OD-CHARACTERIZATION OF ALMOST SIMPLE GROUPS RELATED TO $L_2(49)$

LIANGCAI ZHANG AND WUJIE SHI

Abstract. In the present paper, we classify groups with the same order and degree pattern as an almost simple group related to the projective special linear simple group $L_2(49)$. As a consequence of this result we can give a positive answer to a conjecture of W. J. Shi and J. X. Bi, for all almost simple groups related to $L_2(49)$ except $L_2(49) \cdot 2^2$. Also, we prove that if $M$ is an almost simple group related to $L_2(49)$ except $L_2(49) \cdot 2^2$ and $G$ is a finite group such that $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, then $G \cong M$.

1. Introduction

Throughout this paper, groups under consideration are finite. For any group $G$, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a simple graph called prime graph of $G$, denoted by $\Gamma(G)$. The vertex set of this graph is $\pi(G)$, and two distinct vertices $p$, $q$ are joined by an edge if and only if $pq \in \pi_e(G)$. In this case, we write $p \sim q$. Denote by $t(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i = \pi_i(G)$ ($i = 1, 2, \ldots, t(G)$) the connected components of $\Gamma(G)$. When $|G|$ is even, then by our convention $2 \in \pi_1(G)$. We also denote by $\pi(n)$ the set of all primes dividing $n$, where $n$ is a natural number. Then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{t(G)}$, where $m_i$’s are positive integers with $\pi(m_i) = \pi_i$. These $m_i$’s are called the order components of $G$. In particular, if $m_i$ is an odd number, then we call it an odd component of $G$. Let $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ be the set of order components of $G$, and $T(G) = \{\pi_i(G) \mid i = 1, 2, \ldots, t(G)\}$.

Let $G$ be a group and $p \in \pi(G)$. We denote by $G_p$ and $Syl_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all of its Sylow $p$-subgroups, respectively. We also denote by $\text{Soc}(G)$ the socle of $G$ which is the subgroup generated by the set of all minimal normal subgroups of $G$. We denote by $A: B$ (or $A \cdot B$) a split (or non-split) extension of $A$ by $B$. Also, $\mathbb{N}$ and $\mathbb{P}$ denote the set of natural numbers and the set of primes, respectively.

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In particular, this paper itself is accessible only with the basic knowledge of group theory. All further unexplained notations are standard and can be found in \[4\].

**Definition 1.1.** Let $G$ be a finite group and $|G| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_i \in \mathbb{P}$ and $\alpha_i \in \mathbb{N}$ for $i = 1, 2, \ldots, k$. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) \mid p \sim q\}|$, called the degree of $p$. We also define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call it the degree pattern of $G$.

**Definition 1.2.** A group $M$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $G$ such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.

**Definition 1.3.** A group $G$ is said to be an almost simple related to $S$ if and only if $S \trianglelefteq G \leq \text{Aut}(S)$ for some non-abelian simple group $S$.

**Definition 1.4.** Let $p$ be a prime. A group $G$ is called a $C_{p, p^r}$-group if and only if $p \in \pi(G)$ and the centralizers of its elements of order $p$ in $G$ are $p$-groups.

The significance of the prime graphs of finite groups can be found in many articles, for example \[9, 18, 21\]. Therefore, the characterizations of finite groups by their orders and degree patterns may help us to know certain properties of the almost simple groups more clearly. In a series of articles (see \[10, 11, 22, 23\]), it was shown that many finite almost simple groups are OD-characterizable. We point out some of these results.

**Result 1** (\[10, 11\]). All sporadic simple groups and their automorphism groups except $\text{Aut}(J_2)$ and $\text{Aut}(M^2L)$ are OD-characterizable.

**Result 2** (\[10\]). The alternating groups $A_p$, $A_{p+1}$, $A_{p+2}$ and the symmetric groups $S_p$ and $S_{p+1}$, where $p$ is a prime, are OD-characterizable.

**Result 3** (\[10, 11\]). The simple groups of Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, $^2B_2(q)$ and $^2G_2(q)$ are OD-characterizable for certain $q \in \mathbb{N}$.

**Result 4** (\[10\]). All finite simple $C_{2, 2}$-groups are OD-characterizable.

**Result 5** (\[23\]). All finite simple groups with exactly four prime divisors except $A_{10}$ are OD-characterizable.

2. **Lemmas**

**Lemma 2.1** (\[9, Table 1\]). Let $G$ be an almost simple group related to $L := L_2(49)$. Then $G$ is isomorphic to one of the following groups: $L$, $L : 2_1 (\cong \text{PGL}(2, 49))$, $L : 2_2$, $L : 2_3$, $L : 2^2 (\cong \text{Aut}(L_2(49)))$. Moreover, $\pi(L) = \{25, 24, 7\}$, $\pi_e(L : 2_1) = \{50, 48, 7\}$, $\pi_e(L : 2_2) = \{25, 24, 14\}$, $\pi_e(L : 2_3) = \{25, 24, 16, 7\}$, and $\pi_e(L : 2^2) = \{50, 48, 14\}$. More information about the algorithm can be obtained in \[8\].

**Lemma 2.2** (\[5, Theorem 1\]). Let $G$ be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$. 
Lemma 2.3 ([9 Table 1]). If $S$ is a finite non-abelian simple groups such that $\pi(S) \subseteq \{2, 3, 5, 7\}$, then $S$ is isomorphic to one of the following simple groups in Table 1. In particular, $\{2, 3\} \subseteq \pi(S)$ and $\pi(\text{Out}(S)) \subseteq \{2, 3\}$ if $S \neq S_6(2)$.

Table 1. Finite non-abelian simple groups $S$ such that $\pi(S) \subseteq \{2, 3, 5, 7\}$

<table>
<thead>
<tr>
<th>$S$</th>
<th>Order of $S$</th>
<th>Out($S$)</th>
<th>$S$</th>
<th>Order of $S$</th>
<th>Out($S$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5$</td>
<td>$2^2 \cdot 3 \cdot 5$</td>
<td>2</td>
<td>$L_2(49)$</td>
<td>$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>$L_2(7)$</td>
<td>$2^3 \cdot 3 \cdot 7$</td>
<td>2</td>
<td>$U_3(5)$</td>
<td>$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$2^3 \cdot 3^2 \cdot 5$</td>
<td>$2^2$</td>
<td>$A_9$</td>
<td>$2^6 \cdot 3^4 \cdot 5 \cdot 7$</td>
<td>2</td>
</tr>
<tr>
<td>$L_2(8)$</td>
<td>$2^3 \cdot 3^2 \cdot 7$</td>
<td>3</td>
<td>$J_2$</td>
<td>$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$</td>
<td>2</td>
</tr>
<tr>
<td>$A_7$</td>
<td>$2^3 \cdot 3^2 \cdot 5 \cdot 7$</td>
<td>2</td>
<td>$S_6(2)$</td>
<td>$2^9 \cdot 3^4 \cdot 5 \cdot 7$</td>
<td>1</td>
</tr>
<tr>
<td>$U_3(3)$</td>
<td>$2^5 \cdot 3^3 \cdot 7$</td>
<td>2</td>
<td>$A_10$</td>
<td>$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$</td>
<td>2</td>
</tr>
<tr>
<td>$A_8$</td>
<td>$2^6 \cdot 3^2 \cdot 5 \cdot 7$</td>
<td>2</td>
<td>$U_4(3)$</td>
<td>$2^7 \cdot 3^6 \cdot 5 \cdot 7$</td>
<td>$D_8$</td>
</tr>
<tr>
<td>$L_3(4)$</td>
<td>$2^6 \cdot 3^2 \cdot 5 \cdot 7$</td>
<td>$D_{12}$</td>
<td>$S_4(7)$</td>
<td>$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$</td>
<td>2</td>
</tr>
<tr>
<td>$U_4(2)$</td>
<td>$2^6 \cdot 3^4 \cdot 5$</td>
<td>2</td>
<td>$O_8^+(2)$</td>
<td>$2^{12} \cdot 3^6 \cdot 5^2 \cdot 7$</td>
<td>$S_3$</td>
</tr>
</tbody>
</table>

Now we quote two lemmas on Frobenius groups.

Lemma 2.4 ([10 Theorem 1]). Let $G$ be a Frobenius group of even order with $H$ and $K$ its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$.

Lemma 2.5 ([10]. Let $G$ be a Frobenius group with kernel $F$ and complement $C$. Then the following assertions are true.

(a) $F$ is a nilpotent group.
(b) $|F| \equiv 1 (\text{mod} |C|)$.
(c) Every subgroup of $C$ of order $p \cdot q$, with $p, q$ (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of $C$ of odd order is cyclic and Sylow 2-subgroup of $C$ is either cyclic or generalized quaternion group. If $C$ is a non-solvable group, then $C$ has a subgroup of index at most 2 isomorphic to $\text{SL}(2, 5) \times M$, where $M$ has cyclic Sylow $p$-subgroups and $(|M|, 30) = 1$; in particular, $15, 20 \notin \pi_e(C)$. If $C$ is solvable and $O(C) = 1$, then either $C$ is a 2-group or $C$ has a subgroup of index at most 2 isomorphic to $\text{SL}(2, 3)$.

A group $G$ is a 2-Frobenius group if there exists a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. Now we quote a lemma on 2-Frobenius groups.

Lemma 2.6 ([10 Theorem 2]). Let $G$ be a 2-Frobenius group of even order, which has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. Then
(a) \( t(G) = 2 \) and \( T(G) = \{ \pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H) \} \).

(b) \( G/K \) and \( K/H \) are cyclic, \( |G/K| \mid |\text{Aut}(K/H)| \), and \( (|G/K|, |K/H|) = 1 \).

(c) \( H \) is a nilpotent group and \( G \) is a solvable group.

The structure of a finite group with disconnected prime graph is described in the following lemma. Though this lemma is a useful tool for the groups with disconnected prime graphs, we should not use it if a finite group has only one connected component.

**Lemma 2.7** ([7] [L7] Theorem A]). Let \( G \) be a finite group with \( t(G) \geq 2 \), then \( G \) is one of the following groups:

(a) \( G \) is a Frobenius or 2-Frobenius group;

(b) \( G \) has a normal series \( 1 \leq H \leq K \leq G \) such that \( H \) and \( G/K \) are \( \pi_1 \)-groups and \( K/H \) is a finite non-abelian simple group, where \( \pi_1 \) is the prime graph component containing \( 2 \), \( H \) is a nilpotent group, and \( |G/K| \mid |\text{Aut}(K/H)| \). Moreover, any odd order component of \( G \) is also an odd order component of \( K/H \).

**Lemma 2.8** ([2] Theorem]). Let \( G \) be a finite non-abelian simple \( C_{p,p} \)-group, where \( p \in \mathbb{P} \).

(a) If \( p = 5 \), then \( G \) is isomorphic to one of the following simple groups: \( A_5, A_6, A_7, M_{11}, M_{22}, L_3(4), S_4(3), S_4(7), U_4(3), S_2(8), S_2(32), L_2(49), L_2(5^m), L_2(2 \cdot 5^m \pm 1) \), where \( m \in \mathbb{N} \) and \( 2 \cdot 5^m \pm 1 \in \mathbb{P} \).

(b) If \( p = 7 \), then \( G \) is isomorphic to one of the following simple groups: \( A_7, A_8, A_9, J_1, M_{22}, J_2, HS, L_3(4), S_6(2), O_8^+(2), G_2(3), G_2(13), U_3(3), U_3(5), U_3(19), U_4(3), U_6(2), S_2(8), L_2(8), L_2(7^m), L_2(2 \cdot 7^m - 1) \), where \( m \in \mathbb{N} \) and \( 2 \cdot 7^m - 1 \in \mathbb{P} \).

**Lemma 2.9** ([22] Theorem]). If \( G \) is a finite group such that \( D(G) = D(M) \) and \( |G| = |M| \), where \( M = U_4(3) : 2_2 \) or \( U_4(3) \cdot 2_3 \), then \( G \cong U_4(3) : 2_2 \) or \( U_4(3) \cdot 2_3 \).

3. OD-characterization of almost simple groups related to \( L_2(49) \)

**Theorem 3.1.** If \( G \) is a finite group such that \( D(G) = D(M) \) and \( |G| = |M| \), where \( M \) is an almost simple group related to \( L := L_2(49) \), then the following assertions are true:

(a) If \( M = L, L : 2_1, L : 2_2 \) or \( L \cdot 2_3 \), then \( G \cong M \).

(b) If \( M = L \cdot 2^2 \), then \( G \cong L \cdot 2^2, \mathbb{Z}_2 \times (L : 2_1), \mathbb{Z}_2 \times (L : 2_2), \mathbb{Z}_2 \times (L \cdot 2_3), \mathbb{Z}_2 \cdot (L : 2_1), \mathbb{Z}_2 \cdot (L : 2_2), \mathbb{Z}_2 \cdot (L \cdot 2_3), \mathbb{Z}_4 \times L \) or \( (\mathbb{Z}_2 \times \mathbb{Z}_2) \times L \).

In particular, \( L, L : 2_1, L : 2_2 \) and \( L \cdot 2_3 \) are OD-characterizable; \( L \cdot 2^2 \) is 9-fold OD-characterizable.

**Proof.** By Lemma 2.1, first we list the prime graphs of the almost simple groups related to \( L \) as follows:
Thus, $G_2$ is $3$-saturated if $2$.

Case 3. If $M = L$, then $G \cong L$ by Result 5.

Case 2. If $M = L_2$, then $G \cong L_2$.

If $M = L_2$, then $\Gamma(G) = \Gamma(M)$ by our assumptions.

First let $G$ be a solvable group. Then $G$ has a solvable Hall $\{3, 5, 7\}$-subgroup $H$. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in $H$ are of prime power order. Hence $t(H) \leq 2$ by Lemma 2.2, a contradiction. Thus $G$ is not solvable, which implies that $G$ is not a 2-Frobenius group by Lemma 2.6(c). If $G$ is a non-solvable Frobenius group with $H$ and $K$ being its Frobenius complement and Frobenius kernel, respectively, then, by Lemma 2.3(c), it follows that $H$ has a normal subgroup $H_0$ with $|H : H_0| \leq 2$ such that $H_0 = \text{SL}(2, 5) \times Z_2$, where the Sylow subgroups of $Z$ are cyclic and $(|Z|, 30) = 1$. Thus $7 \in \pi(K)$ since $5 \sim 7$ in $\Gamma(G)$. Since $|G| = |M| = 2^5 \cdot 3 \cdot 5 \cdot 7^2$, it follows that $5 \not\sim 7$ in $\Gamma(K)$ too. Because $K$ is nilpotent by Lemma 2.5(a), it follows that $5 \sim 7$ in $\Gamma(K)$, an obvious contradiction. Hence $G$ is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, $G$ has a normal series $1 \leq N \leq G_1 \leq G$ such that $N$ is a nilpotent $\pi_1$-group, $G_1/N$ is a finite simple $C_7$-group and $G/G_1$ is a solvable $\pi_1$-group. By Lemmas 2.3 and 2.8(b), we obtain that $G_1/N$ must be isomorphic to $L$.

Since $G/N \leq \text{Aut}(G_1/N)$, it follows that $L \leq G/N \leq \text{Aut}(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/G_1(N) \cong \text{Aut}(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_7 \in \text{Syl}_7(G)$. Then $NG_7$ is a subgroup of $G$, which implies that $2 \sim 7$ in $\Gamma(NG_7)$, an obvious contradiction. Therefore $G/N \cong L_2$ since $|G| = 2 |L|$.

It follows that $G \cong L_2$ since $2 \sim 5$ in $\Gamma(L_2)$ and $\Gamma(L_2)$.

Case 3. If $M = L_2$, then $G \cong L_2$.

If $M = L_2$, then $\Gamma(G) = \Gamma(M)$.

First let $G$ be a solvable group. Then $G$ has a solvable Hall $\{3, 5, 7\}$-subgroup $H$. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in $H$ are of prime power order. Hence $t(H) \leq 2$ by Lemma 2.2, a contradiction. Thus $G$ is not solvable, which implies that $G$ is not a 2-Frobenius group by Lemma

\[
\begin{array}{c|ccc}
\Gamma(L) & 2 & 3 & 5 & 7 \\
\hline
\Gamma(L : 2_1) & 3 & 2 & 5 & 7 \\
\Gamma(L : 2_2) & 3 & 2 & 7 & 5 \\
\Gamma(L \cdot 2_3) & 3 & 2 & 5 & 7 \\
\Gamma(L \cdot 2^2) & 3 & 2 & 5 & 7
\end{array}
\]
2.6(c). If $G$ is a non-solvable Frobenius group with $H$ and $K$ being its Frobenius complement and Frobenius kernel, respectively, then, by Lemma 2.5(c), it follows that $H$ has a normal subgroup $H_0$ with $|H : H_0| \leq 2$ such that $H_0 = \text{SL}(2, 5) \times Z$, where the Sylow subgroups of $Z$ are cyclic and $(|Z|, 30) = 1$. Thus $7 \in \pi(K)$ since $5 \sim 7$ in $\Gamma(G)$. Since $|G| = |M| = 2^5 \cdot 5^2 \cdot 7^2$ and $|\text{SL}(2, 5)| = 2^3 \cdot 3 \cdot 5$, it follows that $5 \in \pi(K)$ too. Because $K$ is nilpotent by Lemma 2.5(a), it follows that $5 \sim 7$ in $\Gamma(K)$, an obvious contradiction. Hence $G$ is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, $G$ has a normal series $1 \leq N \leq G_1 \leq G$ such that $N$ is a nilpotent $\pi_1$-group, $G_1/N$ is a finite simple $C_{5,5}$-group and $G/G_1$ is a solvable $\pi_1$-group. By Lemmas 2.3 and 2.8(a), we obtain that $G_1/N$ must be isomorphic to $L$.

Since $G/N \lessdot \text{Aut}(G_1/N)$, it follows that $L \lessdot G/N \lessdot \text{Aut}(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lessdot \text{Aut}(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_5 \in \text{Syl}_5(G)$. Then $NG_5$ is a subgroup of $G$, which implies that $2 \sim 5$ in $\Gamma(NG_5)$, an obvious contradiction. Therefore $G/N \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ since $|G| = 2|L|$. It follows that $G \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ by Lemma 2.1. Obviously, $G \cong L : 2_2$ since $2 \sim 7$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$.

**Case 4.** If $M = L : 2_3$, then $G \cong L : 2_3$.

If $M = L : 2_3$, then $\Gamma(G) = \Gamma(M)$. Thus $t(G) = t(M) = 3$. By Lemmas 2.4 and 2.6(a), $G$ is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, $G$ has a normal series $1 \leq N \leq G_1 \leq G$ such that $N$ is a nilpotent $\pi_1$-group, $G_1/N$ is a finite simple $C_{5,5}$- and $C_{7,7}$-group, and $G/G_1$ is a solvable $\pi_1$-group. By Lemmas 2.3 and 2.8(a), we obtain that $G_1/N$ must be isomorphic to $L$.

Since $G/N \lessdot \text{Aut}(G_1/N)$, it follows that $L \lessdot G/N \lessdot \text{Aut}(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lessdot \text{Aut}(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_5 \in \text{Syl}_5(G)$. Then $NG_5$ is a subgroup of $G$, which implies that $2 \sim 5$ in $\Gamma(NG_5)$, an obvious contradiction. Therefore $G/N \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ since $|G| = 2|L|$. It follows that $G \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ by Lemma 2.1. Obviously, $G \cong L : 2_2$ since $2 \sim 7$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$, respectively.

**Case 5.** If $M = L : 2^2$, then $G \cong L : 2^2$, $Z_2 \times (L : 2_1)$, $Z_2 \times (L : 2_2)$, $Z_2 \times (L : 2_3)$, $Z_2 \cdot (L : 2_1)$, $Z_2 \cdot (L : 2_2)$, $Z_2 \cdot (L : 2_3)$, $Z_4 \times L$ or $(Z_2 \times Z_2) \times L$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-subgroup. In particular, $G$ is non-solvable.

If $M = L : 2^2$, then $\Gamma(G) = \Gamma(M)$.

First assume that $\{5, 7\} \subseteq \pi(K)$. Let $T$ be a Hall $\{5, 7\}$-subgroup of $K$. It is easy to see that $T$ is an abelian subgroup of order $5^i \cdot 7^j$, where $i, j = 1$ or 2. Thus $5 \cdot 7 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Next, we assume that $5 \in \pi(K)$ and $7 \notin \pi(K)$. Then $K$ is a $\{2, 3, 5\}$-group. Let $R \in \text{Syl}_5(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order 7, say $x$. Now $xR > R$ is a subgroup of $G$ of order $5^i \cdot 7$, where $i = 1$ or 2. Hence $\langle x \rangle R$ is an abelian group. Thus $5 \cdot 7 \in \pi_e(\langle x \rangle R) \subseteq \pi_e(G)$, a contradiction. Finally,
we assume \( 7 \in \pi(K) \) and \( 5 \notin \pi(K) \). In this case, \( K \) is a \( \{2,3,7\} \)-subgroup and we consider the Sylow 7-subgroup \( P \) of \( K \). As before, we see that \( G = KN_G(P) \) and by a similar argument we get \( 5 \cdot 7 \in \pi_e(G) \), which is a contradiction. Thus \( K \) is a \( \{2,3\} \)-subgroup.

Let \( G \) be a solvable group. Then \( G \) has a solvable Hall \( \{3,5,7\} \)-subgroup \( H \). Since there exists no edge between 3, 5 and 7 in \( \Gamma(G) \), it implies that all elements in \( H \) are of prime power order. Hence \( t(H) \leq 2 \) by Lemma 2.2, a contradiction. Thus \( G \) is not solvable.

**Step 2.** The quotient \( G/K \) is an almost simple group. In fact, \( S \leq G/K \leq \text{Aut}(S) \) where \( S \) is a finite non-abelian simple group isomorphic to \( A_5 \), \( L_2(7) \) or \( L \).

Let \( G := G/K \). Then \( S := \text{Soc}(G) = P_1 \times P_2 \times \cdots \times P_m \), where \( P_i \)'s are finite non-abelian simple groups. It is obvious that \( \{2,3\} \subseteq \pi(P_i) \subseteq \{2,3,5,7\} \) by Lemma 2.3, where \( i = 1, 2, \ldots, m \). Now we assert that \( C_G(S) \leq \{2,3\} \)-group by Burnside’s Theorem. It follows that \( 5 \leq \pi(C_G(S)) \) or \( 7 \leq \pi(C_G(S)) \), which shows that \( 3 \cdot 5 \leq \pi_e(C_G(S)) \) or \( 3 \cdot 7 \leq \pi_e(C_G(S)) \) since \( \{2,3\} \subseteq \pi(P_1) \subseteq \pi(S) \). It follows that \( 3 \cdot 5 \leq \pi_e(G) \) or \( 3 \cdot 7 \leq \pi_e(G) \), which is a contradiction since \( 3 \sim 5 \) and \( 3 \sim 7 \) in \( \Gamma(G) \). Suppose \( 3 \not\mid t(K) =: \text{Aut}(S) \), which is solvable. Then \( T/K \not\leq G/K \) since \( G/K \) is non-solvable. Thus \( K \not\leq T \not\leq G \), where \( T \) is solvable. This is a contradiction by the choice of \( K \). Hence \( C_G(S) = 1 \). It follows that \( S \leq G/K \cong G/K/C_G(S) \leq \text{Aut}(S) \).

By Lemma 2.3, it is clear that \( m = 1 \) since \( |G|_3 = 3 \), where \( |G|_3 \) is the 3-part of \( |G| \). Using Table 1, \( S \) is isomorphic to one of the following simple groups: \( A_5 \), \( L_2(7) \) or \( L \).

**Step 3.** \( G \cong L \cdot 2^2 \cdot Z_2 \times (L: 2_1) \), \( Z_2 \times (L: 2_2) \), \( Z_2 \times (L: 2_3) \), \( Z_2 \cdot (L: 2_1) \), \( Z_2 \cdot (L: 2_2) \), \( Z_2 \cdot (L: 2_3) \), \( Z_4 \times L \) or \( (Z_2 \times Z_2) \times L \).

By Step 2, \( S \leq G/K \leq \text{Aut}(S) \) where \( S \) is a finite non-abelian simple group isomorphic to \( A_5 \), \( L_2(7) \) or \( L \).

If \( S \cong A_5 \), then \( A_5 \leq \overline{G} \leq \text{Aut}(A_5) \). It follows that \( |K| = 2^4 \cdot 5 \cdot 7^2 \) or \( 2^3 \cdot 5 \cdot 7^2 \) by Lemma 2.3. Obviously, this is a contradiction since \( K \) is a \( \{2,3\} \)-group by Step 1.

If \( S \cong L_2(7) \), then \( L_2(7) \leq \overline{G} \leq \text{Aut}(L_2(7)) \). It follows that \( |K| = 2^3 \cdot 5 \cdot 7 \) or \( 2^2 \cdot 5^2 \cdot 7 \) by Lemma 2.3. Obviously, this is a contradiction since \( K \) is a \( \{2,3\} \)-group by Step 1.

Therefore, \( S \cong L \). Thus \( L \leq \overline{G} \leq \text{Aut}(L) \). Hence \( |K| = 1, 2 \) or \( 2^2 \).

If \( |K| = 1 \), then \( G \cong L \cdot 2^2 \) by Lemma 2.1.

If \( |K| = 2 \), then \( K \leq Z(G) \), i.e., \( G \) is a central extension of \( K \) by \( L : 2_1 \), \( L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), we obtain \( G \cong Z_2 \times (L : 2_1) \), \( Z_2 \times (L : 2_2) \) or \( Z_2 \times (L : 2_3) \). Otherwise we have \( G \cong Z_2 \cdot (L : 2_1) \), \( Z_2 \cdot (L : 2_2) \) or \( Z_2 \cdot (L : 2_3) \).

If \( |K| = 2^2 \), then \( G/K \cong L \). In this case, we have \( G/C_G(K) \leq \text{Aut}(K) \cong Z_2 \) or \( S_3 \). Thus \( |G/C_G(K)| = 1, 2, 3 \) or \( 6 \). If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), i.e., \( G \) is a central extension of \( K \) by \( L \). If \( G \) is a non-split extension of \( K \) by \( L \), then \( |K| \) must divide the Schur multiplier of \( L \), which is 2 (see [3]). But this is a contradiction. So we obtain that \( G \) splits over \( K \). Hence \( G \cong K \times L \). Thus \( G \cong Z_4 \times L \) or
Theorem 3.3. If \(|G/C_G(K)| = 2, 3\) or 6, then \(K < C_G(K)\) and \(1 \neq C_G(K)/K \leq G/K \cong L\). Since \(L\) is simple, we obtain that \(G = C_G(K)\), a contradiction. \(\square\)

Remark 1. W. J. Shi and J. X. Bi in [16] put forward the following conjecture:

Conjecture. Let \(G\) be a finite group and \(M\) a finite simple group. Then \(G \cong M\) if and only if \(|G| = |M|\) and \(\pi_e(G) = \pi_e(M)\).

This conjecture is valid for the sporadic simple groups (see [14]), alternating groups and some simple groups of Lie type (see [13] [15] [16]). As a consequence of Theorem 3.1, we verify the validity of this conjecture for the groups under discussion.

Theorem 3.2. If \(G\) is a finite group such that \(|G| = |M|\) and \(\pi_e(G) = \pi_e(M)\), where \(M\) is an almost simple group related to \(L_2(49)\) except \(L_2(49) \cdot 2^2\), then \(G \cong M\).

Proof. Since \(|G| = |M|\) and \(\pi_e(G) = \pi_e(M)\), we obtain \(|G| = |M|\) and \(\Gamma(G) = \Gamma(M)\). It follows that \(|G| = |M|\) and \(D(G) = D(M)\). By Theorem 3.1, we have \(G \cong M\). \(\square\)

Note that if \(G\) is a finite group such that \(|G| = |M|\) and \(D(G) = D(M)\), where \(M\) is a given finite group, then \(\pi_e(G)\) is not equal to \(\pi_e(M)\) necessarily. Now, we give a counterexample as follows. Let \(L := U_4(3)\), then \(L: 2_2\) is 2-fold \(OD\)-characterizable by Lemma 2.9. However, in this case, \(\pi_e(L: 2_2) = \{18, 12, 10, 8, 7\}\) is not equal to \(\pi_e(L \cdot 2_3) = \{24, 10, 9, 7\}\) (see [9]).

Theorem 3.3. If \(G\) is a finite group such that \(|G| = |M|\) and \(\Gamma(G) = \Gamma(M)\), where \(M\) is an almost simple group related to \(L_2(49)\) except \(L_2(49) \cdot 2^2\), then \(G \cong M\).

Proof. Since \(|G| = |M|\) and \(\Gamma(G) = \Gamma(M)\), we obtain that \(|G| = |M|\) and \(D(G) = D(M)\). By Theorem 3.1, we have \(G \cong M\). \(\square\)

Question. Let \(G\) be a finite group such that \(D(G) = D(M)\) and \(|G| = |M|\), where \(M\) is an almost simple group. Is \(G\) non-solvable, too?

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