of Theorem A. This is an application of Lemma 5 from Section 3. We have a map

$$\alpha : C^\infty(M, N) \rightarrow C^\infty(Y, J^r(D, N))$$

sending f to $f_*|_Y$. This map is continuous for the weak topology on the target and clearly $\mathfrak{X} = \{f \in C^\infty(M, N) \mid \alpha(f) \pitchfork Z\}$. We have to verify the conditions of Lemma 5.



Let $f_0 \in C^\infty(M, N)$ and $K \subseteq Y, L \subseteq Z$ compact discs. We can assume that $\tau(K)$ lies in a coordinate chart $\mathbb{R}^m \cong U \subseteq M$ and that $\tau(L)$ lies in a coordinate chart $\mathbb{R}^n \cong V \subseteq N$. We use these charts to identify U with \mathbb{R}^m and V with \mathbb{R}^n when needed. Let $\lambda : U \rightarrow \mathbb{R}$ be a compactly supported function such that $\lambda = 1$ on a neighborhood of $\tau(K) \cap f_0^{-1}(\tau(L))$ and such that $\lambda = 0$ on $U - f_0^{-1}(V)$. This is summarized in the picture above where we put $W = \text{int } \lambda^{-1}(1)$.

We set $Q := J_0^r(\mathbb{R}^m, \mathbb{R}^n)$ and identify it both with $J_*^r(U, V)$, where $*$ stands for an arbitrary point in U , and also with the space of polynomial mappings $U \rightarrow V$. Then we get a map

$$\beta : Q \rightarrow C^\infty(M, N)$$

sending q to the function $f_0 + \lambda q$ where the operations are interpreted inside V via the chart. It is continuous (in the strong topology) and the adjoint map

$$(2) \quad \gamma = (\alpha\beta)^\sharp : Q \times Y \rightarrow J^r(D, N)$$

is smooth. Thus it is enough to show that (after a suitable restriction) $\gamma \pitchfork Z$. Clearly γ sends $(q, j_x^r(h))$ to $j_x^r((f_0 + \lambda q)h)$. Suppose now that $h(x) \in W$ so that this equals to $j_x^r(f_0 h + qh)$. By restriction we get a map

$$\delta : Q \cong Q \times \{j_x^r(h)\} \xrightarrow{\gamma} J_x^r(D, V).$$

In the affine structure on $J_x^r(D, V)$ inherited from the chart, δ is clearly affine. Identifying Q with $J_{h(x)}^r(U, V)$ the linear part of δ is just a precomposition with h

$$(3) \quad h^* : J_{h(x)}^r(U, V) \rightarrow J_x^r(D, V).$$

The map h , being an immersion, has (locally - near x) a left inverse π which then gives a right inverse π^* of h^* and so the linear part of δ is surjective and hence it is a submersion.

In the horizontal direction our transversality condition $\sigma_Y \pitchfork \sigma_Z$ applies and so $\gamma \pitchfork Z$ on $Q \times \tau_Y^{-1}(W)$. If $f : M \rightarrow N$ is close enough to f_0 then

$$\tau(K) \cap f^{-1}(\tau(L)) \subseteq W$$

(equivalently $f(\tau(K) - W) \subseteq N - \tau(L)$ which is one of the basic open sets for the compact-open topology) and in particular there is a neighbourhood Q' of 0 in Q such that $\beta(Q')$ consists only of such maps. Therefore the restriction of γ to

$$Q' \times K \longrightarrow J^r(D, N)$$

is transverse to L and hence the same is also true in some neighbourhoods of K and L which then form the coverings required in Lemma 5.

If τ_Y happens to be proper then α is continuous even in strong topologies and \mathfrak{X} is a preimage of the open subset of maps $f : Y \rightarrow J^r(D, N)$ transverse to Z . \square

Now we prove two corollaries of the previous theorem.

Proof of Theorem B. We apply Theorem A to $D = M$ and

$$Y = M \xrightarrow{j^r(\text{id})} J^r(M, M) .$$

As $\sigma_Y = \text{id} = \tau_Y$ it is both proper and transverse to σ_Z for any Z . \square

Corollary 4. *Let M, N be manifolds, $Y \subseteq J_{0,\text{imm}}^r(\mathbb{R}^d, M)$ and $Z \subseteq J_0^r(\mathbb{R}^d, N)$ submanifolds. Then the subset*

$$\{f \in C^\infty(M, N) \mid f_*|_Y \pitchfork Z\}$$

is residual in $C^\infty(M, N)$ with the strong topology, and open if Z is closed (as a subset) and $\tau_Y : Y \rightarrow M$ proper.

Proof. Under the natural identification $\mathbb{R}^d \times J_0^r(\mathbb{R}^d, N) \cong J^r(\mathbb{R}^d, N)$ we can apply Theorem A to $D = \mathbb{R}^d$, the same M, N and Y but with $\mathbb{R}^d \times Z \subseteq J^r(\mathbb{R}^d, N)$ in place of Z . \square

Now we are finally able to prove our main theorem.

Proof of Theorem C. Consider first the special case $s = 0$ and

$$A = M \times A_0 \subseteq M \times N = J^0(M, N) .$$

Then $j^0 f \pitchfork A$ iff $f \pitchfork A_0$ and under this assumption Lemma 3 provides a translation between $g|_{f^*A} \pitchfork f^*A[B]$ and $f_*|_Y \pitchfork Z$, where $Z = J_0^r(\mathbb{R}^d, A_0) \subseteq J_0^r(\mathbb{R}^d, N)$. The genericity of $f \pitchfork A_0$ is the usual transversality theorem while the genericity of $f_*|_Y \pitchfork Z$ is the last corollary.

The next step is to replace the transversality to $A_0 \subseteq N$ by a jet transversality condition. Let $A \subseteq J^s(M, N)$ be a submanifold and consider

$$\begin{array}{ccc}
 f^*A & \longrightarrow & A \\
 \downarrow j & & \downarrow \\
 M & \xrightarrow{j^s f} & J^s(M, N) \\
 \downarrow g & & \\
 P & &
 \end{array}$$

It is possible to apply Lemma 3 to this situation and translate $g|_{f^*A} \pitchfork f^*A[B]$ to the transversality of the composition

$$Y \hookrightarrow J_{0, \text{imm}}^r(\mathbb{R}^d, M) \xrightarrow{(j^s f)^*} J_0^r(\mathbb{R}^d, J^s(M, N))$$

to $J_0^r(\mathbb{R}^d, A)$. We cannot however apply Theorem A directly since we are not interested in all maps $M \rightarrow J^s(M, N)$ but only in the holonomic sections (those of the form $j^s f$). This means that in our proof of Theorem A, $Q = J_*^r(\mathbb{R}^m, J^s(\mathbb{R}^m, \mathbb{R}^n))$ has to be replaced by its subspace $J_*^{r+s}(\mathbb{R}^m, \mathbb{R}^n)$ and in general there is no guarantee that the new map γ (see (2)) will be transverse (after restriction) to $Z = J_0^r(\mathbb{R}^d, A)$. This is however easily implied by $\kappa \pitchfork J_0^r(\mathbb{R}^d, A)$ since again we can arrive at the linear part of γ being a composition map (as in (3)) which is then easily identified with κ . □

3. A GENERAL TRANSVERSALITY LEMMA

We will formulate a basic lemma for deciding whether a given family of maps contains a dense subset of maps with a particular transversality property. In a sense this is just the essence of any proof of such a statement. We will be considering maps $\varphi : R \rightarrow C^\infty(S, T)$. We denote by φ^\sharp its adjoint

$$\varphi^\sharp : R \times S \rightarrow T.$$

Lemma 5. *Let S, T be smooth manifolds and $Z \subseteq T$ a submanifold. Let there be given two open coverings: \mathcal{U} of S and \mathcal{V} of T . Let R be a topological space and $\varphi : R \rightarrow C^\infty(S, T)$ a continuous map where $C^\infty(S, T)$ is given the weak topology. Assume that for every $r_0 \in R$ and every $U \in \mathcal{U}, V \in \mathcal{V}$ there is a finite dimensional manifold Q and a continuous map $k : Q \rightarrow R$ with r_0 in its image such that*

$$Q \times U \xrightarrow{k \times \text{incl}} R \times S \xrightarrow{\varphi^\sharp} T$$

is smooth and transverse to V . Then the subset

$$\mathfrak{X} := \{r \in R \mid \varphi(r) \pitchfork Z\} \subseteq R$$

is residual in R .

Proof. Following the proof of the Theorem 4.9. of Chapter 4 of [1], let us cover S by a countable family of compact discs K_i that have a neighbourhood $U_i \in \mathcal{U}$ and at the same time we choose a covering of Z by a countable family of compact discs

L_j that have a neighbourhood $V_j \in \mathcal{V}$. Then the set \mathfrak{X} is a countable intersection of the sets

$$\mathfrak{X}_{ij} := \{r \in R \mid \varphi(r) \pitchfork L_j \text{ on } K_i\}$$

and it is enough to show that each \mathfrak{X}_{ij} is open and dense. The set $\hat{\mathfrak{X}}_{ij}$ of maps $S \rightarrow T$ transverse to L_j on K_i is open in $C^\infty(S, T)$ and $\mathfrak{X}_{ij} = \varphi^{-1}(\hat{\mathfrak{X}}_{ij})$ so it is also open.

To prove the denseness we fix $r_0 \in R$ and choose a map $k : Q \rightarrow R$ with $r_0 = k(q_0)$ such that the map

$$l : Q \times U_i \rightarrow T$$

from the statement is smooth and transverse to V_j . By the parametric transversality theorem (see e.g. Theorem 2.7, Chapter 3 in [2]) the points $q \in Q$ for which $l(q, -) \pitchfork V_j$ is dense in Q . In particular q_0 lies in the closure of this set and hence r_0 lies in the closure of its image in R . But this image certainly lies in \mathfrak{X}_{ij} . \square

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