

UNIQUENESS OF MEROMORPHIC FUNCTIONS
WHEN TWO NON-LINEAR DIFFERENTIAL
POLYNOMIALS SHARE A SMALL FUNCTION

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ABSTRACT. In the paper we deal with the uniqueness of meromorphic functions when two non-linear differential polynomials generated by two meromorphic functions share a small function.

1. INTRODUCTION, DEFINITIONS AND RESULTS

Let f and g be two non-constant meromorphic functions defined in the open complex plane \mathbb{C} . For $a \in \{\infty\} \cup \mathbb{C}$ we say that f and g share the value a CM (counting multiplicities) if f, g have the same a -points with the same multiplicity and we say that f, g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $T(r, f)$ the Nevanlinna characteristic function of the meromorphic function f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$ possibly outside a set of finite linear measure.

A meromorphic function α is said to be a small function of f if $T(r, \alpha) = S(r, f)$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. Also we denote by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ as $r \rightarrow \infty$, possibly outside a set of finite linear measure.

In the recent past a number of authors worked on the uniqueness problem of meromorphic functions when differential polynomials generated by them share certain values (cf. [1], [2], [3], [4], [6], [9], [10], [11]).

In [6] following question was asked:

What can be said if two non-linear differential polynomials generated by two meromorphic functions share 1 CM?

A considerable amount of research has already been done in this direction ([1], [3], [4], [10], [11]). In 2002 Fang-Fang [3] and in 2004 Lin-Yi [11] independently proved the following result.

Theorem A. *Let f and g be two non-constant meromorphic functions and $n (\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f \equiv g$.*

Also in [3] Fang-Fang proved the following theorem.

2000 *Mathematics Subject Classification*: primary 30D35.

Key words and phrases: non-linear differential polynomial, small function, uniqueness.

Received October 31, 2007. Editor O. Došlý.

Theorem B. *Let f and g be two non-constant meromorphic functions and $n (\geq 28)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 IM, then $f \equiv g$.*

In 2001 an idea of gradation of sharing of values was introduced to measure how close a shared value is to being shared CM or to being shared IM. This notion is known as weighted sharing of values and is defined as follows.

Definition 1.1 ([8, 7]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share a value a with weight k , then z_0 is an a -point of f with multiplicity $m (\leq k)$ if and only if it is an a -point of g with multiplicity $m (\leq k)$ and z_0 is an a -point of f with multiplicity $m (> k)$ if and only if it is an a -point of g with multiplicity $n (> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

If $\alpha = \alpha(z)$ is a small function of f and g then f, g share (α, k) means that $f - \alpha$ and $g - \alpha$ share $(0, k)$.

In 2004 Lahiri-Sarkar [10] proved the following theorems.

Theorem C ([10]). *Let f and g be two non-constant meromorphic functions such that $2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} > 4$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $(1, 2)$ then $f \equiv g$, where $n (\geq 7)$ is an integer.*

Theorem D ([10]). *Let f and g be two non-constant meromorphic functions such that $2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} > 4$. If $f^n(f^2-1)f'$ and $g^n(g^2-1)g'$ share $(1, 2)$, then either $f \equiv g$ or $f \equiv -g$, where $n (\geq 8)$ is an integer. If n is an even integer then the possibility $f \equiv -g$ does not arise.*

In the paper we investigate uniqueness of meromorphic functions when two non-linear differential polynomials share a small function. We now state the main result of the paper.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions and $\alpha (\neq 0, \infty)$ be a small function of f and g . Let n and $k (\geq 2)$ be two positive integers such that $f^n(f^k - a)f'$ and $g^n(g^k - a)g'$ share (α, m) , where $a (\neq 0)$ is a finite complex number. Then $f \equiv g$ or $f \equiv -g$ provided one of the following holds:*

- (i) $m \geq 2$ and $n > \max\{4, k + 10 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\}$;
- (ii) $m = 1$ and $n > \max\{4, \frac{3k}{2} + 12 - 3\Theta(\infty; f) - 3\Theta(\infty; g)\}$;
- (iii) $m = 0$ and $n > \max\{4, 4k + 22 - 5\Theta(\infty; f) - 5\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\}$.

Also the possibility $f \equiv -g$ does not arise if n and k are both even or both odd or if n is even and k is odd.

For standard definitions and notations of the value distribution theory we refer the reader to [5].

2. LEMMAS

In this section we present some lemmas which will be needed to prove the theorem.

Lemma 2.1 ([12, 13]). *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1f + a_2f^2 + \dots + a_n f^n$, where $a_0, a_1, a_2, \dots, a_n$ ($\neq 0$) are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([14]). *Let f be a non-constant meromorphic function. Then*

$$N(r, 0; f^{(k)}) \leq k\bar{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

Lemma 2.3 ([8]). *Let f and g be two non-constant meromorphic functions sharing (1, 2). Then one of the following cases holds:*

- (i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,
- (ii) $f \equiv g$,
- (iii) $fg \equiv 1$.

Lemma 2.4 ([1]). *Let f and g be two non-constant meromorphic functions sharing (1, m) and*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \not\equiv \frac{g''}{g'} - \frac{2g'}{g-1}.$$

Now the following hold:

- (i) if $m = 1$ then $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + \frac{1}{2}\bar{N}(r, 0; f) + \frac{1}{2}\bar{N}(r, \infty; f) + S(r, f) + S(r, g)$;
- (ii) if $m = 0$ then $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + 2\bar{N}(r, 0; f) + \bar{N}(r, 0; g) + 2\bar{N}(r, \infty; f) + \bar{N}(r, \infty; g) + S(r, f) + S(r, g)$.

Lemma 2.5 ([15]). *If*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \equiv \frac{g''}{g'} - \frac{2g'}{g-1}$$

and

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{\bar{N}(r, 0; f) + \bar{N}(r, 0; g) + \bar{N}(r, \infty; f) + \bar{N}(r, \infty; g)}{T(r)} < 1$$

then $f \equiv g$ or $fg \equiv 1$, where E is a set of finite linear measure and not necessarily the same at each of its occurrence.

Lemma 2.6. *Let f and g be two non-constant meromorphic functions and α ($\neq 0, \infty$) be a small function of f and g . Let n (≥ 4) and k (≥ 2) be positive integers. Then for any non-zero constant a ,*

$$f^n(f^k - a)f'g^n(g^k - a)g' \not\equiv \alpha^2.$$

Proof. We suppose that

$$(2.1) \quad f^n(f^k - a)f'g^n(g^k - a)g' \equiv \alpha^2.$$

Let z_0 ($\alpha(z_0) \neq 0, \infty$) be a zero of f with multiplicity p . Then z_0 is a pole of g with multiplicity q , say. From (2.1) we get

$$np + p - 1 = nq + kq + q + 1$$

and so

$$(2.2) \quad kq + 2 = (n + 1)(p - q).$$

From (2.2) we get $q \geq \frac{n-1}{k}$ and again from (2.2) we obtain

$$p \geq \frac{1}{n+1} \left[\frac{(n+k+1)(n-1)}{k} + 2 \right] = \frac{n+k-1}{k}.$$

Let z_1 ($\alpha(z_1) \neq 0, \infty$) be a zero of $f^k - a$ with multiplicity p . Then z_1 is a pole of g with multiplicity q , say. So from (2.1) we get

$$\begin{aligned} 2p - 1 &= (n + k + 1)q + 1 \\ &\geq n + k + 2 \end{aligned}$$

i.e.,

$$p \geq \frac{n+k+3}{2}.$$

Since a pole of f (which is not a pole of α) is either a zero of $g^n(g^k - a)$ or a zero of g' , we have

$$\begin{aligned} \bar{N}(r, \infty; f) &\leq \bar{N}(r, 0; g) + \bar{N}(r, a; g^k) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g) \\ &\leq \frac{k}{n+k-1}N(r, 0; g) + \frac{2}{n+k+3}N(r, a; g^k) + \bar{N}_0(r, 0; g') \\ &\quad + S(r, f) + S(r, g) \\ &\leq \left(\frac{k}{n+k-1} + \frac{2k}{n+k+3} \right) T(r, g) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g), \end{aligned}$$

where $\bar{N}_0(r, 0; g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g^k - a)$.

Let $f^k - a = (f - a_1)(f - a_2) \dots (f - a_k)$. Then by the second fundamental theorem we get

$$\begin{aligned} kT(r, f) &\leq \bar{N}(r, \infty; f) + \bar{N}(r, 0; f) + \sum_{j=1}^k \bar{N}(r, a_j; f) - \bar{N}_0(r, 0; f') + S(r, f) \\ &= \bar{N}(r, \infty; f) + \bar{N}(r, 0; f) + \bar{N}(r, a; f^k) - \bar{N}_0(r, 0; f') + S(r, f) \\ &\leq \left(\frac{k}{n+k-1} + \frac{2k}{n+k+3} \right) T(r, g) + \bar{N}_0(r, 0; g') + \frac{k}{n+k-1} N(r, 0; f) \end{aligned}$$

$$\begin{aligned}
 & + \frac{2}{n+k+3}N(r, a; f^k) - \bar{N}_0(r, 0; f') + S(r, f) + S(r, g) \\
 & \leq \left(\frac{k}{n+k-1} + \frac{2k}{n+k+3} \right) \{T(r, f) + T(r, g)\} + \bar{N}_0(r, 0; g') \\
 (2.3) \quad & - \bar{N}_0(r, 0; f') + S(r, f) + S(r, g).
 \end{aligned}$$

Similarly we get

$$\begin{aligned}
 kT(r, g) & \leq \left(\frac{k}{n+k-1} + \frac{2k}{n+k+3} \right) \{T(r, f) + T(r, g)\} + \bar{N}_0(r, 0; f') \\
 (2.4) \quad & - \bar{N}_0(r, 0; g') + S(r, f) + S(r, g).
 \end{aligned}$$

Adding (2.3) and (2.4) we obtain

$$\left(1 - \frac{2}{n+k-1} - \frac{4}{n+k+3} \right) \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which is a contradiction. This proves the lemma. □

Lemma 2.7. *Let f and g be two non-constant meromorphic functions and $F = f^{n+1} \left(\frac{f^k}{n+k+1} - \frac{a}{n+1} \right)$ and $G = g^{n+1} \left(\frac{g^k}{n+k+1} - \frac{a}{n+1} \right)$, where a is a non-zero constant. Further let $F_0 = \frac{F'}{\alpha}$ and $G_0 = \frac{G'}{\alpha}$, where $\alpha (\neq 0, \infty)$ is a small function of f and g . Then $S(r, F_0)$ and $S(r, G_0)$ are replaceable by $S(r, f)$ and $S(r, g)$ respectively.*

Proof. By Lemma 2.1 we get

$$\begin{aligned}
 T(r, F_0) & \leq T(r, F') + S(r, f) \\
 & \leq 2T(r, F) + S(r, f) \\
 & = 2(n+k+1)T(r, f) + S(r, f)
 \end{aligned}$$

and similarly

$$T(r, G_0) \leq 2(n+k+1)T(r, g) + S(r, g).$$

This proves the lemma. □

Lemma 2.8. *Let F, G, F_0 and G_0 be defined as in Lemma 2.7. Then*

- (i) $T(r, F) \leq T(r, F_0) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) - N(r, a; f^k) - N(r, 0; f') + S(r, f),$
- (ii) $T(r, G) \leq T(r, G_0) + N(r, 0; g) + N\left(r, \frac{n+k+1}{n+1}a; g^k\right) - N(r, a; g^k) - N(r, 0; g') + S(r, g).$

Proof. We prove (i) only as the proof of (ii) is similar. By Nevanlinna’s first fundamental theorem and lemma 2.1 we get

$$\begin{aligned} T(r, F) &= T\left(r, \frac{1}{F}\right) + O(1) \\ &= N(r, 0; F) + m\left(r, \frac{1}{F}\right) + O(1) \\ &\leq N(r, 0; F) + m\left(r, \frac{F_0}{F}\right) + m(r, 0; F_0) + O(1) \\ &= N(r, 0; F) + T(r, F_0) - N(r, 0; F_0) + S(r, F) \\ &= T(r, F_0) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) \\ &\quad - N(r, a; f^k) - N(r, 0; f') + S(r, f). \end{aligned}$$

This proves the lemma. □

Following lemma can be proved in the line of Lemma 2.10 [10].

Lemma 2.9. *Let F and G be defined as in Lemma 2.7, where k and n ($\geq 3 + k$) are positive integers. Then $F' \equiv G'$ implies $F \equiv G$.*

Lemma 2.10. *Let F and G be defined as in Lemma 2.7 and $F \equiv G$. If $k \geq 2$ and $n + k \geq 5$ then either $f \equiv g$ or $f \equiv -g$. Also if n and k are both even or both odd or if n is even and k is odd then the possibility $f \equiv -g$ does not arise.*

Proof. Clearly if n and k are both even or both odd or if n is even and k is odd, then $f \equiv -g$ contradicts $F \equiv G$.

Let neither $f \equiv g$ nor $f \equiv -g$. We put $h = \frac{g}{f}$. Then $h \neq 1$ and $h \neq -1$. Also $F \equiv G$ implies

$$f^k = a \frac{n+k+1}{n+1} \frac{h^{n+1} - 1}{h^{n+k+1} - 1}.$$

Since f is non-constant, we see that h is not a constant. Again since f^k has no simple pole, $h - \alpha_m$ has no simple zero, where $\alpha_m = \exp\left(\frac{2m\pi i}{n+k+1}\right)$ and $m = 1, 2, \dots, n+k$. Hence $\Theta(\alpha_m; h) \geq \frac{1}{2}$ for $m = 1, 2, \dots, n+k$, which is impossible. Therefore either $f \equiv g$ or $f \equiv -g$. This proves the lemma. □

3. PROOF OF THE THEOREM

Proof of Theorem 1.1. Let F, G, F_0 and G_0 be defined as in Lemma 2.7. We consider the following three cases of the theorem separately.

Case (i). Since F_0 and G_0 share (1, 2), one of the possibilities of Lemma 2.3 holds. We suppose that

$$\begin{aligned} T_0(r) &\leq N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) \\ (3.1) \quad &\quad + S(r, F_0) + S(r, G_0), \end{aligned}$$

where $T_0(r) = \max \{T(r, F_0), T(r, G_0)\}$. We now choose a number ϵ such that

$$0 < 2\epsilon < n - k - 10 + 2\Theta(\infty; f) + 2\Theta(\infty; g) + \min \{\Theta(\infty; f), \Theta(\infty; g)\}.$$

Now by Lemma 2.2, Lemma 2.7 and Lemma 2.8 we get from (3.1)

$$\begin{aligned}
 T(r, F) &\leq T(r, F_0) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) - N(r, a; f^k) \\
 &\quad - N(r, 0; f') + S(r, f) \\
 &\leq N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) + N(r, 0; f) \\
 &\quad + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) - N(r, a; f^k) - N(r, 0; f') + S(r, f) + S(r, g) \\
 &\leq 2\bar{N}(r, 0; f) + N(r, a; f^k) + N(r, 0; f') + 2\bar{N}(r, \infty; f) + 2\bar{N}(r, 0; g) \\
 &\quad + N(r, a; g^k) + N(r, 0; g') + 2\bar{N}(r, \infty; g) + N(r, 0; f) \\
 &\quad + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) \\
 &\quad - N(r, a; f^k) - N(r, 0; f') + S(r, f) + S(r, g) \\
 &= 2\bar{N}(r, 0; f) + 2\bar{N}(r, \infty; f) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) \\
 &\quad + 2\bar{N}(r, 0; g) + N(r, a; g^k) + N(r, 0; g') + 2\bar{N}(r, \infty; g) + S(r, f) + S(r, g) \\
 &\leq \{5+k-2\Theta(\infty, f) + \epsilon\}T(r, f) + \{6+k-3\Theta(\infty, g) + \epsilon\}T(r, g) \\
 &\quad + S(r, f) + S(r, g).
 \end{aligned}$$

So by Lemma 2.1 we obtain

$$(n+k+1)T(r, f) \leq \{11+2k-2\Theta(\infty, f) - 3\Theta(\infty, g) + 2\epsilon\} \times T(r) + S(r). \quad (3.2)$$

Similarly we get

$$(n+k+1)T(r, g) \leq \{11+2k-3\Theta(\infty, f) - 2\Theta(\infty, g) + 2\epsilon\} \times T(r) + S(r). \quad (3.3)$$

From 3.2 and 3.3 we see that

$$[n-k-10+2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} - 2\epsilon]T(r) \leq S(r),$$

which is a contradiction. Hence 3.1 does not hold. So by Lemma 2.3 either $F_0G_0 \equiv 1$ or $F_0 \equiv G_0$. Since by Lemma 2.6 $F_0G_0 \not\equiv 1$, we get $F_0 \equiv G_0$. Now the result follows from Lemma 2.9 and Lemma 2.10.

Case (ii). We put

$$H = \left(\frac{F_0''}{F_0'} - \frac{2F_0'}{F_0-1}\right) - \left(\frac{G_0''}{G_0'} - \frac{2G_0'}{G_0-1}\right).$$

Also we choose a number ϵ such that

$$0 < 2\epsilon < n - \frac{3k}{2} - 12 + 3\Theta(\infty; f) + 3\Theta(\infty; g).$$

We suppose that $H \neq 0$. Since F_0 and G_0 share $(1, 1)$, by Lemma 2.2, Lemma 2.4(i), Lemma 2.7 and Lemma 2.8 we get

$$\begin{aligned}
 T(r, F) &\leq T(r, F_0) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) - N(r, a; f^k) \\
 &\quad - N(r, 0; f') + S(r, f) \\
 &\leq N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) \\
 &\quad + \frac{1}{2}\overline{N}(r, 0; F_0) + \frac{1}{2}\overline{N}(r, \infty; F_0) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) \\
 &\quad - N(r, a; f^k) - N(r, 0; f') + S(r, f) + S(r, g) \\
 &\leq 2\overline{N}(r, 0; f) + N(r, a; f^k) + N(r, 0; f') + 2\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; g) + \\
 &\quad N(r, a; g^k) + N(r, 0; g') + 2\overline{N}(r, \infty; g) + \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, a; f^k) \\
 &\quad + \frac{1}{2}\overline{N}(r, 0; f') + \frac{1}{2}\overline{N}(r, \infty; f) + N(r, 0; f) + N\left(r, \frac{n+k+1}{n+1}a; f^k\right) \\
 &\quad - N(r, a; f^k) - N(r, 0; f') + S(r, f) + S(r, g) \\
 &\leq \left\{ \frac{3k}{2} + 7 - 3\Theta(\infty, f) + \epsilon \right\} T(r, f) + \{6 + k - 3\Theta(\infty, g) + \epsilon\} T(r, g) \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq \left\{ 13 + \frac{5k}{2} - 3\Theta(\infty, f) - 3\Theta(\infty, g) + 2\epsilon \right\} T(r) + S(r).
 \end{aligned}$$

So by Lemma 2.1 we get

$$(n+k+1)T(r, f) \leq \left\{ 13 + \frac{5k}{2} - 3\Theta(\infty, f) - 3\Theta(\infty, g) + 2\epsilon \right\} T(r) + S(r).$$

Similarly we get

$$(n+k+1)T(r, g) \leq \left\{ 13 + \frac{5k}{2} - 3\Theta(\infty, f) - 3\Theta(\infty, g) + 2\epsilon \right\} T(r) + S(r).$$

Combining the above two inequalities we obtain

$$\left\{ n - \frac{3k}{2} - 12 + 3\Theta(\infty, f) + 3\Theta(\infty, g) - 2\epsilon \right\} T(r) \leq S(r),$$

which is a contradiction. Hence $H \equiv 0$. Now by Lemma 2.1 we get

$$\begin{aligned}
 (n+k)T(r, f) &= T(r, f^n(f^k - a)) + S(r, f) \\
 &\leq T(r, F') + T(r, f') + S(r, f) \\
 &\leq T(r, F_0) + 2T(r, f) + S(r, f)
 \end{aligned}$$

and so

$$T(r, F_0) \geq (n+k-2)T(r, f) + S(r, f).$$

Similarly we get

$$T(r, G_0) \geq (n+k-2)T(r, g) + S(r, g).$$

Also we see by Lemma 2.2 that

$$\begin{aligned}
 & \overline{N}(r, 0; F_0) + \overline{N}(r, \infty; F_0) + \overline{N}(r, 0; G_0) + \overline{N}(r, \infty; G_0) \\
 & \leq \overline{N}(r, 0; f) + \overline{N}(r, a; f^k) + \overline{N}(r, 0; f') + \overline{N}(r, \infty; f) + \overline{N}(r, 0; g) \\
 & \quad + \overline{N}(r, a; g^k) + \overline{N}(r, 0; g') + \overline{N}(r, \infty; g) + S(r, f) + S(r, g) \\
 & \leq (k+2)T(r, f) + 2\overline{N}(r, \infty; f) + (k+2)T(r, g) + 2\overline{N}(r, \infty; g) \\
 & \quad + S(r, f) + S(r, g) \\
 & \leq \{k+4-2\Theta(\infty; f)+\epsilon\}T(r, f) + \{k+4-2\Theta(\infty; g)+\epsilon\}T(r, g) \\
 & \quad + S(r, f) + S(r, g) \\
 & \leq \frac{2k+8-2\Theta(\infty; f)-2\Theta(\infty; g)+2\epsilon}{n+k-2}T_0(r) + S(r),
 \end{aligned}$$

where $S_0(r) = o\{T_0(r)\}$ as $r \rightarrow \infty$ possibly outside a set of finite linear measure and $\epsilon (> 0)$ is sufficiently small.

In view of the hypothesis we get from above

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{\overline{N}(r, 0; F_0) + \overline{N}(r, \infty; F_0) + \overline{N}(r, 0; G_0) + \overline{N}(r, \infty; G_0)}{T_0(r)} < 1.$$

So by Lemma 2.5 we obtain either $F_0G_0 \equiv 1$ or $F_0 \equiv G_0$. Hence the result follows from Lemma 2.6, Lemma 2.9 and Lemma 2.10.

Case (iii). Using Lemma 2.4(ii) this case can be proved as case II. This proves the theorem. \square

REFERENCES

- [1] Banerjee, A., *Meromorphic functions sharing one value*, Internat. J. Math. Math. Sci. **22** (2005), 3587–3598.
- [2] Banerjee, A., *On uniqueness for non-linear differential polynomials sharing the same 1-point*, Ann. Polon. Math. **89** (3) (2006), 259–272.
- [3] Fang, C. Y., Fang, M. L., *Uniqueness of meromorphic functions and differential polynomials*, Comput. Math. Appl. **44** (2002), 607–617.
- [4] Fang, M. L., Hong, W., *A unicity theorem for entire functions concerning differential polynomials*, Indian J. Pure Appl. Math. **32** (2001), 1343–1348.
- [5] Hayman, W. K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.
- [6] Lahiri, I., *Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points*, Ann. Polon. Math. **71** (1999), 113–128.
- [7] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161** (2001), 193–206.
- [8] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Variables Theory Appl. **46** (2001), 241–253.
- [9] Lahiri, I., *Linear differential polynomials sharing the same 1-points with weight two*, Ann. Polon. Math. **79** (2002), 157–170.
- [10] Lahiri, I., Sarkar, A., *Non-linear differential polynomials sharing 1-points with weight two*, Chinese J. Contemp. Math. **25** (2004), 325–334.

- [11] Lin, W. C., Yi, H. X., *Uniqueness theorems for meromorphic functions*, Indian J. Pure Appl. Math. **35** (2004), 121–132.
- [12] Mohonko, A. Z., *On the Nevanlinna characteristics of some meromorphic functions*, *The Theory of Functions, Functional Analysis and Their Applications*, Izd. Kharkovsk. Univ. **14** (1971), 83–87.
- [13] Yang, C. C., *On deficiencies of differential polynomials II*, Math. Z. **125** (1972), 107–112.
- [14] Yi, H. X., *Uniqueness of meromorphic functions and a question of C. C. Yang*, Complex Variables Theory Appl. **14** (1990), 169–176.
- [15] Yi, H. X., *Meromorphic functions that share one or two values*, Complex Variables Theory Appl. **28** (1995), 1–11.

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